Hussain et al. *Journal of Inequalities and Applications* 2014, 2014:348
http://www.journalofinequalitiesandapplications.com/content/2014/1/348

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Generalized fixed point theorems for multi-valued α-ψ-contractive mappings

Nawab Hussain[1*], Jamshaid Ahmad[2] and Akbar Azam[2]

*Correspondence: nhusain@kau.edu.sa
[1]Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

The aim of this paper is to establish certain new fixed point results for multi-valued as well as single-valued maps satisfying an α-ψ-contractive conditions in complete metric space. As an application, we derive some new fixed point theorems for ψ-graphic contractions defined on a metric space endowed with a graph as well as an ordered metric space. The presented results complement and extend some very recent results proved by Asl *et al.* (Fixed Point Theory Appl. 2012:212, 2012) and Samet *et al.* (Nonlinear Anal. 75:2154-2165, 2012) as well as other theorems given by Hussain *et al.* (Fixed Point Theory Appl. 2013:212, 2013). Some comparative examples are constructed which illustrate the superiority of our results to the existing ones in the literature.

MSC: 46S40; 47H10; 54H25

Keywords: metric space; fixed point; α-ψ-contraction

1 Introduction

In metric fixed point theory the contractive conditions on underlying functions play an important role for finding solutions of fixed point problems. The Banach contraction principle [1] is a fundamental result in metric fixed point theory. Over the years, it has been generalized in different directions by several mathematicians (see [1–25]). In particular, there has been a number of studies involving altering distance functions which alter the distance between two points in a metric space. In 2012, Samet *et al.* [25] introduced the concepts of α-ψ-contractive and α-admissible mappings and established various fixed point theorems for such mappings in complete metric spaces.

Denote with Ψ the family of nondecreasing functions $\psi : [0,+\infty) \to [0,+\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$ for all $t > 0$, where ψ^n is the nth iterate of ψ.

The following lemma is well known.

Lemma 1 *If $\psi \in \Psi$, then the following hold*:

(i) *$(\psi^n(t))_{n\in\mathbb{N}}$ converges to 0 as $n \to \infty$ for all $t \in (0,+\infty)$*;

(ii) *$\psi(t) < t$ for all $t > 0$*;

(iii) *$\psi(t) = 0$ iff $t = 0$*.

Samet *et al.* [25] defined the notion of α-admissible mappings as follows.

Definition 2 Let T be a self-mapping on X and $\alpha : X \times X \to [0, +\infty)$ be a function. We say that T is a α-admissible mapping if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq 1.$$

Theorem 3 [25] *Let (X, d) be a complete metric space and T be α-admissible mapping. Assume that*

$$\alpha(x, y)d(Tx, Ty) \leq \psi\big(d(x, y)\big) \tag{1.1}$$

for all $x, y \in X$, where $\psi \in \Psi$. Also, suppose that

(i) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(ii) *either T is continuous or for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

Then T has a fixed point.

Afterwards, Asl *et al.* [21] generalized these notions by introducing the concepts of α_*-ψ-contractive multifunctions, and of α_*-admissibility, and they obtained some fixed point results for these multifunctions.

Definition 4 [21] Let (X, d) be a metric space, $T : X \to 2^X$ be a given closed-valued multifunction. We say that T is called α_*-ψ-contractive multifunction if there exist two functions $\alpha : X \times X \to [0, +\infty)$ and $\psi \in \Psi$ such that

$$\alpha_*(Tx, Ty)H(Tx, Ty) \leq \psi\big(d(x, y)\big)$$

for all $x, y \in X$, where H is the Hausdorff generalized metric, $\alpha_*(A, B) = \inf\{\alpha(a, b) : a \in A, b \in B\}$ and 2^X denotes the family of all nonempty subsets of X.

Definition 5 [21] Let (X, d) be a metric space, $T : X \to 2^X$ be a given closed-valued multifunction and $\alpha : X \times X \to [0, +\infty)$. We say that T is called α_*-admissible whenever $\alpha(x, y) \geq 1$ implies that $\alpha_*(Tx, Ty) \geq 1$.

Very recently Hussain *et al.* [12] modified the notions of α_*-admissible and α_*-ψ-contractive mappings as follows:

Definition 6 Let $T : X \to 2^X$ be a multifunction, $\alpha, \eta : X \times X \to \mathbb{R}_+$ be two functions where η is bounded. We say that T is α_*-admissible mapping with respect to η if

$$\alpha(x, y) \geq \eta(x, y) \quad \text{implies} \quad \alpha_*(Tx, Ty) \geq \eta_*(Tx, Ty), \quad x, y \in X,$$

where

$$\alpha_*(A, B) = \inf_{x \in A, y \in B} \alpha(x, y) \quad \text{and} \quad \eta_*(A, B) = \sup_{x \in A, y \in B} \eta(x, y).$$

If $\eta(x, y) = 1$ for all $x, y \in X$, then this definition reduces to Definition 5. In the case $\alpha(x, y) = 1$ for all $x, y \in X$, T is called η_*-subadmissible mapping.

Hussain *et al.* [12] proved following generalization of the above mentioned results of [21].

Theorem 7 *Let (X,d) be a complete metric space and $T: X \to 2^X$ be a α_*-admissible with respect to η and the closed-valued multifunction on X. Assume that for $\psi \in \Psi$,*

$$\forall x,y \in X, \quad \alpha_*(Tx,Ty) \geq \eta_*(Tx,Ty) \quad \Longrightarrow \quad H(Tx,Ty) \leq \psi\bigl(d(x,y)\bigr). \tag{1.2}$$

Also suppose that the following assertions hold:

(i) *there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq \eta(x_0,x_1)$;*

(ii) *for a sequence $\{x_n\} \subset X$ converging to $x \in X$ and $\alpha(x_n,x_{n+1}) \geq \eta(x_n,x_{n+1})$ for all $n \in \mathbb{N}$, we have $\alpha(x_n,x) \geq \eta(x_n,x)$ for all $n \in \mathbb{N}$.*

Then T has a fixed point.

