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Multiple blowing-up and concentrating solutions for Liouville-type equations with singular sources under mixed boundary conditions

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Abstract

In this article, we mainly construct multiple blowing-up and concentrating solutions for a class of Liouville-type equations under mixed boundary conditions:

$$\begin{cases} -\Delta v = \varepsilon^2 e^v - 4\pi \sum_{i=1}^N \alpha_i \delta_{p_i}, \text{ in } \ \Omega, \\ \varepsilon(1-t)\dfrac{\partial v}{\partial \nu} + tb(x)v = 0, \qquad \text{on } \partial\Omega, \end{cases}$$

for ε small, where $t \in (0,1], N \in \mathbb{N} \cup \{0\}, \{\alpha_1, \alpha_2, \ldots, \alpha_N\} \subset (-1, +\infty) \backslash (\mathbb{N} \cup \{0\})$, Ω is a bounded, smooth domain in $\mathbb{R}^2$, $\Gamma := \{p_1, ..., p_N\} \subset \Omega$ is the set of singular sources, δ_p denotes the Dirac mass at p, ν denotes unit outward normal vector to $\partial\Omega$ and $b(x) > 0$ is a smooth function on $\partial\Omega$.

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Keywords: multiple blowing-up and concentrating solution, Liouville-type equation, singular source, mixed boundary conditions, finite dimensional reduction

1 Introduction

In this article, we mainly investigate the mixed boundary value problem:

$$\begin{cases} -\Delta v = \varepsilon^2 e^v - 4\pi \sum_{i=1}^N \alpha_i \delta_{p_i}, \text{ in } \ \Omega, \\ \varepsilon(1-t)\dfrac{\partial v}{\partial \nu} + tb(x)v = 0, \qquad \text{on } \partial\Omega, \end{cases} \tag{1}$$

for ε small, where $t \in (0,1], N \in \mathbb{N} \cup \{0\}, \{\alpha_1, \alpha_2, \ldots, \alpha_N\} \subset (-1, +\infty) \backslash (\mathbb{N} \cup \{0\})$, Ω is a bounded, smooth domain in $\mathbb{R}^2$, $\Gamma := \{p_1, ..., p_N\} \subset \Omega$ is the set of singular sources, δ_p denotes the Dirac mass at p, ν denotes unit outward normal vector to $\partial\Omega$ and $b(x) > 0$ is a smooth function on $\partial\Omega$.

Such problems occur in conformal geometry [1], statistical mechanics [2-4], Chern-Simons vortex theory [5-11] and several other fields of applied mathematics [12-16]. In all these contexts, an interesting point is how to construct solutions which exactly "blow-up" and "concentrate" at some given points, whose location carries relevant information about the potentially geometrical or physical properties of the problem. However, the authors mainly consider the Dirichlet boundary value problem, and little is known for the problem with singular sources satisfying $\alpha_i \in (-1, 0)$ for some $i = 1, ..., N$. The main purpose of this article is to study how to construct multiple blowing-up and concentrating solutions of the Equation (1) with the mixed boundary conditions and singular sources.

Let $G_{t,\varepsilon}$ denotes the Green's function of $-\Delta$ with mixed boundary conditions on Ω, namely for any $y \in \Omega$,

$$\begin{cases} -\Delta_x G_{t,\varepsilon}(x,y) = 2\pi\delta_y(x), & \text{in } \Omega, \\ \varepsilon(1-t)\dfrac{\partial G_{t,\varepsilon}(x,y)}{\partial \nu} + tb(x)G_{t,\varepsilon}(x,y) = 0, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

and let $H_{t,\varepsilon}(x, y) = G_{t,\varepsilon}(x, y) + \log|x - y|$ be its regular part. Set $G_1 = G_{1,\varepsilon}$ and $H_1 = H_{1,\varepsilon}$. Since ε exactly disappears in the Equation $(2)|_{t=1}$, G_1 and H_1 don't depend on ε. The Equation (1) is equivalent to solving for $u = v + 2\sum_{i=1}^N \alpha_i G_{t,\varepsilon}(x,p_i)$, the regular part of v, the equation

$$\begin{cases} -\Delta u = \varepsilon^2 |x-p_1|^{2\alpha_1} \dots |x-p_N|^{2\alpha_N} e^{-2\sum_{i=1}^N \alpha_i H_{t,\varepsilon}(x,p_i)} e^u, & \text{in } \Omega, \\ \varepsilon(1-t)\dfrac{\partial u}{\partial \nu} + tb(x)u = 0, & \text{on } \partial\Omega. \end{cases}$$

Thus, we consider the more general model problem:

$$\begin{cases} -\Delta u = \varepsilon^2 |x-p_1|^{2\alpha_1} \dots |x-p_N|^{2\alpha_N} f(x) e^u, & \text{in } \Omega, \\ \varepsilon(1-t)\dfrac{\partial u}{\partial \nu} + tb(x)u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $f : \Omega \to \mathbb{R}$ is a smooth function such that $f(pi) > 0$ for any $i = 1, ..., N$. Set $\Omega' = \{x \in \Omega: f(x) > 0\}$, $S(x) = |x-p_1|^{2\alpha_1} \dots |x-p_N|^{2\alpha_N}$ and $\Delta_m = \{p = (p_1, ..., p_m) \in \Omega^m$: $p_i = pj$ for some $i \neq j\}$.

It is known that for $\{\alpha_1, \dots, \alpha_N\} \subset (0, +\infty)\backslash\mathbb{N}$, or $\alpha_i = 0$ for any $i = 1, ..., N$, if u_ε is a family of solutions of the Equation $(3)|_{t=1}$ with $\inf_\Omega f > 0$, which is not uniformly bounded from above for ε small, then u_ε blows up at different points $p_{k_1}, \dots, p_{k_{n+m}}$ with $n + m \geq 1$, $0 \leq n \leq N$, $p = (p_{k_{n+1}}, \dots, p_{k_{n+m}}) \in (\Omega'\backslash\Gamma)^m\backslash\Delta_m$ and $\{p_{k_1}, \dots, p_{k_n}\} \subset \Gamma$, and satisfies the concentration property:

$$\varepsilon^2 |x-p_1|^{2\alpha_1} \dots |x-p_N|^{2\alpha_N} f(x) e^{u_\varepsilon} \rightharpoonup 8\pi \sum_{i=1}^n (1+\alpha_{k_i})\delta_{p_{k_i}} + 8\pi \sum_{i=n+1}^{n+m} \delta_{p_{k_i}}, \tag{4}$$

in the sense of measures in $\bar{\Omega}$. Moreover, $p = (p_{k_{n+1}}, \dots, p_{k_{n+m}})$ is a critical point of the function:

$$\begin{aligned} \varphi_{n,m}(p) = &\sum_{i=n+1}^{n+m} \left[H_1(p_{k_i}, p_{k_i}) + \frac{1}{2}\log f(p_{k_i}) S(p_{k_i}) \right] + \sum_{i,j=n+1, i\neq j}^{n+m} G_1(p_{k_j}, p_{k_i}) \\ &+ 2\sum_{i=1}^n \sum_{j=n+1}^{n+m} (1+\alpha_{k_i}) G_1(p_{k_j}, p_{k_i}), \end{aligned} \tag{5}$$

(see [7,17-23]). An obvious problem for the Equation (3) is the reciprocal, namely the existence of multiple blowing-up solutions with concentration points near critical points of $\phi_{n,m}$.

The earlier result concerning the existence of multiple blowing-up and concentrating solutions of the Equation (3) is given by Baraket and Pacard in [24]. When $t = 1$ and $\alpha_i = 0$ for any $i = 1,2, ..., N$, they prove that any non-degenerate critical point

$p = (p_{k_{n+1}}, \dots, p_{k_{n+m}})$ of the function $\phi_{n,m}$ with $n = 0$ generates a family of the solutions u_ε which blow-up at $p_{k_{n+1}}, \dots, p_{k_{n+m}}$, and concentrate in the sense that (4) holds. Esposito [20] performs a similar asymptotic analysis and extends the previous result by allowing the presence of singular sources in the Equation $(3)|_{t=1}$, that is, $\{\alpha_1, \dots, \alpha_N\} \subset (0, +\infty)\backslash\mathbb{N}$. However, the asymptotic analysis method depends on the non-degenerate assumption of critical point of the function $\phi_{n,m}$ so much that it pays in return at a price of the very complicated and accurate control on the asymptotics of the solutions.

In fact, the finite dimensional reduction method, used successfully in higher dimensional nonlinear elliptic equation involving critical Sobolev exponent (see [6,25]), can avoid the technical difficulty in carrying out the asymptotic analysis method for the Equation (3). It is necessary to point out that the key step of the finite dimensional reduction is the analysis of the bounded invertibility of the corresponding linearized operator L of the Equation (3) at the suitable approximate solution. In [26,27], the authors construct the approximate solution, carry out the finite dimensional reduction and use some stability assumptions of critical points of $\phi_{0,m}$ to get the existence of multiple blowing-up and concentrating solutions for the Equation $(3)|_{t=1}$ with $\Gamma = \emptyset$, namely $\alpha_i = 0$ for any $i = 1,2, \dots, N$. When $\{\alpha_1, \dots, \alpha_N\} \subset (0, +\infty)\backslash\mathbb{N}$, a similar result for the Equation $(3)|_{t=1}$ under C^0-stable assumption of critical point of $\phi_{n,m}$ (see Definition 4.1) is also established in [28].

Here in the spirit of the finite dimensional reduction, we try to extend the result of the Equation (1) in [20,28] by allowing the presence of singular sources $4\pi \sum_{i=1}^{N} \alpha_i \delta_{p_i}$ with some $\alpha_i \in (-1, 0)$ and Robin boundary conditions $\varepsilon(1-t)\dfrac{\partial v}{\partial \nu} + tb(x)v = 0$ with $t \in (0,1)$. When we carry out the finite dimensional reduction, we need to get the invertibility of the desired linearized operator L for the Equation (3) under some $\alpha_i \in (-1, 0)$. Obviously, the linearized operator L easily produces the singularities at some singular sources with $\alpha_i \in (-1, 0)$, which makes trouble for the analysis of the bounded invertibility of L. But we can successfully get rid of it by introducing a suitable L^∞-weighted norm (see (30) below) related with a "gap interval" $(-1, \alpha_0)$, where $\alpha_0 = \min\{0, \alpha_1, \dots, \alpha_N\}$. On the other hand, the presence of the term $\varepsilon(1-t)\dfrac{\partial u}{\partial \nu}$ in the Equation $(3)|_{0<t<1}$ brings some new technical difficulties. A flexible approach exactly helps us overcome the difficulties by making use of the maximum principle. In addition, a weaker stable assumption of critical points of the function $\phi_{n,m}$ also helps us construct multiple blowing-up and concentrating solutions of the Equation (3). As a consequence, we have the following result.

Theorem 1.1 *Let* $0 \le n \le N$ *and* $m \in \mathbb{N} \cup \{0\}$ *such that* $n + m \ge 1$. *Assume that* $p^* = (p^*_{n+1}, \dots, p^*_{n+m})$ *and* $p^* = (p^*_{n+1}, \dots, p^*_{n+m})$ *is a* C^0*-stable critical point for* $\phi_{n,m}$ *in* $(\Omega' \setminus \Gamma)^m \setminus \Delta_m$ *with* $m \ge 1$ *(see Definition 4.1). Then there exists a family of solutions* u_ε *for the Equation* (3) *with the concentration property* (4), *which blow up at n-different points* $(p_{k_1}, \dots, p_{k_n})$ *in* Γ, *and m-points* $p = (p_{k_{n+1}}, \dots, p_{k_{n+m}})$ *in* $(\Omega' \setminus \Gamma)^m \setminus \Delta_m$ *with* $\phi_{n,m}(p^*) = \phi_{n,m}(p)$. *Moreover,* u_ε *remains uniformly bounded on* $\Omega \setminus \cup_{i=1}^{n+m} B_\lambda(p_{k_i})$, *and* $\sup_{B_\lambda(p_{k_i})} u_\varepsilon \to +\infty$ *for any* $\lambda > 0$.

Let us point out that from the proof of Theorem 1.1 Robin boundary condition can be considered as a perturbation of Dirichlet boundary condition for the problem (3) in using perturbation techniques to construct multiple blowing-up and concentrating solutions. Based on this point, we also consider the Dirichlet-Robin boundary value problem:

$$\begin{cases} -\Delta u = \varepsilon^2 \left|x - p_1\right|^{2\alpha_1} \dots \left|x - p_N\right|^{2\alpha_N} f(x) e^u, \text{ in } \Omega, \\ \varepsilon \dfrac{\partial u}{\partial \nu} + b(x) u = 0, \quad \text{on } T, \qquad u = 0, \qquad \text{on } \partial\Omega \backslash T, \end{cases} \tag{6}$$

where $T \subseteq \partial\Omega$ is a relatively closed subset and $\partial\Omega \backslash T \neq \emptyset$. This together with other similar mixed boundary value problems can be founded in [29,30]. For the problem (6), we obtain the following result.

Theorem 1.2 *Under the assumption of Theorem 1.1, then there exists a family of solutions* u_ε *for the Equation* (6) *with the concentration property* (4), *which blow up at n-different points* $(p_{k_1}, \dots, p_{k_n})$ *in* Γ, *and m-points* $p = (p_{k_{n+1}}, \dots, p_{k_{n+m}})$ *in* $(\Omega' \setminus \Gamma)^m \setminus \Delta_m$ *with* $\phi_{n,m}(p^*) = \phi_{n,m}(p)$. *Moreover,* u_ε *remains uniformly bounded on* $\Omega \backslash \cup_{i=1}^{n+m} B_\lambda(p_{k_i})$, *and* $\sup_{B_\lambda(p_{k_i})} u_\varepsilon \to +\infty$ *for any* $\lambda > 0$.

Finally, it is very interesting to mention that to prove the above results we need to choose the classification solutions of the following Liouville-type equation to construct concentrating solutions of the Equation (1) or (3):

$$\begin{cases} -\Delta u = |z|^{2\gamma} e^u, \quad \text{in} \quad \mathbb{R}^2, \\ \int\limits_{\mathbb{R}^2} |z|^{2\gamma} e^u < +\infty, \ \gamma > -1, \end{cases} \tag{7}$$

given by

$$u(z) = \log \frac{8(1+\gamma)^2 \mu^2}{\left(\mu^2 + \left|z^{\gamma+1} - c\right|^2\right)^2}, \tag{8}$$

with $\mu > 0, c \in \mathbb{C}$ if $\gamma \in \mathbb{N} \cup \{0\}$, $c = 0$ if $\gamma \in (-1, +\infty) \backslash (\mathbb{N} \cup \{0\})$ (see [5,11,31,32]). Using these classification solutions scaled up and projected to satisfy the mixed boundary conditions up to a right order, the initial approximate solutions can be built up. Then through the finite dimensional reduction procedure and the notions of stability of critical points of the asymptotic reductional functional $\phi_{n,m}$, multiple blowing-up and concentrating solutions can be constructed as a small additive perturbation of the initial approximations.

This article is organized as follows. In Section 2, we will construct the approximate solution and rewrite the Equation (3) in terms of a linearized operator L. In Section 3, we give the invertibility of the linearized operator L, carry out the finite dimensional reduction and get the asymptotical expansion of the functional of the Equation (3) with respect to the suitable approximate solution. In Section 4, we give the proofs of Theorems 1.1 and 1.2.

2 Construction of the approximate solution

In this section, we will construct the approximate solution for the Equation (3). Let μ_i, $i = 1, \dots, N + m$, be positive numbers and set

$$\alpha_i = 0, \quad \forall \ i = N+1, \dots, N+m,$$

and

$$Q_i(x) = \frac{S(x)}{\left|x - p_i\right|^{2\alpha_i}}.$$

Obviously, $Q_i(x) = S(x)$ for any $i = N + 1, ..., N + m$. Then the function

$$u_i(x) = \log \frac{8\mu_i^2(1+\alpha_i)^2}{\left(\mu_i^2\varepsilon^2 + \left|x - p_i\right|^{2(1+\alpha_i)}\right)^2 f(p_i)Q_i(p_i)}, \tag{9}$$

satisfies

$$-\Delta u_i = \varepsilon^2 \left|x - p_i\right|^{2\alpha_i} f(p_i)Q_i(p_i)e^{u_i}, \quad \text{in } \mathbb{R}^2. \tag{10}$$

Set $\{k_1, ..., k_n\} \subset \{1, ..., N\}$ and $k_{n+i} = N + i$ for any $i = 1, ..., m$.

We hope to take $\sum_{i=1}^{n+m} u_{k_i}$ as an initial approximate solution of the problem (3). So we modify it to be

$$U(x) := \sum_{i=1}^{n+m} U_{k_i} = \sum_{i=1}^{n+m} (u_{k_i} + H_{k_i}^t), \tag{11}$$

where $H_{k_i}^t(x)(x)$ is the solution of

$$\begin{cases} \Delta H_{k_i}^t = 0 & \text{in } \Omega, \\ \varepsilon(1-t)\dfrac{\partial}{\partial \nu} H_{k_i}^t + tb(x)H_{k_i}^t = -\left[\varepsilon(1-t)\dfrac{\partial u_{k_i}}{\partial \nu} + tb(x)u_{k_i}\right], & \text{on } \partial\Omega. \end{cases} \tag{12}$$

Then $U_{k_i} := u_{k_i} + H_{k_i}^t$ satisfies

$$\begin{cases} -\Delta U_{k_i} = \varepsilon^2 \left|x - p_{k_i}\right|^{2\alpha_{k_i}} f(p_{k_i})Q_{k_i}(p_{k_i})e^{u_{k_i}}, & \text{in } \Omega, \\ \varepsilon(1-t)\dfrac{\partial}{\partial \nu} U_{k_i} + tb(x)U_{k_i} = 0, & \text{on } \partial\Omega. \end{cases} \tag{13}$$

Lemma 2.1 *For* $t \in (0, 1]$ *and* $p_{k_i} \in \Omega'$,

$$H_{k_i}^t(x) = 4(1+\alpha_{k_i})H_{t,\varepsilon}(x, p_{k_i}) - \log \frac{8\mu_{k_i}^2(1+\alpha_{k_i})^2}{f(p_{k_i})Q_{k_i}(p_{k_i})} + O(\varepsilon^2), \tag{14}$$

uniformly in $C(\bar{\Omega})$ *and in* $C_{\text{loc}}^2(\Omega)$*for* ε *small.*

Proof. Set $z_t(x) = H_{k_i}^t(x) - 4(1+\alpha_{k_i})H_{t,\varepsilon}(x, p_{k_i}) + \log \dfrac{8\mu_{k_i}^2(1+\alpha_{k_i})^2}{f(p_{k_i})Q_{k_i}(p_{k_i})}$. Then $z_t(x)$ satisfies

$$\begin{cases} \Delta z_t(x) = 0, & \text{in } \Omega, \\ \varepsilon(1-t)\dfrac{\partial z_t(x)}{\partial \nu} + tb(x)z_t(x) = F_t(x), & \text{on } \partial\Omega, \end{cases}$$

where

$$\begin{aligned} F_t(x) = 4\varepsilon(1-t)(1+\alpha_{k_i}) &\left[\left|x - p_{k_i}\right|^{2\alpha_{k_i}} \frac{\nu(x)\cdot(x - p_{k_i})}{\mu_{k_i}^2\varepsilon^2 + \left|x - p_{k_i}\right|^{2(1+\alpha_{k_i})}} - \frac{\nu(x)\cdot(x - p_{k_i})}{\left|x - p_{k_i}\right|^2}\right] \\ &+ tb(x)\left[2\log\left(\mu_{k_i}^2\varepsilon^2 + \left|x - p_{k_i}\right|^{2(1+\alpha_{k_i})}\right) - 4(1+\alpha_{k_i})\log\left|x - p_{k_i}\right|\right]. \end{aligned}$$

For any $t \in (0, 1]$, it is easy to check $\|F_t(x)\|_{L^\infty(\partial\Omega)} = O(\varepsilon^2)$. If $t = 1$, from the maximum principle and smooth function $b(x) > 0$, it follows

$$\max_{\bar{\Omega}} |z_1(x)| = \max_{\partial\Omega} |z_1(x)| \le C(b) \|F_1(x)\|_{L^\infty(\partial\Omega)} = O(\varepsilon^2).$$

If $0 < t < 1$, from the maximum principle with the Robin boundary condition (see [[33], Lemma 2.6]), it also follows

$$\max_{\bar{\Omega}} |z_t(x)| \le \frac{1}{t} C(b) \|F_t(x)\|_{L^\infty(\partial\Omega)} = O(\varepsilon^2).$$

Thus using the interior estimate of derivative of harmonic function (see [[34], Theorem 2.10]), there holds

$$\max_{K} |D^\alpha z_t(x)| \le \left(\frac{2\,|\alpha|}{\mathrm{dist}(K, \partial\Omega)}\right)^{|\alpha|} \max_{\bar{\Omega}} |z_t(x)| = O(\varepsilon^2),$$

for any compact subset K of Ω, any $t \in (0, 1]$ and any multi-index α with $|\alpha| \le 2$, which derives (14) uniformly in $C(\bar{\Omega})$ and in $C^2_{\mathrm{loc}}(\Omega)$ for ε small. □

From this lemma we can construct the approximate solution $U(x) = \sum_{i=1}^{n+m} (u_{k_i} + H^t_{k_i})$, which satisfies the mixed boundary conditions. On the other hand, we hope that the error $U(x) - u_{k_i}$ is smaller near every p_{k_i}. In fact, we can realize this point by further choosing positive number μ_{k_i} such that

$$\log \frac{8\mu_{k_i}^2 (1 + \alpha_{k_i})^2}{f(p_{k_i}) Q_{k_i}(p_{k_i})} = 4(1 + \alpha_{k_i}) H_{t,\varepsilon}(p_{k_i}, p_{k_i}) + 4 \sum_{j=1, j\neq i}^{j=n+m} (1 + \alpha_{k_j}) G_{t,\varepsilon}(p_{k_i}, p_{k_j}). \tag{15}$$

Consider the scaling of the solution of the Equation (3)

$$v(y) = u(\varepsilon y) + 4 \log \varepsilon,$$

then $v(y)$ satisfies

$$\begin{cases} -\Delta v = S(\varepsilon y) f(\varepsilon y) e^v, & \text{in } \Omega_\varepsilon, \\ (1 - t)\dfrac{\partial v}{\partial \nu} + t b(\varepsilon y) v = 4 t b(\varepsilon y) \log \varepsilon, & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{16}$$

where $\Omega_\varepsilon = \frac{1}{\varepsilon}\Omega$.. We also set $p'_{k_i} = \frac{1}{\varepsilon} p_{k_i}$ and define the new approximation in expanded variables as $V(y) = U(\varepsilon y) + 4 \log \varepsilon$. Furthermore, set

$$\rho_{k_i} = \varepsilon^{\frac{1}{1 + \alpha_{k_i}}}, \tag{17}$$

and

$$W(y) = S(\varepsilon y) f(\varepsilon y) e^{V(y)}. \tag{18}$$

Obviously, $\rho_{k_{n+i}} = \varepsilon$ for all $i = 1, ..., m$.