For more details on α-ψ-contractions and fixed point theory, we refer the reader to [3, 6, 10, 13, 14, 22, 23, 26–29].

The aim of this paper is to unify the concepts of α-ψ-contractive type mappings and establish some new fixed point theorems in complete metric spaces for such mappings.

Let (X,d) be a complete metric space, $x_0 \in X$ and $r > 0$. We denote by $B(x_0,r) = \{x \in X : d(x_0,x) < r\}$ the open ball with center x_0 and radius r and by $\overline{B(x_0,r)} = \{x \in X : d(x_0,x) \leq r\}$ the closed ball with center x_0 and radius r.

The following lemmas of Nadler will be needed in the sequel.

Lemma 8 [19] *Let A and B be nonempty, closed and bounded subsets of a metric space (X,d) and $0 < h \in \mathbb{R}$. Then, for every $b \in B$, there exists $a \in A$ such that $d(a,b) \leq H(A,B) + h$.*

Lemma 9 [4] *Let (X,d) be a metric space and B be nonempty, closed subsets of X and $q > 1$. Then, for each $x \in X$ with $d(x,B) > 0$ and $q > 1$, there exists $b \in B$ such that $d(x,b) < qd(x,B)$.*

2 Main result

The following result, regarding the existence of the fixed point of the mapping satisfying an α-ψ-contractive condition on the closed ball, is very useful in the sense that it requires the contractiveness of the mapping only on the closed ball instead of the whole space.

Theorem 10 *Let (X,d) be a complete metric space and $T: X \to 2^X$ be an α_*-admissible and closed-valued multifunction on X. Assume that for $\psi \in \Psi$,*

$$\alpha_*(Tx,Ty)H(Tx,Ty) \leq \psi\bigl(d(x,y)\bigr) \tag{2.1}$$

for all $x,y \in \overline{B(x_0,r)}$ and for $x_0 \in X$, there exists $x_1 \in Tx_0$ such that

$$\sum_{i=0}^{n} \psi^i\bigl(d(x_0,x_1)\bigr) < r \tag{2.2}$$

for all $n \in \mathbb{N}$ and $r > 0$. Also suppose that the following assertions hold:

(i) $\alpha(x_0,x_1) \geq 1$ *for $x_0 \in X$ and $x_1 \in Tx_0$;*

(ii) *for a sequence* $\{x_n\}$ *in* $\overline{B(x_0,r)}$ *converging to* $x \in \overline{B(x_0,r)}$ *and* $\alpha(x_n,x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$, *we have* $\alpha(x_n,x) \geq 1$ *for all* $n \in \mathbb{N}$.

Then T has a fixed point.

Proof Since $\alpha(x_0,x_1) \geq 1$ and T is α_*-admissible, so $\alpha_*(Tx_0,Tx_1) \geq 1$. From (2.2), we get

$$d(x_0,x_1) < \sum_{i=0}^{n} \psi^i\bigl(d(x_0,x_1)\bigr) < r.$$

It follows that

$$x_1 \in \overline{B(x_0,r)}.$$

If $x_0 = x_1$, then

$$\alpha_*(Tx_0,Tx_1)H(Tx_0,Tx_1) \leq \psi\bigl(d(x_0,x_1)\bigr) = 0$$

implies that

$$Tx_0 = Tx_1,$$

and we have finished. Assume that $x_0 \neq x_1$. By Lemmas 1 and 8, we take $x_2 \in Tx_1$ and $h > 0$ as $h = \psi^2(d(x_0,x_1))$. Then

$$\begin{aligned} 0 &< d(x_1,x_2) \leq H(Tx_0,Tx_1) + h \\ &\leq \psi\bigl(d(x_0,x_1)\bigr) + \psi^2\bigl(d(x_0,x_1)\bigr) \\ &= \sum_{i=1}^{2} \psi^i\bigl(d(x_0,x_1)\bigr). \end{aligned}$$

Note that $x_2 \in \overline{B(x_0,r)}$, since

$$\begin{aligned} d(x_0,x_2) &\leq d(x_0,x_1) + d(x_1,x_2) \\ &\leq d(x_0,x_1) + \psi\bigl(d(x_0,x_1)\bigr) + \psi^2\bigl(d(x_0,x_1)\bigr) \\ &= \sum_{i=0}^{2} \psi^i\bigl(d(x_0,x_1)\bigr) < r. \end{aligned}$$

By repeating this process, we can construct a sequence x_n of points in $\overline{B(x_0,r)}$ such that $x_{n+1} \in Tx_n$, $x_n \neq x_{n-1}$, $\alpha(x_n,x_{n+1}) \geq 1$ with

$$d(x_n,x_{n+1}) \leq \sum_{i=1}^{n+1} \psi^i\bigl(d(x_0,x_1)\bigr). \tag{2.3}$$

Now, for each $n, m \in N$ with $m > n$ using the triangular inequality, we obtain

$$d(x_n,x_m) \leq \sum_{k=n}^{m-1} d(x_k,x_{k+1}) \leq \sum_{k=n}^{m} \psi^k\bigl(d(x_0,x_1)\bigr). \tag{2.4}$$

Thus we proved that $\{x_n\}$ is a Cauchy sequence. Since $\overline{B(x_0,r)}$ is closed. So there exists $x^* \in \overline{B(x_0,r)}$ such that $x_n \to x^*$ as $n \to \infty$. Now we prove that $x^* \in Tx^*$. Since $\alpha(x_n,x^*) \geq 1$ for all n and T is α_*-admissible with respect to η, so $\alpha_*(Tx_n, Tx^*) \geq 1$ for all n. Then

$$\begin{aligned} d\bigl(x^*, Tx^*\bigr) &\leq \alpha_*\bigl(Tx_n, Tx^*\bigr)H\bigl(Tx_n, Tx^*\bigr) + d\bigl(x_n, x^*\bigr) \\ &\leq \psi\bigl(d\bigl(x_n, x^*\bigr)\bigr) + d\bigl(x_n, x^*\bigr) \\ &\leq \psi\bigl(d\bigl(x_n, x^*\bigr)\bigr) + d\bigl(x_n, x^*\bigr). \end{aligned} \tag{2.5}$$