Here, we want to see how well $-\Delta V(y)$ match with $W(y)$ through $V(y)$. A simple computation shows

$$
\begin{aligned}
-\Delta V(y) &= -\varepsilon^2 \Delta_x U(x) \\
&= -\varepsilon^2 \sum_{i=1}^{n+m} \Delta_x \left[u_{k_i}(x) + H_{k_i}^t(x) \right] \\
&= \sum_{i=1}^{n+m} \left(\frac{\varepsilon}{\rho_{k_i}} \right)^2 \frac{8\mu_{k_i}^2 (1+\alpha_{k_i})^2 \left| \frac{\varepsilon y - p_{k_i}}{\rho_{k_i}} \right|^{2\alpha_{k_i}}}{\left[\mu_{k_i}^2 + \left| \frac{\varepsilon y - p_{k_i}}{\rho_{k_i}} \right|^{2(1+\alpha_{k_i})} \right]^2}.
\end{aligned}
$$

Then given a small number $\delta > 0$, if $\left| \frac{\varepsilon y - p_{k_i}}{\rho_{k_i}} \right| > \frac{\delta}{\rho_{k_i}}$ for all $i = 1, ..., n + m$,

$$
-\Delta V(y) = O(\varepsilon^4), \tag{19}
$$

and if $\left| \frac{\varepsilon y - p_{k_i}}{\rho_{k_i}} \right| \le \frac{\delta}{\rho_{k_i}}$ for some i,

$$
-\Delta V(y) = \left(\frac{\varepsilon}{\rho_{k_i}} \right)^2 \frac{8\mu_{k_i}^2 (1+\alpha_{k_i})^2 \left| \frac{\varepsilon y - p_{k_i}}{\rho_{k_i}} \right|^{2\alpha_{k_i}}}{\left[\mu_{k_i}^2 + \left| \frac{\varepsilon y - p_{k_i}}{\rho_{k_i}} \right|^{2(1+\alpha_{k_i})} \right]^2} + O(\varepsilon^4). \tag{20}
$$

On the other hand, if $\left| \frac{\varepsilon y - p_{k_i}}{\rho_{k_i}} \right| > \frac{\delta}{\rho_{k_i}}$ for all $i = 1, ..., n + m$, obviously,

$$
W(y) = O(\varepsilon^4), \tag{21}
$$

and if $\left| \frac{\varepsilon y - p_{k_i}}{\rho_{k_i}} \right| \le \frac{\delta}{\rho_{k_i}}$ for some i,

$$
\begin{aligned}
W(y) &= \left| \varepsilon y - p_{k_i} \right|^{2\alpha_{k_i}} Q_{k_i}(\varepsilon y) f(\varepsilon y) e^{V(y)} \\
&= \varepsilon^4 \frac{8\mu_{k_i}^2 (1+\alpha_{k_i})^2 \left| \varepsilon y - p_{k_i} \right|^{2\alpha_{k_i}}}{\left[\varepsilon^2 \mu_{k_i}^2 + \left| \varepsilon y - p_{k_i} \right|^{2(1+\alpha_{k_i})} \right]^2} \cdot \frac{f(\varepsilon y) Q_{k_i}(\varepsilon y)}{f(p_{k_i}) Q_{k_i}(p_{k_i})} \\
&\quad \times \exp \left\{ H_{k_i}^t(\varepsilon y) + \sum_{j=1, j \ne i}^{n+m} \left[\log \frac{8\mu_{k_j}^2 (1+\alpha_{k_j})^2}{\left(\varepsilon^2 \mu_{k_j}^2 + \left| \varepsilon y - p_{k_j} \right|^{2(1+\alpha_{k_j})} \right)^2 f(p_{k_j}) Q_{k_j}(p_{k_j})} + H_{k_j}^t(\varepsilon y) \right] \right\}.
\end{aligned}
$$

Now from (14), (15) and (17), we have

$$
W(y) = \left(\frac{\varepsilon}{\rho_{k_i}} \right)^2 \frac{8\mu_{k_i}^2 (1+\alpha_{k_i})^2 \left| \frac{\varepsilon y - p_{k_i}}{\rho_{k_i}} \right|^{2\alpha_{k_i}}}{\left[\mu_{k_i}^2 + \left| \frac{\varepsilon y - p_{k_i}}{\rho_{k_i}} \right|^{2(1+\alpha_{k_i})} \right]^2} \left[1 + O\left(\rho_{k_i} \left| \frac{\varepsilon y - p_{k_i}}{\rho_{k_i}} \right| \right) + O(\varepsilon^2) \right]. \tag{22}
$$

In summary, we set

$$
R(y) = \Delta V(y) + W(y), \tag{23}
$$

and if $\left|\frac{\varepsilon y - p_{k_i}}{\rho_{k_i}}\right| > \frac{\delta}{\rho_{k_i}}$ for all $i = 1, ..., n + m$,

$$R(y) = O(\varepsilon^4), \tag{24}$$

while $\left|\frac{\varepsilon y - p_{k_i}}{\rho_{k_i}}\right| \le \frac{\delta}{\rho_{k_i}}$ for some i,

$$R(y) = \left(\frac{\varepsilon}{\rho_{k_i}}\right)^2 \frac{8\mu_{k_i}^2(1+\alpha_{k_i})^2\left|\frac{\varepsilon y - p_{k_i}}{\rho_{k_i}}\right|^{2\alpha_{k_i}}}{\left[\mu_{k_i}^2 + \left|\frac{\varepsilon y - p_{k_i}}{\rho_{k_i}}\right|^{2(1+\alpha_{k_i})}\right]^2}\left[O\left(\rho_{k_i}\left|\frac{\varepsilon y - p_{k_i}}{\rho_{k_i}}\right|\right) + O(\varepsilon^2)\right]. \tag{25}$$

In the rest of this article, we try to find a solution v of the form $v = V + \varphi$ of the Equation (16). In terms of φ, the problem (3) becomes

$$\begin{cases} L\phi = \Delta\phi + W\phi = -[R + N(\phi)], \text{ in } \Omega_\varepsilon, \\ (1-t)\dfrac{\partial\phi}{\partial\nu} + tb(\varepsilon y)\phi = 0, \qquad \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{26}$$

where

$$N(\phi) = W[e^\phi - 1 - \phi]. \tag{27}$$

3 The finite dimensional reduction

In this section, we will carry out the finite dimensional reduction to solve the Equation (26). First of all, we need to get the desired invertibility of linearized operation L. Set

$$z_{i0}(z) = \frac{|z|^{2(1+\alpha_{k_i})-\mu_{k_i}^2}}{|z|^{2(1+\alpha_{k_i})+\mu_{k_i}^2}}, \qquad \text{for } i = 1, 2, \dots, n+m,$$

$$z_{ij}(z) = \frac{4z_j}{|z|^2 + \mu_{k_i}^2}, \quad \text{for } i = n+1, \dots, n+m, \quad j = 1, 2,$$

$$L_i\phi = \Delta\phi + \frac{8\mu_{k_i}^2(1+\alpha_{k_i})^2|z|^{2\alpha_{k_i}}}{\left[\mu_{k_i}^2 + |z|^{2(1+\alpha_{k_i})}\right]^2}\phi, \quad \text{for } i = 1, \dots, n+m.$$

A basic fact to get the needed invertibility is that the linearized operator L formally approaches to the operator L_i under suitable dilations and translations, which have some well-known properties that any bounded solution of $L_t\varphi = 0$ is

- a linear combination of z_{i0} and z_{ij} for $i = n + 1, ..., n + m, j = 1, 2$ (see [24,35]);
- proportional to z_{i0} for $0 < \alpha_{k_i} \notin \mathbb{N}$ and $i = 1, 2, ..., n$ (see [20,28,36]).

Remark 3.1 These properties of the operator L_i have been discussed in the above papers only if $0 < \alpha_{k_i} \notin \mathbb{N}$ for $i = 1, ..., n$, or $\alpha_{k_i} = 0$ for $i = n + 1, ..., n + m$. In fact, if $-1 < \alpha_{k_i} < 0$ for some $i = 1, ..., n$, the operator L_i has also the corresponding properties.

Lemma 3.2 *For* $-1 < \alpha \notin \mathbb{N} \cup \{0\}$, *any bounded solution* φ *of*

$$\Delta\phi + \frac{8\mu^2(1+\alpha)^2|z|^{2\alpha}}{\left(\mu^2 + |z|^{2(1+\alpha)}\right)^2}\phi = 0, \quad \forall z \in \mathbb{C}\backslash\{0\}, \tag{28}$$

is proportional to $\dfrac{|z|^{2(1+\alpha)} - \mu^2}{|z|^{2(1+\alpha)} + \mu^2}$.

Proof. If we express the bounded solution φ of the Equation (28) in Fourier expansion form as follow

$$\phi(z) = \sum_{n=-\infty}^{+\infty} u_n(r)e^{-in\theta}, \quad z = re^{i\theta},$$

$u_n(r)$ is a bounded nontrivial solution of the equation

$$u''(r) + \frac{1}{r}u'(r) - \frac{n^2}{r^2}u + \frac{8\mu^2(1+\alpha)^2 r^{2\alpha}}{\left(\mu^2 + r^{2(1+\alpha)}\right)^2}u = 0. \tag{29}$$

Since any solution of $-\Delta u = e^u$ in $\mathbb{C}$ is given by the Liouville formula

$$\ln \frac{8\left|F'(z)\right|^2}{\left(1 + \left|F(z)\right|^2\right)^2},$$

for any meromorphic function F defined on $\{z \in \mathbb{C} : F'(z) \neq 0\}$, the function

$$\ln \frac{8\mu^2(1+\alpha)^2\left|1 + \dfrac{n+\alpha+1}{\alpha+1}az^n\right|^2}{\left(\mu^2 + |z|^{2(1+\alpha)}|1 + az^n|^2\right)^2},$$

with any $n \in \mathbb{Z}$ and $|a| < \dfrac{\alpha+1}{|n|+\alpha+1}$, is the solution of $-\Delta u = |z|^{2\alpha}e^u$ in $\mathbb{C}\backslash\{0\}$. Moreover, its derivative with respect to a at $a = 0$

$$\phi_n(z) = \frac{1}{\alpha+1}\frac{(n+\alpha+1)\mu^2 + (n-\alpha-1)|z|^{2(1+\alpha)}}{\mu^2 + |z|^{2(1+\alpha)}}z^n, \quad n \in \mathbb{Z},$$

is a solution of the Equation (29) with $r = |z|$.

For $|n| \geq 1$, since $\{\varphi_n(r), \varphi_{-n}(r)\}$ is a set of linearly independent solutions of the second order linear homogeneous ODE (29), any bounded solution is a linear combination of $\varphi_n(r)$ and $\varphi_{-n}(r)$. However, $\varphi_{|n|}(r)$ (resp. $\varphi_{-n}(r)$) tends to 0 (resp. ∞) as $r \mapsto 0$ and $\varphi_{|n|}(r)$ (resp. $\varphi_{-|n|}(r)$) tends to ∞ (resp. 0) as $r \mapsto +\infty$, which implies that the Equation (29) $|_{|n|\geq 1}$ has no bounded nontrivial solution.