Taking the limit as $n \to \infty$ in (2.5), we get $d(x^*, Tx^*) = 0$. Thus $x^* \in Tx^*$. □

Example 11 Let $X = [0,\infty)$ and $d(x,y) = |x-y|$. Define the multi-valued mapping $T : X \to 2^X$ by

$$Tx = \begin{cases} [0, \frac{x}{2}] & \text{if } x \in [0,1], \\ [\frac{4x}{5}, \frac{5x}{6}] & \text{if } x \in (1,\infty). \end{cases}$$

Considering, $x_0 = \frac{1}{2}$ and $x_1 = \frac{1}{4}$, $r = \frac{1}{2}$, then $\overline{B(x_0,r)} = [0,1]$ and

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x,y \in [0,1], \\ \frac{3}{2} & \text{otherwise.} \end{cases}$$

Clearly T is an α-ψ-contractive mapping with $\psi(t) = \frac{t}{2}$. Now

$$\begin{aligned} &d(x_0,x_1) = \frac{1}{4}, \\ &\sum_{i=1}^{n} \psi^n\bigl(d(x_0,x_1)\bigr) = \frac{1}{4}\sum_{i=0}^{n}\frac{1}{2^n} < 2\left(\frac{1}{4}\right) = \frac{1}{2} = r. \end{aligned}$$

We prove that all the conditions of our Theorem 10 are satisfied only for $x,y \in \overline{B(x_0,r)}$. Without loss of generality, we suppose that $x \leq y$. The contractive condition of theorem is trivial for the case when $x = y$. So we suppose that $x < y$. Then

$$\alpha_*(Tx, Ty)H(Tx, Ty) = \frac{1}{2}|y-x| = \psi\bigl(d(x,y)\bigr).$$

Put $x_0 = \frac{1}{2}$ and $x_1 = \frac{1}{4}$. Then $\alpha(x_0,x_1) \geq 1$. Then T has a fixed point 0.

Now we prove that the contractive condition is not satisfied for $x,y \notin \overline{B(x_0,r)}$. We suppose $x = \frac{3}{2}$ and $y = 2$, then

$$\alpha_*(Tx, Ty)H(Tx, Ty) = \frac{3}{5} \geq \frac{1}{4} = \psi\bigl(d(x,y)\bigr).$$

Similarly we can deduce the following corollaries.

Corollary 12 *Let (X,d) be a complete metric space and $T : X \to 2^X$ be an α_*-admissible and closed-valued multifunction on X. Assume that for $\psi \in \Psi$, we have*

$$\bigl(\alpha_*(Tx, Ty) + 1\bigr)^{H(Tx,Ty)} \leq 2^{\psi(d(x,y))} \tag{2.6}$$

for all $x, y \in \overline{B(x_0, r)}$ *and for* $x_0 \in X$, *there exists* $x_1 \in Tx_0$ *such that*

$$\sum_{i=0}^{n} \psi^i\bigl(d(x_0, x_1)\bigr) < r$$

for all $n \in \mathbb{N}$ *and* $r > 0$. *Also suppose that the following assertions hold*:
- (i) $\alpha(x_0, x_1) \geq 1$ *for* $x_0 \in X$ *and* $x_1 \in Tx_0$;
- (ii) *for a sequence* $\{x_n\}$ *in* $\overline{B(x_0, r)}$ *converging to* $x \in \overline{B(x_0, r)}$ *and* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$, *we have* $\alpha(x_n, x) \geq 1$ *for all* $n \in \mathbb{N}$.

Then T *has a fixed point.*

Corollary 13 *Let* (X, d) *be a complete metric space and* $T : X \to 2^X$ *be an* α_**-admissible and closed-valued multifunction on* X. *Assume that for* $\psi \in \Psi$, *we have*

$$\bigl(H(Tx, Ty) + l\bigr)^{\alpha_*(Tx,Ty)} \leq \psi\bigl(d(x, y)\bigr) + l$$

for all $x, y \in \overline{B(x_0, r)}$ *and* $l > 0$ *and for* $x_0 \in X$, *there exists* $x_1 \in Tx_0$ *such that*

$$\sum_{i=0}^{n} \psi^i\bigl(d(x_0, x_1)\bigr) < r$$

for all $n \in \mathbb{N}$ *and* $r > 0$. *Also suppose that the following assertions hold*:
- (i) $\alpha(x_0, x_1) \geq 1$ *for* $x_0 \in X$ *and* $x_1 \in Tx_0$;
- (ii) *for a sequence* $\{x_n\}$ *in* $\overline{B(x_0, r)}$ *converging to* $x \in \overline{B(x_0, r)}$ *and* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$, *we have* $\alpha(x_n, x) \geq 1$ *for all* $n \in \mathbb{N}$.

Then T *has a fixed point.*

Theorem 14 *Let* (X, d) *be a complete metric space and* $T : X \to 2^X$ *be an* α_**-admissible and closed-valued multifunction on* X. *Assume that for* $\psi \in \Psi$, *we have*

$$\alpha_*(Tx, Ty)H(Tx, Ty) \leq \psi\left(\max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)}\right\}\right) \tag{2.7}$$

for all $x, y \in X$. *Also suppose that the following assertions hold*:
- (i) *there exist* $x_0 \in X$ *and* $x_1 \in Tx_0$ *with* $\alpha(x_0, x_1) \geq 1$;
- (ii) *for a sequence* $\{x_n\}$ *in* X *converging to* $x \in X$ *and* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$, *we have* $\alpha(x_n, x) \geq 1$ *for all* $n \in \mathbb{N}$.