For $n = 0$, $\phi_0(z) = -\dfrac{|z|^{2(1+\alpha)} - \mu^2}{|z|^{2(1+\alpha)} + \mu^2}$ is a bounded solution of the Equation $(29)|_{n=0}$, that is, of the Equation (28). We claim that there does not exist the second linearly independent bounded solution of the Equation $(29)|_{n=0}$. Otherwise, let ω be another linearly independent bounded solution of $(29)|_{n=0}$. Writing $\omega(r) = c(r)\varphi_0(r)$, we get that

$$c''(r)\phi_0 + c'(r)\left(2\phi_0' + \frac{1}{r}\phi_0\right) = 0.$$

Then there exists a constant $C > 0$ such that

$$c'(r) = \frac{C}{r\phi_0^2(r)} = \frac{C(r^{2(1+\alpha)} + \mu^2)^2}{r(r^{2(1+\alpha)} - \mu^2)^2} \sim \frac{C}{r} \quad \text{for r small,}$$
$$c(r) \sim C \log r \quad \text{for r small.}$$

Hence, $\omega(r) \sim C \log r$ for r small, which implies $\omega(r)$ is unbounded on $(0, +\infty)$. It contradicts the assumption that ω is bounded. □

Let us denote

$$Z_{i0}(y) = z_{i0}\left(\frac{\varepsilon y - p_{k_i}}{\rho_{k_i}}\right), \quad \text{for } i = 1, 2, \ldots, n+m,$$
$$Z_{ij}(y) = z_{ij}\left(\frac{\varepsilon y - p_{k_i}}{\rho_{k_i}}\right), \quad \text{for } i = n+1, \ldots, n+m,\ j = 1, 2,$$
$$\chi_\iota(y) = \chi\left(\left|\frac{\varepsilon y - p_{k_i}}{\rho_{k_i}}\right|\right), \quad \text{for } i = 1, 2, \ldots, n+m,$$

where $\chi(r)$ is a smooth, non-increasing cut-off function such that for a large but fixed number $R_0 > 0$, $\chi(r) = 1$ if $r \le R_0$, and $\chi(r) = 0$ if $r \ge R_0 + 1$. Additionally, set $\alpha_0 = \min\{0, \alpha_1, ..., \alpha_N\}$. For any $\alpha \in (-1, \alpha_0)$, we introduce the Banach space

$$\mathcal{C}_{n,m} := \{\psi \in L^\infty(\Omega_\varepsilon) : \|\psi\|_{n,m} < +\infty\},$$

with the norm

$$\|\psi\|_{n,m} = \sup_{y\in\Omega_\varepsilon} \frac{|\psi(y)|}{\varepsilon^2 + \sum_{i=1,\alpha_i<0}^{n} \frac{\left(\frac{\varepsilon}{\rho_i}\right)^2 \left|\frac{\varepsilon y - p_i}{\rho_i}\right|^{2\alpha_i}}{\left(1 + \left|\frac{\varepsilon y - p_i}{\rho_i}\right|\right)^{4+2\alpha+2\alpha_i}} + \sum_{i=1,\alpha_i\ge 0}^{n+m} \frac{\left(\frac{\varepsilon}{\rho_i}\right)^2}{\left(1 + \left|\frac{\varepsilon y - p_i}{\rho_i}\right|\right)^{4+2\alpha}}}. \tag{30}$$

Now to get the invertibility of the linearized operator L, we only need to solve the following linear problems: given h of class $\mathcal{C}_{n,m} \cap C^{0,\beta}(\Omega_\varepsilon)$ with $\beta \in (0,1)$, for $m \ge 1$ and $0 \le n \le N$, we find a function φ and scalars c_{ij}, $i = n + 1, ..., n + m$, $j = 1, 2$, such that

$$\begin{cases} L\phi = \Delta\phi + W\phi = h + \sum_{j=1}^{2}\sum_{i=n+1}^{n+m} c_{ij}\chi_i Z_{ij}, & \text{in} \quad \Omega_\varepsilon, \\ (1-t)\frac{\partial}{\partial\nu}\phi + tb(\varepsilon y)\phi = 0, & \text{on} \quad \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \chi_i Z_{ij}\phi dy = 0, \quad \forall\, i = n+1, \ldots, n+m,\ j = 1, 2, \end{cases} \tag{31}$$

and for $m = 0$ and $1 \le n \le N$, we find a function φ such that

$$\begin{cases} L\phi = \Delta\phi + W\phi = h, & \text{in } \Omega_\varepsilon, \\ (1-t)\frac{\partial}{\partial\nu}\phi + tb(\varepsilon y)\phi = 0, \text{ on } \partial\Omega_\varepsilon. \end{cases} \tag{32}$$

Proposition 3.1 *(i) If $m \ge 1$ and $0 \le n \le N$, given a fixed number $\delta > 0$, there exist positive numbers ε_0 and C such that for any points p_{k_l}, $l = n + 1, ..., n + m$, in Ω', with*

$$\operatorname{dist}(p_{k_l}, \partial\Omega) \geq \delta, \quad \left|p_{k_l} - p_{k_i}\right| \geq \delta, \quad \textit{for } l \neq i \textit{ and } i = 1, \ldots, n+m, \tag{33}$$

there is a unique solution $\phi \in L^{\infty}(\Omega_\varepsilon)$, $c_{n+1}, \ldots, c_{n+m} \in \mathbb{R}$, *of the Equation* (31), *which satisfies*

$$\|\phi\|_{\infty} \leq C\left(\log\frac{1}{\varepsilon}\right)\|h\|_{n,m}, \tag{34}$$

for all $\varepsilon < \varepsilon_0$ *and* $t \in (0, 1]$. *Moreover, the map* $p' \mapsto \varphi$ *is* C^1 *and*

$$\left\|D_{p'}\phi\right\|_{\infty} \leq C\left(\log\frac{1}{\varepsilon}\right)^2\|h\|_{n,m}, \tag{35}$$

where $p' := \left(\frac{1}{\varepsilon}p_{k_{n+1}}, \ldots, \frac{1}{\varepsilon}p_{k_{n+1}}\right)$.

(ii) If $m = 0$ *and* $1 \leq n \leq N$, *there exist positive numbers* ε_0 *and* C *such that there is a unique solution* $\varphi \in L^{\infty}(\Omega_\varepsilon)$ *of the Equation* (32), *which satisfies*

$$\|\phi\|_{\infty} \leq C\left(\log\frac{1}{\varepsilon}\right)\|h\|_{n,0}, \tag{36}$$

for all $\varepsilon < \varepsilon_0$ *and* $t \in (0, 1]$.

These results can be established through some technical lemmas. First for the linear Equation (32) under the additional orthogonality conditions with respect to Z_{i0}, $i = 1, ..., n + m$, and Z_{ij}, $i = n + 1, ..., n + m$, $j = 1, 2$, we prove the following priori estimates.

Lemma 3.3 *(i) If* $m \geq 1$ *and* $0 \leq n \leq N$, *given a fixed number* $\delta > 0$, *there exist positive numbers* ε_0 *and* C *such that for any points* p_{k_l}, $l = n + 1, ..., n + m$, *in* Ω', *which satisfy the relation* (33), *and any solution* φ *of the Equation* (32) *with* $t \in (0, 1]$ *under the orthogonality conditions*

$$\begin{cases} \displaystyle\int_{\Omega_\varepsilon} \chi_i Z_{i0}\phi\, dy = 0, & \forall\ i = 1, \ldots, n+m, \\ \displaystyle\int_{\Omega_\varepsilon} \chi_i Z_{ij}\phi\, dy = 0, & \forall\ i = n+1, \ldots, n+m,\ j = 1, 2, \end{cases} \tag{37}$$

one has

$$\|\phi\|_{\infty} \leq C\|h\|_{n,m}, \tag{38}$$

for all $\varepsilon < \varepsilon_0$.

(ii) If $m = 0$ *and* $1 \leq n \leq N$, *there exist positive numbers* ε_0 *and* C *such that for any solution* φ *of the Equation* (32) *with* $t \in (0, 1]$ *under the orthogonality conditions*

$$\int_{\Omega_\varepsilon} \chi_i Z_{i0}\phi\, dy = 0, \quad \forall\ i = 1, \ldots, n, \tag{39}$$

one has

$$\|\phi\|_{\infty} \leq C\|h\|_{n,0}, \tag{40}$$

for all $\varepsilon < \varepsilon_0$.

Remark 3.4 The idea behind these estimates partly comes from observing the linear Equation (32) with $h = 0$ on bounded set $B_{i,R} := \left\{ y \in \Omega_\varepsilon : \left| \frac{\varepsilon y - p_{k_i}}{\rho_{k_i}} \right| < R \right\}$ for ε small. After a translation and a rotation so that Ω_ε converges to the whole plan $\mathbb{R}^2$, the Equation (32) approaches $L_i\varphi = 0$ in $\mathbb{R}^2$. As a result, the solution of the Equation (32) under the additional orthogonality conditions (37) should be zero.

Proof. Case (i): First consider the "inner norm" $\|\phi\|_I = \sup_{\cup_{i=1}^{n+m} \overline{B_{i,R}}} |\phi|$ and the "boundary norm" $\|\phi\|_o = \sup_{\partial\Omega_\varepsilon} |\phi|$, we claim that there is a constant $C > 0$ such that if $L\varphi = h$ in Ω_ε, then

$$\|\phi\|_\infty \le C\left(\|\phi\|_I + \|\phi\|_o + \|h\|_{n,m}\right). \tag{41}$$

We will establish it with the help of suitable barrier.

Consider that the function $g(z) = \frac{|z|^{2(1+\alpha)} - 1}{|z|^{2(1+\alpha)} + 1}$ is a radial solution in $\mathbb{R}^2$ of

$$\Delta g(z) + \frac{8(1+\alpha)^2 |z|^{2\alpha}}{\left(1 + |z|^{2(1+\alpha)}\right)^2} g(z) = 0,$$

we define a bounded comparison function

$$Z(y) = \sum_{i=1}^{n+m} g\left(a \left| \frac{\varepsilon y - p_{k_i}}{\rho_{k_i}} \right|\right), \quad y \in \Omega_\varepsilon,$$

with $a > 0$. Set $R_a = \frac{1}{a} 3^{\frac{1}{2(1+\alpha)}}$. While $\left| \frac{\varepsilon y - p_{k_i}}{\rho_{k_i}} \right| \ge R_a$ for all $i = 1, \ldots, n+m$,

$$\begin{aligned} -\Delta Z(y) &= \sum_{i=1}^{n+m} \left(\frac{\varepsilon}{\rho_{k_i}}\right)^2 a^2 \frac{8(1+\alpha)^2 \left| a\frac{\varepsilon y - p_{k_i}}{\rho_{k_i}} \right|^{2\alpha}}{\left(1 + \left| a\frac{\varepsilon y - p_{k_i}}{\rho_{k_i}} \right|^{2(1+\alpha)}\right)^2} g\left(a \left| \frac{\varepsilon y - p_{k_i}}{\rho_{k_i}} \right|\right) \\ &\ge \sum_{i=1}^{n+m} \left(\frac{\varepsilon}{\rho_{k_i}}\right)^2 a^2 \frac{4(1+\alpha)^2 \left| a\frac{\varepsilon y - p_{k_i}}{\rho_{k_i}} \right|^{2\alpha}}{\left(1 + \left| a\frac{\varepsilon y - p_{k_i}}{\rho_{k_i}} \right|^{2(1+\alpha)}\right)^2} \\ &> \sum_{i=1}^{n+m} \left(\frac{\varepsilon}{\rho_{k_i}}\right)^2 a^{-2(1+\alpha)} \frac{(1+\alpha)^2}{\left| \frac{\varepsilon y - p_{k_i}}{\rho_{k_i}} \right|^{2\alpha+4}}. \end{aligned} \tag{42}$$

Moreover, according to (21) and (22), on the same region,

$$W(y)Z(y) \le C \sum_{i=1}^{n+m} \left(\frac{\varepsilon}{\rho_{k_i}}\right)^2 \frac{1}{\left| \frac{\varepsilon y - p_{k_i}}{\rho_{k_i}} \right|^{2\alpha_{k_i}+4}}. \tag{43}$$

So if a is small enough to satisfy $(1+\alpha)^2 a^{-2(1+\alpha)} > C + 1$, R_a *is* sufficiently large. As a result, by (42) and (43), for any $R \ge R_a$, we have $Z(y) > 0$ and $L(Z) < 0$ in $\Omega_{R,\varepsilon}^c := \Omega_\varepsilon \setminus \cup_{i=1}^{n+m} B_{i,R}$.

Let M be a large number such that for all $i = 1, ..., n + m$, $\Omega \subset B(p_{k_i}, M)$. Consider now the solution of the problem

$$\begin{cases} -\Delta\psi_i = \left(\dfrac{\varepsilon}{\rho_{k_i}}\right)^2 \dfrac{4}{\left|\dfrac{\varepsilon y - p_{k_i}}{\rho_{k_i}}\right|^{2\alpha+4}} + 4\varepsilon^2, & R < \left|\dfrac{\varepsilon y - p_{k_i}}{\rho_{k_i}}\right| < \dfrac{M}{\rho_{k_i}}, \\ \psi_i(y) = 0, \quad \text{for} \quad \left|\dfrac{\varepsilon y - p_{k_i}}{\rho_{k_i}}\right| = R & \text{and} \quad \left|\dfrac{\varepsilon y - p_{k_i}}{\rho_{k_i}}\right| = \dfrac{M}{\rho_{k_i}}. \end{cases}$$

A direct computation shows

$$\psi_i(y) = \varphi_i(r_i) - \varphi_i\left(\frac{M}{\rho_{k_i}}\right)\frac{\log\dfrac{r_i}{R}}{\log\dfrac{M}{R\rho_{k_i}}},$$

where

$$r_i = \left|\frac{\varepsilon y - p_{k_i}}{\rho_{k_i}}\right|,$$

and

$$\varphi_i(t) = \frac{1}{(1+\alpha)^2}\left(\frac{1}{R^{2(1+\alpha)}} - \frac{1}{t^{2(1+\alpha)}}\right) + \rho_{k_i}^2(R^2 - t^2).$$

For the sake of the convenience, we choose R larger if necessary. Then it easily see that these functions ψ_i, $i = 1, ..., n + m$, have a uniform bound independent of ε.