Then T *has a fixed point.*

Proof Since $\alpha(x_0, x_1) \geq 1$ and T is α_*-admissible, so $\alpha_*(Tx_0, Tx_1) \geq 1$. If $x_0 = x_1$, then we have nothing to prove. Let $x_0 \neq x_1$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T. Assume that $x_1 \notin Tx_1$, then from (2.7), we get

$$\begin{aligned} 0 &< d(x_1, Tx_1) \\ &\leq H(Tx_0, Tx_1) \\ &\leq \psi\left(\max\left\{d(x_0, x_1), d(x_0, Tx_0), d(x_1, Tx_1), \frac{d(x_0, Tx_0)d(x_1, Tx_1)}{1 + d(x_0, x_1)}\right\}\right) \end{aligned}$$

$$\begin{aligned}
&\leq \psi\left(\max\left\{d(x_0,x_1),d(x_0,x_1),d(x_1,Tx_1),\frac{d(x_0,x_1)d(x_1,Tx_1)}{1+d(x_0,x_1)}\right\}\right)\\
&= \psi\left(\max\left\{d(x_0,x_1),d(x_1,Tx_1)\right\}\right).
\end{aligned}$$

If $\max\{d(x_1,Tx_1),d(x_0,x_1)\} = d(x_1,Tx_1)$, then $d(x_1,Tx_1) \leq \psi(d(x_1,Tx_1))$. Since $\psi(t) < t$ for all $t > 0$. Then we get a contradiction. Hence, we obtain $\max\{d(x_1,Tx_1),d(x_0,x_1)\} = d(x_0,x_1)$. So

$$d(x_1,Tx_1) \leq \psi\left(d(x_0,x_1)\right).$$

Let $q > 1$, then from Lemma 9 we take $x_2 \in Tx_1$ such that

$$d(x_1,x_2) < qd(x_1,Tx_1) \leq q\psi\left(d(x_0,x_1)\right). \tag{2.8}$$

It is clear that $x_1 \neq x_2$. Put $q_1 = \frac{\psi(q\psi(d(x_0,x_1)))}{\psi(d(x_1,x_2))}$. Then $q_1 > 1$ and $\alpha(x_1,x_2) \geq 1$. Since T is α_*-admissible, so $\alpha_*(Tx_1,Tx_2) \geq 1$. If $x_2 \in Tx_2$, then x_2 is fixed point of T. Assume that $x_2 \notin Tx_2$. Then from (2.7), we get

$$\begin{aligned}
0 &< d(x_2,Tx_2) \leq \alpha_*(Tx_1,Tx_2)H(Tx_1,Tx_2)\\
&\leq \psi\left(\max\left\{d(x_1,x_2),d(x_1,Tx_1),d(x_2,Tx_2),\frac{d(x_1,Tx_1)d(x_2,Tx_2)}{1+d(x_1,x_2)}\right\}\right)\\
&\leq \psi\left(\max\left\{d(x_1,x_2),d(x_1,x_2),d(x_2,Tx_2),\frac{d(x_1,x_2)d(x_2,Tx_2)}{1+d(x_1,x_2)}\right\}\right)\\
&= \psi\left(\max\left\{d(x_1,x_2),d(x_2,Tx_2)\right\}\right).
\end{aligned}$$

If $\max\{d(x_2,Tx_2),d(x_1,x_2)\} = d(x_2,Tx_2)$, we get contradiction to the fact $d(x_2,Tx_2) < d(x_2,Tx_2)$. Hence we obtain

$$\max\left\{d(x_2,Tx_2),d(x_1,x_2)\right\} = d(x_1,x_2).$$

So $d(x_2,Tx_2) \leq \psi(d(x_1,x_2))$. Since $q_1 > 1$, so by Lemma 9 we can find $x_3 \in Tx_2$ such that

$$\begin{aligned}
&d(x_2,x_3) < q_1 d(x_2,Tx_2) \leq q_1\psi\left(d(x_1,x_2)\right),\\
&d(x_2,x_3) < q_1\psi\left(d(x_1,x_2)\right) \leq q_1\psi\left(d(x_1,x_2)\right) = \psi\left(q\psi\left(d(x_0,x_1)\right)\right).
\end{aligned} \tag{2.9}$$

It is clear that $x_2 \neq x_3$. Put $q_2 = \frac{\psi^2(q\psi(d(x_0,x_1)))}{\psi(d(x_2,x_3))}$. Then $q_2 > 1$ and $\alpha(x_2,x_3) \geq 1$. Since T is α_*-admissible, so $\alpha_*(Tx_2,Tx_3) \geq 1$. If $x_3 \in Tx_3$, then x_3 is fixed point of T. Assume that $x_3 \notin Tx_3$. From (2.7), we have

$$\begin{aligned}
0 &< d(x_3,Tx_3) \leq \alpha_*(Tx_2,Tx_3)H(Tx_2,Tx_3)\\
&\leq \psi\left(\max\left\{d(x_2,x_3),d(x_2,Tx_2),d(x_3,Tx_3)\frac{d(x_2,Tx_2)d(x_3,Tx_3)}{1+d(x_2,x_3)}\right\}\right)\\
&\leq \psi\left(\max\left\{d(x_2,x_3),d(x_2,x_3),d(x_3,Tx_3)\frac{d(x_2,x_3)d(x_3,Tx_3)}{1+d(x_2,x_3)}\right\}\right)\\
&= \psi\left(\max\left\{d(x_2,x_3),d(x_3,Tx_3)\right\}\right).
\end{aligned}$$

If $\max\{d(x_3,Tx_3),d(x_2,x_3)\}=d(x_3,Tx_3)$. Then we get a contradiction. So $\max\{d(x_3,Tx_3), d(x_2,x_3)\}=d(x_2,x_3)$. Thus

$$d(x_3,Tx_3)\leq\psi\bigl(d(x_2,x_3)\bigr).$$

Since $q_2>1$, so by Lemma 9 we can find $x_4\in Tx_3$ such that

$$d(x_3,x_4)<q_2d(x_3,Tx_3)\leq q_2\psi\bigl(d(x_2,x_3)\bigr)=\psi^2\bigl(q\psi\bigl(d(x_0,x_1)\bigr)\bigr). \tag{2.10}$$