Now we can construct the needed barrier:

$$\tilde{\phi}(y) = 2(\|\phi\|_l + \|\phi\|_o)Z(y) + \|h\|_{n,m}\sum_{i=1}^{n+m}\psi_i(y).$$

It is easy to check that $L\tilde{\phi} < h = L\phi$ in $\Omega^c_{R,\varepsilon}$, and $\tilde{\phi} \geq \phi$ on $\partial\Omega^c_{R,\varepsilon}$. Since $Z(y) > 0$ and $LZ(y) < 0$ in $\Omega^c_{R,\varepsilon}$, from the maximum principle (see [[37], Theorem 10, Chap. 2]), it follows that $\tilde{\phi} \geq \phi$ in $\Omega^c_{R,\varepsilon}$. Similarly, $-\tilde{\phi} \leq \phi$ in $\Omega^c_{R,\varepsilon}$, which derives the estimate (41).

We prove the priori estimate (38) by contradiction. Assume that there exist a sequence $\varepsilon_k \to 0$, points $p^k_{k_l}$, $l = n + 1, ..., n + m$, in Ω' which satisfy relation (33), functions h_k with $\|h_k\|_{n,m} \to 0$, solutions φ_k with $\|\varphi_k\|_\infty = 1$, such that

$$\begin{cases} L\phi_k = \Delta\phi_k + W\phi_k = h_k, & \text{in} \quad \Omega_\varepsilon, \\ (1-t)\dfrac{\partial}{\partial\nu}\phi_k + tb(\varepsilon y)\phi_k = 0, & \text{on} \quad \partial\Omega_\varepsilon, \\ \displaystyle\int_{\Omega_\varepsilon} \chi_i Z_{i0}\phi_k dy = 0, & \forall \ i = 1, \ldots, n+m, \\ \displaystyle\int_{\Omega_\varepsilon} \chi_i Z_{i0}\phi_k dy = 0, & \forall \ i = n+1, \ldots, n+m, \ j = 1, 2. \end{cases}$$

Then from the estimate (41), $\|\varphi_k\|_l \geq \kappa$ or $\|\varphi_k\|_o \geq \kappa$ for some $\kappa > 0$. Briefly set $\varepsilon := \varepsilon_k$, $p_{k_i} : p^k_{k_i}$. If $\|\varphi_k\|_l \geq \kappa$, with no loss of generality, we assume that $\sup_{B_{i,R}} |\phi_k| \geq k$ for some i Then if we set $\hat{\phi}_k(z) = \phi_k\left(\dfrac{\rho_{k_i}}{\varepsilon}z + \dfrac{p_{k_i}}{\varepsilon}\right)$ and $\hat{h}_k(z) = h_k\left(\dfrac{\rho_{k_i}}{\varepsilon}z + \dfrac{p_{k_i}}{\varepsilon}\right)$, $\hat{\phi}_k$ satisfies

$$\Delta\hat{\phi}_k + \frac{8\mu_{k_i}^2(1+\alpha_{k_i})^2|z|^{2\alpha_{k_i}}}{[\mu_{k_i}^2 + |z|^{2(1+\alpha_{k_i})}]^2}[1 + O(\rho_{k_i}|z|) + O(\varepsilon^2)]\hat{\phi}_k = \left(\frac{\rho_{k_i}}{\varepsilon}\right)^2 \hat{h}_k,$$

for $z \in B_R(0)$. Obviously, for any $q \in \left[1, -\frac{1}{\alpha}\right]$ we easily get $\left(\frac{\rho_{k_i}}{\varepsilon}\right)^2 \hat{h}_k \to 0$ in $L^q(B_R(0))$. Since $\frac{8\mu_{k_i}^2(1+\alpha_{k_i})^2|z|^{2\alpha_{k_i}}}{[\mu_{k_i}^2 + |z|^{2(1+\alpha_{k_i})}]^2}$ is bounded in $L^q(B_R(0))$ and $\left\|\hat{\phi}_k\right\|_\infty = 1$, elliptic regularity theory readily implies that $\hat{\phi}_k$ converges uniformly over compact subsets near the origin to a bounded nontrivial solution $\hat{\phi}$ of the equation

$$L_i\hat{\phi} = \Delta\hat{\phi} + \frac{8\mu_{k_i}^2(1+\alpha_{k_i})^2|z|^{2\alpha_{k_i}}}{[\mu_{k_i}^2 + |z|^{2(1+\alpha_{k_i})}]^2}\hat{\phi} = 0, \quad \text{in} \quad \mathbb{R}^2.$$

From Lemma 3.2, this equation implies that $\hat{\phi}$ is proportional to z_{i0} for $i = 1, ..., n$, or a linear combination of z_{i0} and z_{ij} for $i = n + 1, ..., n + m, j = 1, 2$. However, our assumed orthogonality conditions (37) on φ_k pass to limit and yield the corresponding conditions (37) on $\hat{\phi}$, which means $\hat{\phi} \equiv 0$. Hence, it is absurd because $\hat{\phi}$ is nontrivial.

If $\|\varphi_k\|_o \geq \kappa$ and $\|\varphi_k\|_l \to 0$, there exists a point $q \in \partial\Omega$ and a number $R_1 > 0$ such that $\sup_{\partial\Omega_\varepsilon \cap B_{R_1}(q')} |\phi_k(y)| \geq k > 0$ with $q' = \frac{1}{\varepsilon}q$. Consider $\hat{\phi}_k(y) = \phi_k(y - q')$ and let us translate and rotate Ω_ε so that $q' = 0$ and Ω_ε approaches the upper half-plan $\mathbb{R}^2_+$. Since $\left|\frac{\varepsilon q' - p_{k_i}}{\rho_{k_i}}\right| > \frac{\delta}{\rho_{k_i}}$ for all $i = 1, ..., n + m$, $\hat{\phi}_k(z)$ satisfies

$$\begin{cases} \Delta\hat{\phi}_k(y) + O(\varepsilon^4)\hat{\phi}_k(y) = h_k(y), \text{ in } \Omega_\varepsilon \setminus \cup_{i=1}^{n+m} B_{i,\delta\rho_{k_i}^{-1}}, \\ (1-t)\frac{\partial}{\partial\nu}\hat{\phi}_k + tb(\varepsilon y)\hat{\phi}_k = 0, \quad \text{on} \quad \partial\Omega_\varepsilon, \end{cases}$$

with $(1-t)\int_{\Omega_\varepsilon} \left|\nabla\hat{\phi}_k\right|^2 + t\int_{\partial\Omega_\varepsilon} b(\varepsilon y)\hat{\phi}_k^2 < C$. Moreover, we easily get $h_k(y) \to 0$ in $\Omega_\varepsilon \setminus \cup_{i=1}^{n+m} B_{i,\delta\rho_{k_i}^{-1}}$. While $t = 1$, it is obvious to see that $\hat{\phi}_k(y) = 0$ on $\partial\Omega_\varepsilon$. So it is absurd because of $\sup_{\partial\Omega_\varepsilon \cap B_{R_1}(q')} |\phi_k(y)| \geq k > 0$. On the other hand, for any $t \in (0,1)$, elliptic regularity theory with the Robin boundary condition (see [30,34,38] and the references therein) implies that $\hat{\phi}_k$ converges uniformly on compact subsets near the origin to a bounded nontrivial solution $\hat{\phi}$ of the equation

$$\begin{cases} \Delta\hat{\phi} = 0, & \text{in } \mathbb{R}^2_+, \\ (1-t)\frac{\partial}{\partial\nu}\hat{\phi} + tb(0)\hat{\phi} = 0, & \text{on } \partial\mathbb{R}^2_+, \end{cases}$$

with $(1-t)\int_{\mathbb{R}^2_+} \left|\nabla\hat{\phi}\right|^2 + tb(0)\int_{\partial\mathbb{R}^2_+} \hat{\phi}^2 < C$. It follows that its bounded solution $\hat{\phi}$ is zero. Hence, it is also absurd because $\hat{\phi}$ is nontrivial, which derives the priori estimate (38) of the case (i). Since the proof of the case (ii) is similar to that of the case (i), we omit it.□

We will give next the priori estimate for the solution of the Equation (32) that satisfies orthogonality conditions with respect to Z_{ij}, $i = n + 1, ..., n + m, j = 1, 2$, only.

Lemma 3.5. *(i) If $m \geq 1$ and $0 \leq n \leq N$, given a fixed number $\delta > 0$, there exist positive numbers ε_0 and C such that for any points p_{k_l}, $l = n + 1, ..., n + m$, in Ω', which satisfy the relation* (33), *and any solution φ of the Equation* (32) *with $t \in (0, 1]$ under the orthogonality conditions*

$$\int\limits_{\Omega_\varepsilon} \chi_i Z_{ij} \phi dy = 0, \quad \forall\ i = n+1, \ldots, n+m,\ j = 1, 2, \tag{44}$$

one has

$$\|\phi\|_\infty \leq C\left(\log \frac{1}{\varepsilon}\right) \|h\|_{n,m}, \tag{45}$$

for all $\varepsilon < \varepsilon_0$.

(ii) If $m = 0$ and $1 \leq n \leq N$, there exist positive numbers ε_0 and C such that for any solution φ of the Equation (32) *with $t \in (0, 1]$, one has*

$$\|\phi\|_\infty \leq C\left(\log \frac{1}{\varepsilon}\right) \|h\|_{n,0}, \tag{46}$$

for all $\varepsilon < \varepsilon_0$.

Proof. Case (i): Let φ satisfy the Equation (32) under the orthogonality conditions (44). We will modify φ to satisfy the orthogonality conditions (37). To realize this point, we consider some related modifications with compact support of the functions Z_{i0}, $i = 1, ..., n + m$.

Let $R > R_0 + 1$ be large and fixed, and let $\hat{z}_{i0}$ be the solution of the equation

$$\begin{cases} \Delta \hat{z}_{i0} + \dfrac{8\mu_{k_i}^2 (1+\alpha_{k_i})^2 |z|^{2\alpha_{k_i}}}{(\mu_{k_i}^2 + |z|^{2(1+\alpha_{k_i})})^2} \hat{z}_{i0} = 0, \qquad \text{for} \quad R < |z| < \dfrac{\delta}{3\rho_{k_i}}, \\ \hat{z}_{i0}(z) = z_{i0}(R) \quad \text{on} \quad |z| = R, \qquad \hat{z}_{i0}(z) = 0 \quad \text{on} \quad |z| = \dfrac{\delta}{3\rho_{k_i}}. \end{cases}$$

A simple computation shows that this solution is explicitly given by

$$\hat{z}_{i0}(z) = z_{i0}(r) \left[1 - \frac{\int_R^r \frac{ds}{s z_{i0}^2(s)}}{\int_R^{\frac{\delta}{3\rho_{k_i}}} \frac{ds}{s z_{i0}^2(s)}} \right], \quad r = |z|.$$

Set

$$\hat{Z}_{i0}(y) = \hat{z}_{i0}\left(\frac{\varepsilon y - p_{k_i}}{\rho_{k_i}}\right), \quad \eta_{1i}(y) = \eta_1\left(\left|\frac{\varepsilon y - p_{k_i}}{\rho_{k_i}}\right|\right), \quad \eta_{2i}(y) = \eta_2\left(4\rho_{k_i}\left|\frac{\varepsilon y - p_{k_i}}{\rho_{k_i}}\right|\right),$$

where $\eta_1(r)$ and $\eta_2(r)$ are smooth cut-off functions with the properties: $\eta_1(r) = 1$ for $r < R$, $\eta_1(r) = 0$ for $r > R + 1$, $|\eta_1'(r)| \leq 2$; $\eta_2(r) = 1$ for $r < \delta$, $\eta_2(r) = 0$ for $r > \frac{4\delta}{3}$, $|\eta_2'(r)| \leq C|\eta_2''(r)| \leq C$. We define a test function

$$\tilde{Z}_{i0} = \eta_{1i} Z_{i0} + (1 - \eta_{1i}) \eta_{2i} \hat{Z}_{i0}.$$

Obviously, $\tilde{Z}_{i0}(y) = Z_{i0}(y)$ if $\left|\frac{\varepsilon y - p_{k_i}}{\rho_{k_i}}\right| < R$, and $\tilde{Z}_{i0}(y) = 0$ if $\left|\frac{\varepsilon y - p_{k_i}}{\rho_{k_i}}\right| > \frac{\delta}{3\rho_{k_i}}$, in particular, $\tilde{Z}_{i0}(y) = 0$ near $\partial\Omega_\varepsilon$.