Continuing in this way, we can generate a sequence $\{x_n\}$ in X such that $x_n\in Tx_{n-1}$ and $x_n\neq x_{n-1}$, and

$$d(x_n,x_{n+1})\leq\psi^{n-1}\bigl(q\psi\bigl(d(x_0,x_1)\bigr)\bigr) \tag{2.11}$$

for all n. Now, for each $m>n$, we have

$$d(x_n,x_m)\leq\sum_{i=n}^{m-1}d(x_i,x_{i+1})\leq\sum_{i=n}^{m-1}\psi^{i-1}\bigl(q\psi\bigl(d(x_0,x_1)\bigr)\bigr).$$

This implies that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $x^*\in X$ such that $x_n\longrightarrow x^*$ as $n\longrightarrow\infty$. We now show that $x^*\in Tx^*$. Since $\alpha(x_n,x^*)\geq 1$ for all n and T is α_*-admissible, so $\alpha_*(Tx_n,Tx^*)\geq 1$ for all n. Then

$$\begin{aligned}d\bigl(x^*,Tx^*\bigr)&\leq\alpha_*\bigl(Tx_n,Tx^*\bigr)H\bigl(Tx^*,Tx_n\bigr)+d\bigl(x_n,x^*\bigr)\\&\leq\psi\left(\max\left\{d\bigl(x_n,x^*\bigr),d(x_n,Tx_n),d\bigl(x^*,Tx^*\bigr),\frac{d(x_n,Tx_n)d(x^*,Tx^*)}{1+d(x_n,x^*)}\right\}\right)\\&\quad+d\bigl(x_n,x^*\bigr)\\&\leq\psi\left(\max\left\{d\bigl(x_n,x^*\bigr),d(x_n,x_{n+1}),d\bigl(x^*,Tx^*\bigr),\frac{d(x_n,x_{n+1})d(x^*,Tx^*)}{1+d(x_n,x^*)}\right\}\right)\\&\quad+d\bigl(x_n,x^*\bigr),\end{aligned}$$

and taking the limit as $n\to\infty$, we get $d(x^*,Tx^*)=0$. Thus $x^*\in Tx^*$. □

Example 15 Let $X=[0,1]$ and $d(x,y)=|x-y|$. Define $T:X\to 2^X$ by $Tx=[0,\frac{x}{10}]$ for all $x\in X$ and

$$\alpha(x,y)=\begin{cases}\frac{1}{|x-y|} & \text{if } x\neq y,\\ 1 & \text{if } x=y.\end{cases}$$

Then $\alpha(x,y)\geq 1\Longrightarrow\alpha^*(Tx,Ty)=\inf\{\alpha(a,b):a\in Tx,b\in Ty\}\geq 1$. Then clearly T is α^*-admissible. Now, for x, y and $x\leq y$, it is easy to check that

$$\alpha_*(Tx,Ty)H(Tx,Ty)\leq\psi\left(\max\left\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\right\}\right),$$

where $\psi(t)=\frac{t}{5}$, for all $t\geq 0$. Put $x_0=1$ and $x_1=\frac{1}{2}$. Then $\alpha(x_0,x_1)=2>1$. Then T has fixed point 0.

Corollary 16 *Let (X,d) be a complete metric space and $T : X \to 2^X$ be an α_*-admissible and closed-valued multifunction on X. Assume that for $\psi \in \Psi$, we have*

$$\left(\alpha_*(Tx, Ty) + 1\right)^{H(Tx,Ty)} \leq 2^{\psi(R(x,y))},$$

where

$$R(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1 + d(x,y)}\right\}$$

for all $x, y \in X$. Also suppose that the following assertions hold:

(i) $\alpha(x_0, x_1) \geq 1$ *for $x_0 \in X$ and $x_1 \in Tx_0$;*

(ii) *for a sequence $\{x_n\}$ in X converging to $x \in X$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

Then T has a fixed point.

Corollary 17 *Let (X,d) be a complete metric space and $T : X \to 2^X$ be an α_*-admissible and closed-valued multifunction on X. Assume that for $\psi \in \Psi$, we have*

$$\left(H(Tx, Ty) + l\right)^{\alpha_*(Tx,Ty)} \leq \psi\left(R(x,y)\right) + l,$$

where

$$R(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1 + d(x,y)}\right\}$$

for all $x, y \in X$ and $l > 0$. Also suppose that the following assertions hold:

(i) $\alpha(x_0, x_1) \geq 1$ *for $x_0 \in X$ and $x_1 \in Tx_0$;*

(ii) *for a sequence $\{x_n\}$ in X converging to $x \in X$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

Then T has a fixed point.

If T is single-valued in Theorem 14, we obtain the following fixed point results.

Theorem 18 *Let (X,d) be a complete metric space and $T : X \to X$ be an α-admissible mapping. Assume that for $\psi \in \Psi$, we have*

$$\alpha(Tx, Ty)d(Tx, Ty) \leq \psi\left(\max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1 + d(x,y)}\right\}\right) \tag{2.12}$$

for all $x, y \in X$. Also suppose that the following assertions hold:

(i) *there exists $x_0 \in X$ with $\alpha(x_0, Tx_0) \geq 1$;*

(ii) *for a sequence $\{x_n\}$ in X converging to $x \in X$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

Then T has a fixed point.

Corollary 19 *Let (X,d) be a complete metric space and $T : X \to X$ be an α-admissible mapping. Assume that for $\psi \in \Psi$, we have*

$$\left(\alpha(Tx, Ty) + 1\right)^{d(Tx,Ty)} \leq 2^{\psi(R(x,y))},$$

where

$$R(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\right\}$$

for all $x, y \in X$. Also suppose that the following assertions hold:

(i) $\alpha(x_0, Tx_0) \geq 1$ *for some* $x_0 \in X$;

(ii) *for a sequence $\{x_n\}$ in X converging to $x \in X$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

Then T has a fixed point.

Now, we give the following result about a fixed point of self-maps on complete metric spaces.

Theorem 20 *Let (X,d) be a complete metric space, $\alpha : X \times X \to [0,+\infty)$ be a mapping, $\psi \in \Psi$ and T be a self-mapping on X such that*

$$\alpha(x,y)d(Tx,Ty) \leq \begin{cases} \psi(\max\{\frac{d(x,Tx)d(y,Ty)}{d(x,y)}, d(x,y)\}) & \text{for } x \neq y, \\ 0 & \text{for } x = y, \end{cases} \tag{2.13}$$

for all $x, y \in X$. Suppose that T is α-admissible and there exist $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0, Tx_0) \geq 1$. If T is continuous. Then T has a unique fixed point.

Proof Take $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, and define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \geq 0$. If $x_n = x_{n+1}$ for some n, then $x^* = x_n$ is a fixed point of T. Assume that $x_n \neq x_{n+1}$ for all n. Since T is α-admissible, so it is easy to check that $\alpha(x_n, x_{n+1}) \geq 1$ for all natural numbers n. Thus for each natural number n, we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq \alpha(x_n, x_{n-1})d(Tx_n, Tx_{n-1}) \\ &\leq \psi\left(\max\left\{\frac{d(x_n, Tx_n)d(x_{n-1}, Tx_{n-1})}{d(x_n, x_{n-1})}, d(x_n, x_{n-1})\right\}\right) \\ &\leq \psi\left(\max\left\{\frac{d(x_n, x_{n+1})d(x_{n-1}, x_n)}{d(x_n, x_{n-1})}, d(x_n, x_{n-1})\right\}\right) \\ &\leq \psi\bigl(\max\bigl\{d(x_n, x_{n+1}), d(x_n, x_{n-1})\bigr\}\bigr). \end{aligned}$$

If $\max\{d(x_n, x_{n+1}), d(x_n, x_{n-1})\} = d(x_n, x_{n+1})$, then $d(x_{n+1}, x_n) \leq \psi(d(x_{n+1}, x_n))$ a contradiction. So we get $d(x_{n+1}, x_n) \leq \psi(d(x_n, x_{n-1}))$. Since ψ is nondecreasing, so we have

$$d(x_{n+1}, x_n) \leq \psi\bigl(d(x_n, x_{n-1})\bigr) \leq \psi^2\bigl(d(x_{n-1}, x_{n-2})\bigr) \leq \cdots \leq \psi^n\bigl(d(x_1, x_0)\bigr) \tag{2.14}$$

for all n. It is easy to check that $\{x_n\}$ is a Cauchy sequence. Since X is complete, so there exists $x^* \in X$ such that $x_n \to x^*$. Further the continuity of T implies that

$$Tx^* = T\Bigl(\lim_{n\to\infty} x_n\Bigr) = \lim_{n\to\infty} T(x_n) = x^*. \tag{2.15}$$

Therefore x^* is a fixed point of T in X. Now, if there exists another point $u \neq x^*$ in X such that $Tu = u$, then

$$\begin{aligned} d(x^*, u) &= d(Tx^*, Tu) \leq \alpha(x^*, u) d(Tx^*, Tu) \\ &\leq \psi\left(\max\left\{\frac{d(x^*, Tx^*)d(u, Tu)}{d(x^*, u)}, d(x^*, u)\right\}\right) \\ &\leq \psi\left(\max\{0, d(x^*, u)\}\right) = \psi\left(d(x^*, u)\right), \end{aligned}$$

a contradiction. Hence x^* is a unique fixed point of T in X. □

Example 21 Let $X = [0, \infty)$ and $d(x, y) = |x - y|$. Define $T : X \to X$ by $Tx = x + 1$ whenever $x, y \in [0, 1]$, $Tx = \frac{4}{3}$ whenever $x, y \in (1, 2)$ and $Tx = x^2 + 3x + 2$ whenever $x \in [2, \infty)$. Also define the mappings $\psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = \frac{t}{3}$ and

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in (1, 2), \\ 0 & \text{otherwise.} \end{cases}$$

By a routine calculation one can easily show that

$$\alpha(x, y) d(Tx, Ty) \leq \psi\left(\max\left\{\frac{d(x, Tx)d(y, Ty)}{d(x, y)}, d(x, y)\right\}\right)$$

for all $x, y \in X$ and $\frac{4}{3}$ is unique fixed point of the mapping T.

3 Fixed point results for graphic contractions

Consistent with Jachymski [15], let (X, d) be a metric space and Δ denote the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops, *i.e.*, $E(G) \supseteq \Delta$. We assume G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph (see [15]) by assigning to each edge the distance between its vertices. If x and y are vertices in a graph G, then a path in G from x to y of length N ($N \in \mathbb{N}$) is a sequence $\{x_i\}_{i=0}^{N}$ of $N + 1$ vertices such that $x_0 = x$, $x_N = y$ and $(x_{n-1}, x_n) \in E(G)$ for $i = 1, \ldots, N$. A graph G is connected if there is a path between any two vertices. G is weakly connected if $\tilde{G}$ is connected (see for details [7, 9, 13, 15]).