Now we modify φ to satisfy the orthogonality conditions with respect to Z_{i0}, $i = 1, ..., n+m$, and set

$$\tilde{\phi} = \phi + \sum_{i=1}^{m+n} d_i \tilde{Z}_{i0},$$

where the numbers d_i are chosen such that

$$d_i \int_{\Omega_\epsilon} \chi_i |Z_{i0}|^2 + \int_{\Omega_\epsilon} \chi_i Z_{i0} \phi = 0, \quad \forall\, i = 1, \dots, m+n.$$

Thus

$$\begin{cases} L\tilde{\phi} = h + \sum\limits_{i=1}^{m+n} d_i L\tilde{Z}_{i0}, & \text{in} \quad \Omega_\varepsilon, \\ (1-t)\dfrac{\partial}{\partial \nu}\tilde{\phi} + tb(\varepsilon y)\tilde{\phi} = 0, \text{ on} & \partial\Omega_\varepsilon, \end{cases} \tag{47}$$

and $\tilde{\phi}$ satisfies all the orthogonality conditions in (37). From Lemma 3.3 (i), we have

$$\left\|\tilde{\phi}\right\|_\infty \le C\left[\|h\|_{n,m} + \sum_{i=1}^{n+m} |d_i| \cdot \left\|L\tilde{Z}_{i0}\right\|_{n,m}\right]. \tag{48}$$

In order to get the estimate (45) of φ, we need to give the sizes of d_i and $\left\|L\tilde{Z}_{i0}\right\|_{n,m}$ for any $t \in (0, 1]$. Multiplying the first equation of (47) by $\tilde{Z}_{j0}$, integrating by parts and using the mixed boundary conditions of (47), we get

$$\left\langle L\tilde{Z}_{j0}, \tilde{\phi}\right\rangle = \left\langle \tilde{Z}_{j0}, h\right\rangle + d_j \left\langle L\tilde{Z}_{j0}, \tilde{Z}_{j0}\right\rangle, \tag{49}$$

where $\langle f, g\rangle = \int_{\Omega_\varepsilon} fg$. A simple computation shows that $\left|\left\langle \tilde{Z}_{j0}, h\right\rangle\right| \le C\|h\|_{n,m}$, which in combination with (48) and (49) yields

$$|d_j| \cdot \left|\left\langle L\tilde{Z}_{j0}, \tilde{Z}_{j0}\right\rangle\right| \le C\|h\|_{n,m}\left[1 + \left\|L\tilde{Z}_{j0}\right\|_{n,m}\right] + C\left\|L\tilde{Z}_{j0}\right\|_{n,m} \sum_{i=1}^{m+n} |d_i| \cdot \left\|L\tilde{Z}_{i0}\right\|_{n,m}. \tag{50}$$

From some similar computations (see [[26], Lemma 3.2] and [[28], Lemma 3.3]), there exists a constant $C > 0$ independent of ε such that

$$\left\|L\tilde{Z}_{j0}\right\|_{n,m} \le C\frac{1}{|\log \rho_{k_j}|}, \tag{51}$$

and

$$\left\langle L\tilde{Z}_{j0}, \tilde{Z}_{j0}\right\rangle \le -\frac{C}{|\log \rho_{k_j}|}\left[1 + O\left(\frac{1}{|\log \rho_{k_j}|}\right)\right], \tag{52}$$

which combined with (50) yields

$$\left|d_j\right| \leq C\left(\log\frac{1}{\rho_{k_j}}\right)\|h\|_{n,m}. \tag{53}$$

Furthermore, from (48), (51), (53), and the definitions of $\tilde{\phi}$ and ρ_{k_j}, we have

$$\|\phi\|_\infty \leq C\left(\log\frac{1}{\varepsilon}\right)\|h\|_{n,m}. \tag{54}$$

Similarly, the proof of the case (ii) can also be done, we omit it. □

Proof of Proposition 3.1. Case (i): From Lemma 3.5 (i), and the Fredholm's alternative theory with Robin boundary condition instead of Dirichlet boundary condition if necessary (see [34,38] and the references therein), the proof can be similarly given through those in [[26], pp. 61-63].

Case (ii): Since the priori estimate (36) of the solution of the Equation (32) has been established in Lemma 3.5 (ii), we can use the Fredholm's alternative and obtain the unique solution of the Equation (32). □

Let us now introduce the auxiliary nonlinear problems: for $m \geq 1$ and $0 \leq n < N$, we find the function φ and scalars c_{ij}, $i = n + 1, ..., n + m$, $j = 1, 2$, such that

$$\begin{cases} L\phi = -\left[R + N(\phi)\right] + \displaystyle\sum_{j=1}^{2}\sum_{i=n+1}^{n+m} c_{ij}\chi_i Z_{ij}, & \text{in} \quad \Omega_\varepsilon, \\ (1-t)\dfrac{\partial}{\partial\nu}\phi + tb(\varepsilon y)\phi = 0, & \text{on} \quad \partial\Omega_\varepsilon, \\ \displaystyle\int_{\Omega_\varepsilon}\chi_i Z_{ij}\phi dy = 0, \qquad \forall\, i = n+1, \ldots, n+m,\, j = 1, 2, & \end{cases} \tag{55}$$

and for $m = 0$ and $1 \leq n \leq N$, we find the solution φ of the nonlinear Equation (26).

The following result can be proved using standard arguments as in [26,28].

Proposition 3.2 *(i) If* $m \geq 1$ *and* $0 \leq n \leq N$*, given a fixed number* $\delta > 0$*, there exist positive numbers* ε_0 *and C such that for any points* p_{k_l}*,* $l = n + 1, ..., n + m$*, in* Ω' *satisfying the relation* (33)*, there is a unique solution for the Equation* (55) *which satisfies*

$$\|\phi\|_\infty \leq C\rho\left|\log\varepsilon\right|, \tag{56}$$

for all $\varepsilon < \varepsilon_0$ *and* $t \in (0, 1]$*. Moreover, the map* $p' \to \varphi$ *is* C^1 *and*

$$\left\|D_{p'}\phi\right\|_\infty \leq C\rho\left|\log\varepsilon\right|^2, \tag{57}$$

where $\rho := \max_{1\leq i\leq n+m}\rho_{k_i}$ *and* $p' := \left(\frac{1}{\varepsilon}p_{k_{n+1}}, \ldots, \frac{1}{\varepsilon}p_{k_{n+m}}\right)$.

(ii) If $m = 0$ *and* $1 \leq n \leq N$*, there exist positive numbers* ε_0 *and C such that there is a unique solution for the Equation* (26) *which also satisfies the estimate* (56) *for all* $\varepsilon < \varepsilon_0$ *and* $t \in (0, 1]$.

Now we only need to find a solution to the Equation (26) with $m \geq 1$ and $0 \leq n \leq N$, and hence to the Equation (55) if $p' = \left(\frac{1}{\varepsilon}p_{k_{n+1}}, \ldots, \frac{1}{\varepsilon}p_{k_{n+m}}\right)$ is such that

$$c_{ij}(p') = 0, \quad \forall\, i = n+1, \ldots, n+m,\ j = 1, 2. \tag{58}$$

Let us introduce the energy functional of the Equation (3), namely for $t = 1$,

$$J_{\varepsilon,1}(u) = \frac{1}{2}\int_{\Omega} |\nabla u|^2 - \varepsilon^2 \int_{\Omega} S(x)f(x)e^u,$$

and for $t \in (0,1)$,

$$J_{\varepsilon,t}(u) = (1-t)\left(\frac{1}{2}\int_{\Omega} |\nabla u|^2 - \varepsilon^2 \int_{\Omega} S(x)f(x)e^u\right) + \frac{t}{2\varepsilon}\int_{\partial\Omega} b(x)u^2.$$

Furthermore, we define

$$F_{\varepsilon,t}(p) = J_{\varepsilon,t}(U(p) + \tilde{\phi}(p)), \tag{59}$$

where $p = (p_{k_{n+1}},\ldots,p_{k_{n+m}}) \in (\Omega')^m$ and $\tilde{\phi}(p)(x) = \phi\left(\frac{x}{\varepsilon}, \frac{p}{x}\right)$ with the solution φ of the Equation (55).

The finite dimensional variational reduction is meaningful in view of the following property.

Proposition 3.3 *If $p = (p_{k_{n+1}},\ldots,p_{k_{n+m}}) \in (\Omega')^m$ satisfying the relation* (33) *is a critical point of $F_{\varepsilon,t}$ with $t \in (0, 1]$, then $U(p) + \tilde{\phi}(p)$is a critical point of $J_{\varepsilon,t}$, namely a solution of the Equation* (3)*. Besides, on any compact subsets S of $(\Omega' \setminus \Gamma)^m \setminus \Delta_m$ the following expansion holds*

$$F_{\varepsilon,t}(p) = J_{\varepsilon,t}(U(p)) + \theta_{\varepsilon,t}(p), \tag{60}$$

where

$$\left|\theta_{\varepsilon,t}(p)\right| \to 0, \quad \text{uniformly on } \mathcal{S}, \tag{61}$$

for ε small.

Proof. Step 1: Let us define for $t = 1$,

$$I_{\varepsilon,1}(v) = \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla v|^2 - \int_{\Omega_\varepsilon} S(\varepsilon y)f(\varepsilon y)e^v,$$

and for $t \in (0,1)$,

$$I_{\varepsilon,t}(v) = (1-t)\left(\frac{1}{2}\int_{\Omega_\varepsilon} |\nabla v|^2 - \int_{\Omega_\varepsilon} S(\varepsilon y)f(\varepsilon y)e^v\right) + \frac{t}{2}\int_{\partial\Omega_\varepsilon} b(\varepsilon y)(v - 4\varepsilon\log\varepsilon)^2.$$

Then $F_{\varepsilon,t}(p) = J_{\varepsilon,t}(U(p) + \tilde{\phi}(p)) = I_{\varepsilon,t}(V(p') + \phi(p'))$, where $p' = \frac{1}{\varepsilon}p$. Moreover, for $k = n + 1, \ldots, n + m$, $l = 1, 2$, it holds

$$\partial_{pkl}F_{\varepsilon,t}(p) = \varepsilon^{-1}DI_{\varepsilon,t}(V+\phi)\left[\partial_{p'_{kl}}V + \partial_{p'_{kl}}\phi\right]. \tag{62}$$

Since $\varphi(p')$ is a solution of the Equation (55), $v = V(p') + \varphi(p')$ satisfies

$$\begin{cases} \nabla v + S(\varepsilon y)f(\varepsilon y)e^v = \sum_{j=1}^{2}\sum_{i=n+1}^{n+m} c_{ij}\chi_i Z_{ij}, \text{ in } \quad \Omega_\varepsilon, \\ (1-t)\dfrac{\partial v}{\partial \nu} + tb(xy)v = 4tb(\varepsilon y)\log\varepsilon, \text{ on } \quad \partial\Omega_\varepsilon. \end{cases} \tag{63}$$

By (62), (63), $D_pF_{\varepsilon,t}(p) = 0$ implies for $t \in (0, 1]$,

$$\sum_{j=1}^{2}\sum_{i=n+1}^{n+m} c_{ij}\int_{\Omega_\varepsilon} \chi_i Z_{ij}(\partial_{p'_{kl}}V + \partial_{p'_{kl}}\phi) = 0.$$

From the definition of V, it can be directly checked $\partial_{p'_{kl}}V = -Z_{kl} + o(1)$, where $o(1)$ is in the sense of the L^∞-norm for ε small. Since $\|D_{p'}\phi\|_\infty \le C\rho|\log\varepsilon|^2$, $\partial_{p'_{kl}}V + \partial_{p'_{kl}}\phi = -Z_{kl} + o(1)$. Hence it follows

$$\sum_{j=1}^{2}\sum_{i=n+1}^{n+m} c_{ij}\int_{\Omega_\varepsilon} \chi_i Z_{ij}(Z_{kl} + o(1)) = 0, \quad \forall\, k = n+1,\dots,n+m,\ l = 1,2,$$

which is a strictly diagonal dominant system. This implies that $c_{ij} = 0$, $\forall\ i = n + 1, \dots, n + m$, $j = 1, 2$. By (63), $U(p) + \tilde{\phi}(p)$ is a critical point of $J_{\varepsilon,t}$, that is, a solution of the Equation (3).