Definition 22 [15] We say that a mapping $T : X \to X$ is a Banach G-contraction or simply G-contraction if T preserves edges of G, *i.e.*,

$$\forall x, y \in X \quad \big((x, y) \in E(G) \quad \Rightarrow \quad \big(T(x), T(y)\big) \in E(G)\big)$$

and T decreases weights of edges of G in the following way:

$$\exists k \in [0, 1), \forall x, y \in X \quad \big((x, y) \in E(G) \quad \Rightarrow \quad d\big(T(x), T(y)\big) \leq k d(x, y)\big)$$

Definition 23 [15] A mapping $T : X \to X$ is called G-continuous, if given $x \in X$ and the sequence $\{x_n\}$

$$x_n \to x \text{ as } n \to \infty \text{ and } (x_n, x_{n+1}) \in E(G) \quad \text{for all } n \in \mathbb{N} \quad \text{imply} \quad Tx_n \to Tx.$$

Theorem 24 *Let (X,d) be a complete metric space endowed with a graph G and T be a self-mapping on X. Suppose the following assertions hold*:

(i) $\forall x,y \in X$, $(x,y) \in E(G) \Rightarrow (T(x),T(y)) \in E(G)$;

(ii) *there exists* $x_0 \in X$ *such that* $(x_0, Tx_0) \in E(G)$;

(iii) *there exists* $\psi \in \Psi$ *such that*

$$d(Tx,Ty) \leq \psi\bigl(R(x,y)\bigr)$$

for all $(x,y) \in E(G)$ *where*

$$R(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\right\};$$

(iv) *if* $\{x_n\}$ *is a sequence in* X *such that* $(x_n, x_{n+1}) \in E(G)$ *for all* $n \in \mathbb{N}$ *and* $x_n \to x$ *as* $n \to +\infty$, *then* $(x_n, x) \in E(G)$ *for all* $n \in \mathbb{N}$.

Then T has a fixed point.

Proof Define, $\alpha : X^2 \to [0,+\infty)$ by $\alpha(x,y) = \begin{cases} 1, & \text{if } (x,y) \in E(G), \\ 0, & \text{otherwise.} \end{cases}$ First we prove that T is an α-admissible mapping. Let, $\alpha(x,y) \geq 1$, then $(x,y) \in E(G)$. From (i), we have $(Tx,Ty) \in E(G)$. That is, $\alpha(Tx,Ty) \geq 1$. Thus T is an α-admissible mapping. From (ii) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$. That is, $\alpha(x_0, Tx_0) \geq 1$. If $(x,y) \in E(G)$, then $(Tx,Ty) \in E(G)$ and hence $\alpha(Tx,Ty) = 1$. Thus, from (iii) we have $\alpha(Tx,Ty)d(Tx,Ty) = d(Tx,Ty) \leq \psi(M(x,y))$. Condition (iv) implies condition (ii) of Theorem 18. Hence, all conditions of Theorem 18 are satisfied and T has a fixed point. □

Corollary 25 *Let (X,d) be a complete metric space endowed with a graph G and T be a self-mapping on X. Suppose the following assertions hold*:

(i) *T is a Banach G-contraction*;

(ii) *there exists* $x_0 \in X$ *such that* $(x_0, Tx_0) \in E(G)$;

(iii) *if* $\{x_n\}$ *is a sequence in* X *such that* $(x_n, x_{n+1}) \in E(G)$ *for all* $n \in \mathbb{N}$ *and* $x_n \to x$ *as* $n \to +\infty$, *then* $(x_n, x) \in E(G)$ *for all* $n \in \mathbb{N}$.

Then T has a fixed point.

As an application of Theorem 20, we obtain;

Theorem 26 *Let (X,d) be a complete metric space endowed with a graph G and T be a self-mapping on X. Suppose the following assertions hold*:

(i) $\forall x,y \in X$, $(x,y) \in E(G) \Rightarrow (T(x),T(y)) \in E(G)$;

(ii) *there exists* $x_0 \in X$ *such that* $(x_0, Tx_0) \in E(G)$;

(iii) *there exists* $\psi \in \Psi$ *such that*

$$d(Tx,Ty) \leq \begin{cases} \psi(\max\{\frac{d(x,Tx)d(y,Ty)}{d(x,y)}, d(x,y)\}) & \textit{for all } (x,y) \in E(G) \textit{ with } x \neq y, \\ 0 & \textit{for } x = y; \end{cases}$$

(iv) *T is G-continuous*.

Then T has a fixed point.

Let $(X, d, \preceq)$ be a partially ordered metric space. Define the graph G by

$$E(G) = \{(x, y) \in X \times X : x \preceq y\}.$$

For this graph, condition (i) in Theorem 24 means T is nondecreasing with respect to this order [8]. From Theorems 24-26 we derive the following important results in partially ordered metric spaces.

Theorem 27 *Let $(X, d, \preceq)$ be a complete partially ordered metric space and T be a self-mapping on X. Suppose the following assertions hold*:

(i) *T is nondecreasing map*;
(ii) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$*;
(iii) *there exists $\psi \in \Psi$ such that*

$$d(Tx, Ty) \le \psi\bigl(R(x, y)\bigr)$$

for all $x \preceq y$ where

$$R(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)}\right\};$$

(iv) *if $\{x_n\}$ is a sequence in X such that $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*

Then T has a fixed point.

Corollary 28 [20] *Let $(X, d, \preceq)$ be a complete partially ordered metric space and $T : X \to X$ be nondecreasing mapping such that*

$$d(Tx, Ty) \le rd(x, y)$$

for all $x, y \in X$ with $x \preceq y$ where $0 \le r < 1$. Suppose that the following assertions hold:

(i) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$*;
(ii) *if $\{x_n\}$ is a sequence in X such that $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*

Then T has a fixed point.

Theorem 29 *Let $(X, d, \preceq)$ be a complete partially ordered metric space and T be a self-mapping on X. Suppose the following assertions hold*:

(i) *T is nondecreasing map*;
(ii) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$*;
(iii) *there exists $\psi \in \Psi$ such that*

$$d(Tx, Ty) \le \begin{cases} \psi(\max\{\frac{d(x,Tx)d(y,Ty)}{d(x,y)}, d(x, y)\}) & \text{for all } x \preceq y \text{ with } x \ne y, \\ 0 & \text{for } x = y; \end{cases}$$

(iv) *T is continuous.*

Then T has a fixed point.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final paper.

Author details
[1]Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia. [2]Department of Mathematics, COMSATS Institute of Information Technology, Chack Shahzad, Islamabad, 44000, Pakistan.

Acknowledgements
This article was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. Therefore, the first author acknowledges with thanks DSR, KAU for financial support.

Received: 14 May 2014 Accepted: 9 August 2014 Published: 04 Sep 2014

References
1. Banach, S: Sur les opérations dans les ensembles abstraits et leur application aux equations itegrales. Fundam. Math. **3**, 133-181 (1922)
2. Agarwal, RP, Hussain, N, Taoudi, M-A: Fixed point theorems in ordered Banach spaces and applications to nonlinear integral equations. Abstr. Appl. Anal. **2012**, Article ID 245872 (2012)
3. Aghajani, A, Radenović, S, Roshan, JR: Common fixed point results for four mappings satisfying almost generalized (S, T)-contractive condition in partially ordered metric spaces. Appl. Math. Comput. **218**, 5665-5670 (2012)
4. Ali, MU, Kamran, T: On $(\alpha^* - \psi)$-contractive multi-valued mappings. Fixed Point Theory Appl. **2013**, Article ID 137 (2013)
5. Azam, A, Arshad, M: Fixed points of a sequence of locally contractive multi-valued maps. Comput. Math. Appl. **57**, 96-100 (2009)
6. Azam, A, Mehmood, N, Ahmad, J, Radenović, S: Multivalued fixed point theorems in cone b-metric spaces. J. Inequal. Appl. **2013**, Article ID 582 (2013)
7. Bojor, F: Fixed point theorems for Reich type contraction on metric spaces with a graph. Nonlinear Anal. **75**, 3895-3901 (2012)
8. Ćirić, L, Abbas, M, Saadati, R, Hussain, N: Common fixed points of almost generalized contractive mappings in ordered metric spaces. Appl. Math. Comput. **217**, 5784-5789 (2011)
9. Espnola, R, Kirk, WA: Fixed point theorems in R-trees with applications to graph theory. Topol. Appl. **153**, 1046-1055 (2006)
10. Hussain, N, Karapınar, E, Salimi, P, Akbar, F: α-Admissible mappings and related fixed point theorems. J. Inequal. Appl. **2013**, Article ID 114 (2013)
11. Hussain, N, Kutbi, MA, Salimi, P: Best proximity point results for modified α-ψ-proximal rational contractions. Abstr. Appl. Anal. **2013**, Article ID 927457 (2013)
12. Hussain, N, Salimi, P, Latif, A: Fixed point results for single and set-valued α-η-ψ-contractive mappings. Fixed Point Theory Appl. **2013**, Article ID 212 (2013)
13. Hussain, N, Al-Mezel, S, Salimi, P: Fixed points for ψ-graphic contractions with application to integral equations. Abstr. Appl. Anal. **2013**, Article ID 575869 (2013)
14. Hussain, N, Karapınar, E, Salimi, P, Vetro, P: Fixed point results for G^m-Meir-Keeler contractive and G-(α, ψ)-Meir-Keeler contractive mappings. Fixed Point Theory Appl. **2013**, Article ID 34 (2013)
15. Jachymski, J: The contraction principle for mappings on a metric space with a graph. Proc. Am. Math. Soc. **1**(136), 1359-1373 (2008)
16. Kutbi, MA, Ahmad, J, Azam, A: On fixed points of α-ψ-contractive multi-valued mappings in cone metric spaces. Abstr. Appl. Anal. **2013**, Article ID 313782 (2013)
17. Lakzian, H, Aydi, H, Rhoades, BE: Fixed points for (φ, ψ, p)-weakly contractive mappings in metric spaces with w-distance. Appl. Math. Comput. **219**(12), 6777-6782 (2013)
18. Mizoguchi, N, Takahashi, W: Fixed point theorems for multi-valued mappings on complete metric spaces. J. Math. Anal. Appl. **141**, 177-188 (1989)
19. Nadler, SB Jr: Multi-valued contraction mappings. Pac. J. Math. **30**, 475-478 (1969)
20. Nieto, JJ, Rodríguez-López, R: Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. Order **22**, 223-229 (2005)
21. Hasanzade Asl, J, Rezapour, S, Shahzad, N: On fixed points of α-ψ contractive multifunctions. Fixed Point Theory Appl. **2012**, Article ID 212 (2012)
22. Long, W, Khaleghizadeh, S, Selimi, P, Radenović, S, Shukla, S: Some new fixed point results in partial ordered metric spaces via admissible mappings. Fixed Point Theory Appl. **2014**, Article ID 117 (2014)
23. Roshan, JR, Parvaneh, V, Sedghi, S, Shobkolaei, N, Shatanawi, W: Common fixed points of almost generalized $(\psi, \varphi)_s$-contractive mappings in ordered b-metric spaces. Fixed Point Theory Appl. **2013**, Article ID 159 (2013)
24. Salimi, P, Latif, A, Hussain, N: Modified α-ψ-contractive mappings with applications. Fixed Point Theory Appl. **2013**, Article ID 151 (2013)
25. Samet, B, Vetro, C, Vetro, P: Fixed point theorem for α-ψ contractive type mappings. Nonlinear Anal. **75**, 2154-2165 (2012)
26. Shobkolaei, N, Sedghi, S, Roshan, JR, Altun, I: Common fixed point of mappings satisfying almost generalized (S, T)-contractive condition in partially ordered partial metric spaces. Appl. Math. Comput. **219**, 443-452 (2012)
27. Shukla, S, Radojević, S, Veljković, Z, Radenović, S: Some coincidence and common fixed point theorems for ordered Prešić-Reich type contractions. J. Inequal. Appl. **2013**, Article ID 520 (2013)
28. Shukla, S, Radenović, S, Vetro, C: Set-valued Hardy-Rogers type contraction in 0-complete partial metric spaces. Int. J. Math. Math. Sci. **2014**, Article ID 652925 (2014)
29. Shukla, S, Sen, R, Radenović, S: Set-valued Presić type contraction in metric spaces. An. Ştiinţ. Univ. 'Al.I. Cuza' Iaşi, Mat. (2014). doi:10.2478/aicu-2014-0011

10.1186/1029-242X-2014-348
Cite this article as: Hussain et al.: Generalized fixed point theorems for multi-valued α-ψ-contractive mappings. *Journal of Inequalities and Applications* 2014, 2014:348