Step 2: Set $\tilde{\theta}_{\varepsilon,t}(p') = I_{\varepsilon,t}(V(p') + \phi(p')) - I_{\varepsilon,t}(V(p'))$. Using $DI_{\varepsilon,t}(V + \varphi)[\varphi] = 0$, a Taylor expansion and an integration by parts, it follows that for $t \in (0, 1)$,

$$\begin{aligned}\tilde{\theta}_{\varepsilon,t}(p') &= I_{\varepsilon,t}(V+\phi) - I_{\varepsilon,t}(V)\\ &= \int_0^1 D^2 I_{\varepsilon,t}(V+s\phi)[\phi]^2(1-s)ds\\ &= \int_0^1\left((1-t)\int_{\Omega_\varepsilon}|\nabla\phi|^2 - S(\varepsilon y)f(\varepsilon y)e^{V+s\phi}\phi^2 + t\int_{\partial\Omega_\varepsilon} b(\varepsilon y)\phi^2\right)(1-s)ds\\ &= (1-t)\int_0^1\left(\int_{\Omega_\varepsilon}\left[N(\phi)+R\right]\phi + W\left[1-e^{s\phi}\right]\phi^2\right)(1-s)ds,\end{aligned} \tag{64}$$

and similarly, for $t = 1$,

$$\tilde{\theta}_{\varepsilon,1}(p') = \int_0^1\left(\int_{\Omega_\varepsilon}\left[N(\phi)+R\right]\phi + W\left[1-e^{s\phi}\right]\phi^2\right)(1-s)ds. \tag{65}$$

Note that $\|\varphi\|_\infty = O(\rho|\log\varepsilon|)$, $\|N(\varphi)\|_{n,m} = O(\rho^2|\log\varepsilon|^2)$, $\|R\|_{n,m} = O(\rho)$, and $\|W\|_{n,m} = O(1)$. Then from (64), (65), it is easy to deduce for $t \in (0, 1]$,

$$\tilde{\theta}_{\varepsilon,t}(p') = O(\rho^2|\log\varepsilon|). \tag{66}$$

Hence, from (66), the expansion (60) satisfies the property (61).□

Finally, we need to write the precisely asymptotical expansion of $J_{\varepsilon,t}(U)$. To realize it, we first establish the following result:

Lemma 3.6 *Assume that points* $p_{k_{n+1}},\dots,p_{k_{n+m}}$, *satisfy the relation* (33), *then fort* $\in$ (0,1) *and* $i = 1, \dots, n + m$, *there hold*

$$G_{t,\varepsilon}(x, p_{k_i}) = G_1(x, p_{k_i}) + O(\varepsilon), \tag{67}$$

and

$$H_{t,\varepsilon}(x, p_{k_i}) = H_1(x, p_{k_i}) + O(\varepsilon), \tag{68}$$

uniformly in $C(\bar{\Omega})$ *and in* $C^2_{\mathrm{loc}}(\Omega)$ *for* ε *small.*

Proof. Set $z_i^t(x) = G_{t,\varepsilon}(x, p_{k_i}) - G_1(x, p_{k_i})$. Then $z_i^t(x)$ satisfies

$$\begin{cases} \nabla z_i^t(x) = 0, & \text{in} \quad \Omega, \\ \varepsilon(1-t)\dfrac{\partial z_i^t(x)}{\partial \nu} + tb(x)z_i^t(x) = F_i^t(x), & \text{on} \quad \partial\Omega, \end{cases}$$

where $F_i^t(x) = -\varepsilon(1-t)\dfrac{\partial}{\partial \nu}G_1(x, p_{k_i})$. Since $H_1(x, p_{k_i})$ has the gradient estimate on $\partial\Omega$ (see [[26], p. 76])

$$\left|\nabla_x H_1(x, p_{k_i})\right| \le C_1 \min\left\{\frac{1}{\left|x - p_{k_i}\right|}, \frac{1}{\mathrm{dist}(p_{k_i}, \partial\Omega)}\right\} + C_2 \le C_1\frac{1}{\delta} + C_2,$$

we can easily get $\left\|F_i^t(x)\right\|_{L^\infty(\partial\Omega)} = O(\varepsilon)$. Using the same technique with the proof of Lemma 2.1, we can also get $z_i^t(x) = O(\varepsilon)$ uniformly in $C(\bar{\Omega})$ and in $C^2_{\mathrm{loc}}(\Omega)$ for ε small, which means (67). From the definition of the regular part of Green function, we can also derive (68). □

Proposition 3.4 *The following asymptotical expansions hold for* $t = 1$,

$$\begin{aligned} J_{\varepsilon,1}(U) = {} & 8\pi m \log 8 + 8\pi \sum_{i=1}^{n}(1+\alpha_{k_i})\log\frac{8(1+\alpha_{k_i})^2}{f(p_{k_i})Q_{k_i}(p_{k_i})} \\ & - 16\pi \sum_{i,j=1, i\ne j}^{n}(1+\alpha_{k_i})(1+\alpha_{k_j})G_1(p_{k_j}, p_{k_i}) - 16\pi \log\varepsilon \sum_{i=1}^{n+m}(1+\alpha_{k_i}) \\ & - 16\pi \sum_{i=1}^{n}(1+\alpha_{k_i})^2 H_1(p_{k_i}, p_{k_i}) - 16\pi \sum_{i=1}^{n+m}(1+\alpha_{k_i}) + \rho\Theta_\varepsilon(p) - 16\pi\varphi_{n,m}(p), \end{aligned} \tag{69}$$

and for $t \in$ (0,1),

$$\begin{aligned} J_{\varepsilon,t}(U) = {} & 8\pi(1-t)m\log 8 - 16\pi(1-t)\sum_{i=1}^{n+m}(1+\alpha_{k_i}) - 16\pi(1-t)\log\varepsilon\sum_{i=1}^{n+m}(1+\alpha_{k_i}) \\ & + 8\pi(1-t)\sum_{i=1}^{n}(1+\alpha_{k_i})\log\frac{8(1+\alpha_{k_i})^2}{f(p_{k_i})Q_i(p_{k_i})} - 16\pi(1-t)\sum_{i=1}^{n}(1+\alpha_{k_i})^2 H_1(p_{k_i}, p_{k_i}) \\ & - 16\pi(1-t)\sum_{i,j=1, i\ne j}^{n}(1+\alpha_{k_i})(1+\alpha_{k_j})G_1(p_{k_i}, p_{k_j}) + \rho\Theta_\varepsilon(p) - 16\pi(1-t)\varphi_{n,m}(p), \end{aligned} \tag{70}$$

where $\phi_{n,m}(p)$ *is defined by* (5), *and for* ε *small,* Θ_ε *is a bounded, smooth function of* $p = (p_{k_{n+1}}, \dots, p_{k_{n+m}})$, *uniformly on points* $p_{k_{n+1}}, \dots, p_{k_{n+m}}$ *in* Ω' *satisfying the relation* (33).

Proof. According to [[26,28], Lemma 6.1], it only remains to discuss the asymptotical expansion of the energy $J_{\varepsilon,t}(U)$ with respect to $t \in$ (0,1). By (11), it follows

$$\int_\Omega |\nabla U|^2 = \sum_{i,j=1}^{n+m}\int_\Omega \nabla U_i \nabla U_j = \sum_{i,j=1}^{n+m}\left(\int_{\partial\Omega} U_i\frac{\partial U_j}{\partial \nu} - \int_\Omega U_i \Delta U_j\right),$$

and

$$\int_{\partial\Omega} b(x)U^2 = \sum_{i,j=1}^{n+m}\int_{\partial\Omega} b(x)U_iU_j,$$

which together with the Equation (13) yields

$$\frac{1-t}{2}\int_{\Omega}|\nabla U|^2+\frac{t}{2\varepsilon}\int_{\partial\Omega}b(x)U^2=-\frac{1-t}{2}\sum_{i,j=1}^{n+m}\int_{\Omega}U_i\Delta U_j.$$

Furthermore

$$\begin{aligned}J_{\varepsilon,t}(U)&=(1-t)\left(\frac{1}{2}\int_{\Omega}|\nabla U|^2-\varepsilon^2\int_{\Omega}S(x)f(x)e^{U}\right)+\frac{t}{2\varepsilon}\int_{\partial\Omega}b(x)U^2\\&=-(1-t)\left(\frac{1}{2}\sum_{i,j=1}^{n+m}\int_{\Omega}U_i\Delta U_j+\varepsilon^2\int_{\Omega}S(x)f(x)e^{U}\right).\end{aligned}\tag{71}$$

From (9), (13), and (14), it implies

$$\begin{aligned}-\int_{\Omega}U_i\Delta U_j&=\int_{\Omega}\varepsilon^2\left|x-p_{k_j}\right|^{2\alpha_{k_j}}f(p_{k_j})Q_{k_j}(p_{k_j})(u_{k_i}+H^t_{k_i})e^{u_{k_j}}\\&=\int_{\Omega}\frac{8\mu_{k_j}^2\varepsilon^2(1+\alpha_{k_j})^2\left|x-p_{k_j}\right|^{2\alpha_{k_j}}}{(\mu_{k_j}^2\varepsilon^2+\left|x-p_{k_j}\right|^{2(1+\alpha_{k_j})})^2}\cdot\\&\quad\left[\log\frac{1}{(\mu_{k_i}^2\varepsilon^2+\left|x-p_{k_i}\right|^{2(1+\alpha_{k_i})})^2}+4(1+\alpha_{k_i})H_{t,\varepsilon}(x,p_{k_i})+O(\varepsilon^2)\right]\\&=\int_{\Omega\rho_{k_j}}\frac{16\mu_{k_j}^2(1+\alpha_{k_j})^{2\alpha_{k_j}}}{(\mu_{k_j}^2+|y|^{2(1+\alpha_{k_j})})^2}\log\frac{1}{\mu_{k_i}^2\varepsilon^2+\left|\rho_{k_j}y+p_{k_j}-p_{k_i}\right|^{2(1+\alpha_{k_i})}}\\&\quad+\int_{\Omega\rho_{k_j}}\frac{32\mu_{k_j}^2(1+\alpha_{k_i})(1+\alpha_{k_j})^2|y|^{2\alpha_{k_j}}}{(\mu_{k_j}^2+|y|^{2(1+\alpha_{k_j})})^2}H_{t,\varepsilon}(\rho_{k_j}y+p_{k_j},p_{k_i},)+O(\varepsilon^2).\end{aligned}$$

Then if $i\neq j$ for any i and j,

$$-\int_{\Omega}U_i\Delta U_j=32\pi(1+\alpha_{k_i})(1+\alpha_{k_j})G_{t,\varepsilon}(p_{k_i},p_{k_j})+O(\rho),\tag{72}$$

and if $i=j$ for any i,

$$-\int_{\Omega}U_i\Delta U_i=32\pi(1+\alpha_{k_i})^2H_{t,\varepsilon}(p_{k_i},p_{k_i})-32\pi(1+\alpha_{k_i})\log(\mu_{k_i}\varepsilon)-16\pi(1+\alpha_{k_i})+O(\rho).\tag{73}$$

On the other hand, from (21) and (22), it follows

$$\begin{aligned}\int_{\Omega_\varepsilon}W&=\sum_{i=1}^{n+m}\int_{B_{i,\delta\rho_{k_i}^{-1}}}W+\int_{\Omega_\varepsilon\setminus\cup_{i=1}^{n+m}B_{i,\delta\rho_{k_i}^{-1}}}W\\&=\sum_{i=1}^{n+m}\int_{B_{i,\delta\rho_{k_i}^{-1}}}\left(\frac{\varepsilon}{\rho_{k_i}}\right)^2\frac{8\mu_{k_i}^2(1+\alpha_{k_i})^2\left|\frac{\varepsilon y-p_{k_i}}{p_{k_i}}\right|^{2\alpha_{k_i}}}{[\mu_{k_i}^2+\left|\frac{\varepsilon y-p_{k_i}}{p_{k_i}}\right|^{2(1+\alpha_{k_i})}]^2}\left[1+O\left(\rho_{k_i}\left|\frac{\varepsilon y-p_{k_i}}{p_{k_i}}\right|\right)+O(\varepsilon^2)\right]+O(\varepsilon^2)\\&=\sum_{i=1}^{n+m}\int_{|z|\leq\frac{\delta}{\rho_{k_i}}}\frac{8\mu_{k_i}^2(1+\alpha_{k_i})^2|z|^{2\alpha_{k_i}}}{\left[\mu_{k_i}^2+|z|^{2(1+\alpha_{k_i})}\right]^2}\left[1+O(\rho_{k_i}|z|)+O(\varepsilon^2)\right]dz+O(\varepsilon^2)\\&=\sum_{i=1}^{n+m}8\pi(1+\alpha_{k_i})+O(\rho).\end{aligned}$$

As a result, it derives

$$\varepsilon^2 \int_{\Omega} S(x)f(x)e^U = \int_{\Omega_\varepsilon} W(y) = \sum_{i=1}^{n+m} 8\pi(1+\alpha_{k_i}) + O(\rho). \tag{74}$$

Now using the choice for μ_{k_i}'s by (15), together with (71)-(74), it holds

$$\begin{aligned} J_{\varepsilon,t}(U) &= 8\pi(1-t)\sum_{i=1}^{n+m}(1+\alpha_{k_i})\log\frac{8(1+\alpha_{k_i})^2}{f(p_{k_i})Q_{k_i}(p_{k_i})} \\ &- 16\pi(1-t)\sum_{i=1}^{n+m}(1+\alpha_{k_i})^2 H_{t,\varepsilon}(p_{k_i},p_{k_i}) - 16\pi(1-t)\sum_{i=1}^{n+m}(1+\alpha_{k_i}) \\ &- 16\pi(1-t)\sum_{i,j=1,i\neq j}^{n+m}(1+\alpha_{k_i})(1+\alpha_{k_j})G_{t,\varepsilon}(p_{k_j},p_{k_i}) - 16\pi(1-t)\log\varepsilon\sum_{i=1}^{n+m}(1+\alpha_{k_i}) + O(\rho), \end{aligned}$$

which derives the asymptotical expansion (70) by (5), (67), and (68). □

4 Proofs of theorems

In this section, we carry out the proofs of Theorems 1.1 and 1.2 basing on the finite dimensional reduction. Now we introduce the definition of C^0-stable critical point of the function $\phi_{n,m}$ just like in [28,36,39].

Definition 4.1. We say that p is a C^0-stable critical point of $\phi_{n,m}$ in $(\Omega' \setminus \Gamma)^m \setminus \Delta_m$, which says that if for any sequence of the functions ψ_j such that $\psi_j \to \phi_{n,m}$ uniformly on the compact subsets of $(\Omega' \setminus \Gamma)^m \setminus \Delta_m$, ψ_j has a critical point ξ_j such that $\psi_j(\xi_j) \to + \phi_{n,m}$.

In particular, if p is a strict local maximum or minimum point of $\phi_{n,m}$, p is a C^0-stable critical point of $\phi_{n,m}$.

Proof of Theorem 1.1. Case (i): $m \geq 1$ and $0 \leq n \leq N$. Let

$$v(y) = V(p')(y) + \phi(p')(y), \quad \forall\, y \in \bar{\Omega}_\varepsilon,$$

where φ is the unique solution of the problem (55), which is established in Proposition 3.2. From Proposition 3.3, $v(y)$ is a solution of the Equation (16), namely $\tilde{\phi}(p)(x)$ is a solution of the Equation (3) if $p_\varepsilon = (p_{k_{n+1},\varepsilon}, \ldots, p_{k_{n+m},\varepsilon})$ satisfying the relation (33) is a critical point of the function $F_{\varepsilon,t}(p)$ with $t \in (0, 1]$. This implies that we only need to find a critical point p_ε of the following function in $(\Omega' \setminus \Gamma)^m \setminus \Delta_m$

$$\tilde{F}_{\varepsilon,t}(p) = \begin{cases} F_{\varepsilon,1}(p) - \beta, & \text{for} \quad t = 1, \\ F_{\varepsilon,t}(p) - (1-t)\beta, & \text{for} \quad t \in (0,1), \end{cases} \tag{75}$$

where

$$\begin{aligned} \beta &= 8\pi m\log 8 + 8\pi\sum_{i=1}^{n}(1+\alpha_{k_i})\log\frac{8(1+\alpha_{k_i})^2}{f(p_{k_i})Q_{k_i}(p_{k_i})} - 16\pi\sum_{i,j=1,i\neq j}^{n}(1+\alpha_{k_i})(1+\alpha_{k_j})G_1(p_{k_j}p_{k_i}) \\ &- 16\pi\log\varepsilon\sum_{i=1}^{n+m}(1+\alpha_{k_i}) - 16\pi\sum_{i=1}^{n}(1+\alpha_{k_i})^2 H_1(p_{k_i},p_{k_i}) - 16\pi\sum_{i=1}^{n+m}(1+\alpha_{k_i}). \end{aligned}$$

From Propositions 3.3 and 3.4, it follows that

$$\tilde{F}_{\varepsilon,t}(p) = \begin{cases} \theta_{\varepsilon,1}(p) + \rho\Theta_\varepsilon(p) - 16\pi\varphi_{n,m}(p), & \text{for} \quad t = 1, \\ \theta_{\varepsilon,t}(p) + \rho\Theta_\varepsilon(p) - 16\pi(1-t)\varphi_{n,m}(p), & \text{for} \quad t \in (0,1), \end{cases} \tag{76}$$

where $\theta_{\varepsilon,t}(p) = O(\rho^2|\log\varepsilon|)$ for any $t \in (0, 1]$, and $\Theta_\varepsilon(p)$ is uniformly bounded on any compact subset S of $(\Omega' \setminus \Gamma)^m \setminus \Delta_m$ for ε small. Then $\tilde{F}_{\varepsilon,1}(p) \to -16\pi\varphi_{n,m}(p)$, and $\tilde{F}_{\varepsilon,t}(p) \to -16\pi(1-t)\varphi_{n,m}(p)$ for any $t \in (0,1)$, uniformly on $\mathcal{S}$ for ε small. By Definition 4.1, there exists a critical point $p_\varepsilon = (p_{k_{n+1},\varepsilon}, \ldots, p_{k_{n+m},\varepsilon})$ of the function $\tilde{F}_{\varepsilon,t}$ such that $\tilde{F}_{\varepsilon,1}(p\varepsilon) \to -16\pi\varphi_{n,m}(p^*)$, and $\tilde{F}_{\varepsilon,t}(p\varepsilon) \to -16\pi(1-t)\varphi_{n,m}(p^*)$ for any $t \in (0,1)$. Moreover, up to a subsequence, there exists $p = (p_{k_{n+1}}, \ldots, p_{k_{n+m}}) \in (\Omega' \backslash \Gamma)^m \backslash \Delta_m$ such that $p_\varepsilon \to p$ for ε small, and $\phi_{n,m}(p^*) = \phi_{n,m}(p)$. Hence, $u_\varepsilon = U(p_\varepsilon) + \tilde{\phi}(p_\varepsilon)$ is a family of solutions of the Equation (3). As a consequence, from the related properties of $U(p_\varepsilon)$ and $\tilde{\phi}(p_\varepsilon)$, we easily know that for any $\lambda > 0$, u_ε is uniformly bounded on $\Omega \backslash \bigcup_{i=1}^{n+m} B_\lambda(p_{k_i})$, and $\sup_{B_\lambda(p_{k_i})} u_\varepsilon \to +\infty$ for ε small.

Finally, we show that u_ε satisfies the concentration property:

$$I = \varepsilon^2 \int_\Omega |x - p_1|^{2\alpha_1} \cdots |x - p_N|^{2\alpha_N} f(x) e^{u_\varepsilon} \Psi \to 8\pi \sum_{i=1}^{n+m} (1 + \alpha_{k_i}) \Psi(p_{k_i}), \quad \forall \Psi \in C(\bar{\Omega}), \tag{77}$$

for ε small. In fact, using the inequality $|e^s - 1| \le e^{|s|}|s|$ for any $s \in \mathbb{R}$ and the estimate (56), we obtain

$$\begin{aligned} I &= \varepsilon^2 \int_\Omega |x - p_1|^{2\alpha_1} \cdots |x - p_N|^{2\alpha_N} f(x) e^{U(p_\varepsilon)(x)} \Psi + o(1) \\ &= \int_{\Omega_\varepsilon} W(p_\varepsilon)(y) \Psi(\varepsilon y) + o(1). \end{aligned}$$

Then from the asymptotical expression (21) and (22) of $W(p_\varepsilon)(y)$, we can easily get

$$\int_{\Omega_\varepsilon} W(p_\varepsilon)(y)\Psi(\varepsilon y) = 8\pi \sum_{i=1}^{n+m} (1 + \alpha_{k_i}) \Psi(p_{k_i}) + o(1),$$

which implies (77).

Case (ii): $m = 0$ and $1 \le n \le N$. From Proposition 3.2 (ii), we find that $u_\varepsilon(x) = U(p)(x) + \phi(\frac{1}{\varepsilon}x)$ is a family of solutions of the Equation (3) with $p = (p_{k_1}, \ldots, p_{k_n})$. Then we can get the needed multiple blowing-up and concentrating properties of u_ε through the similar proof of Case (i). □

In order to give the proof of Theorem 1.2, we need a version of the maximum principle under Dirichlet-Robin boundary conditions, which is the extension of the corresponding one with respect to Dirichlet or Robin boundary condition only.

Lemma 4.2 *Assume that $T \subseteq \partial\Omega$ is a relatively closed subset, $b > 0$ is a smooth function on $T, F : T \to \mathbb{R}$is a smooth function. If u is a solution of the equation*

$$\begin{cases} \Delta u = 0, & in\ \Omega, \\ \dfrac{\partial u}{\partial \nu} + \lambda b(x) u = F, & on\ T, \\ u = 0, & on\ \partial\Omega \backslash T, \end{cases}$$

where $\lambda > 0$, there exists a constant $C(b) > 0$ only depending on $b(x)$ such that

$$\|u\|_{L^\infty(\Omega)} + \left\|\operatorname{dist}(x, \partial\Omega)\nabla u\right\|_{L^\infty(\Omega)} \le \frac{C(b)}{\lambda} \|F\|_{L^\infty(T)}.$$

Proof. The proof is similar to that of Lemma 2.6 in [33]. □

Proof of Theorem 1.2. Using the maximum principle with Dirichlet-Robin boundary conditions instead of Robin boundary condition if necessary (see Lemma 4.2), the proof can be similarly given out through that of Theorem 1.1. □

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Authors' contributions
All authors typed, read and approved the final manuscript.

Competing interests
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