STABILITY ANALYSIS OF A RATIO-DEPENDENT PREDATOR-PREY SYSTEM WITH DIFFUSION AND STAGE STRUCTURE

XINYU SONG, ZHIHAO GE, AND JINGANG WU

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A two-species predator-prey system with diffusion term and stage structure is discussed, local stability of the system is studied using linearization method, and global stability of the system is investigated by strong upper and lower solutions. The asymptotic behavior of solutions and the negative effect of stage structure on the permanence of populations are given.

1. Introduction

Predator-prey models have been studied by many authors (see [6, 21]), but the stage structure of species has been ignored in the existing literature. In the natural world, however, there are many species whose individual members have a life history that take them through two stages: immature and mature (see [1–3, 7–9, 18–20]). In particular, we have in mind mammalian populations and some amphibious animals, which exhibit these two stages. In these models, the age to maturity is represented by a time delay, leading to a system of retarded functional differential equations. For general models one can see [11].

Specifically, the standard Lotka-Volterra type models, on which nearly all existing theories are built, assume that the per capita rate predation depends on the prey numbers only. An alternative assumption is that, as the numbers of predators change slowly (relative to prey change), there is often competition among the predators and the per capita rate of predation depends on the numbers of both preys and predators, most likely and simply on their ratio. A ratio-dependent predator-prey model has been investigated by [10].

Recently, a model of ratio-dependent two species predator-prey with stage structure was derived in [19]. The model takes the form

$$
\begin{aligned}
\frac{dX_1(t)}{dt} &= \alpha X_2(t) - \gamma X_1(t) - \alpha e^{-\gamma\tau} X_2(t-\tau), \\
\frac{dX_2(t)}{dt} &= \alpha e^{-\gamma\tau} X_2(t-\tau) - \beta {X_2}^2(t) - \frac{cX_2(t)Y(t)}{X_2(t)+mY(t)},
\end{aligned}
$$

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$$\frac{dY(t)}{dt} = Y(t)\Big(-d + \frac{fX_2(t)}{X_2(t)+mY(t)}\Big),$$
$$x_1(0) > 0, \quad y(0) > 0, \quad x_2(t) = \varphi(t) \geq 0, \quad -\tau \leq t \leq 0, \tag{1.1}$$

where $X_1(t)$, $X_2(t)$ represent, respectively, the immature and mature prey populations densities; $Y(t)$ represents the density of predator population; $f > 0$ is the transformation coefficient of mature predator population; $\alpha e^{-\gamma\tau}X_2(t-\tau)$ represents the immatures who were born at time $t-\tau$ and survive at time t (with the immature death rate γ), and τ represents the transformation of immatures to matures; $\alpha > 0$ is the birth rate of the immature prey population; $\gamma > 0$ is the death rate of the immature prey population; and $\beta > 0$ represents the mature death and overcrowding rate. The model is derived under the following assumptions.

(H1) The birth rate of the immature prey population is proportional to the existing mature population with a proportionality constant $\alpha > 0$; the death rate of the immature prey population is proportional to the existing immature population with a proportionality constant $\gamma > 0$; we assume for the mature population that the death rate is of a logistic nature.

(H2) In the absence of prey spaces, the population of the predator decreased, and $d > 0$, $f > 0$, $m > 0$.

Note that the first equation of system (1.1) can be rewritten to

$$X_1(t) = \int_{t-\tau}^{t} \alpha e^{-\gamma(t-s)}X_2(s)ds, \tag{1.2}$$

so we have

$$X_1(0) = \int_{-\tau}^{0} \alpha e^{\gamma s}X_2(s)ds. \tag{1.3}$$

This suggests that if we know the properties of $X_2(t)$, then the properties of $X_1(t)$ can be obtained from $X_2(t)$ and $Y(t)$. Therefore, in the following we need only to consider the following model:

$$\frac{dX_2(t)}{dt} = \alpha e^{-\gamma\tau}X_2(t-\tau) - \beta X_2{}^2(t) - \frac{cX_2(t)Y(t)}{X_2(t)+mY(t)},$$
$$\frac{dY(t)}{dt} = Y(t)\Big(-d + \frac{fX_2(t)}{X_2(t)+mY(t)}\Big), \tag{1.4}$$
$$x_1(0) > 0, \quad y(0) > 0, \quad x_2(t) = \varphi(t) \geq 0, \quad -\tau \leq t \leq 0.$$

In [19], the effect of delay on the populations and the global asymptotic attractivity of the system (1.4) were considered, for detailed results we refer to [19]. However, the diffusion of the species which is in addition to the species' natural tendency to diffuse to areas of smaller population concentration is not considered. For the details of diffusion in different areas, we can see [4, 12–17, 22]. In this paper, we study the system (1.1) with diffusion terms, taking into account the diffusion of the species in different areas. The role of diffusion in the following system of nonlinear pdes with diffusion terms and stage structure will be studied:

$$
\begin{gathered}
\frac{\partial u_1}{\partial t} - D_1\Delta u_1 = \alpha u_2(x,t) - \gamma u_1(x,t) - \alpha e^{-\gamma\tau}u_2(x,t-\tau), \\
\frac{\partial u_2}{\partial t} - D_1\Delta u_2 = \alpha e^{-\gamma\tau}u_2(x,t-\tau) - \beta {u_2}^2(x,t) - \frac{cu_2(x,t)v(x,t)}{u_2(x,t)+mv(x,t)}, \\
\frac{\partial v}{\partial t} - D_2\Delta v = v(x,t)\left(-d + \frac{fu_2(x,t)}{u_2(x,t)+mv(x,t)}\right), \quad x\in\Omega,\ t>0, \\
\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} = \frac{\partial v}{\partial n} = 0, \quad x\in\partial\Omega,\ t>0, \\
u_1(x,t) = \varphi_1(x,t), \quad u_2(x,t) = \varphi_2(x,t), \quad v(x,0) = \varphi_3(x,0), \quad x\in\bar{\Omega},\ t\in[-\tau,0],
\end{gathered}
\tag{1.5}
$$

where $\partial/\partial n$ is differentiation in the direction of the outward unit normal to the boundary $\partial\Omega$, we assume $\Omega\subset\mathbb{R}^N$ is open, bounded and $\partial\Omega$ is smooth. The diffusion coefficients D_1, D_2, and D_3 are positive. The homogeneous Neumann boundary condition indicates that the predator-prey system is self-contained with zero population flux across the boundary. The initial functions $\varphi_1(x,t)$, $\varphi_2(x,t)$, and $\varphi_3(x,t)$ are Hölder continuous, and satisfy the compatible condition

$$
\frac{\partial \varphi_i}{\partial n} = 0 \quad \text{on } \partial\Omega,\ i=1,2,3. \tag{1.6}
$$

Denote $u_2(x,t)$ and $v(x,t)$ as $u_1(x,t)$ and $u_2(x,t)$, respectively, so we get the following subsystem of the system (1.5):

$$
\begin{gathered}
\frac{\partial u_1}{\partial t} - D_1\Delta u_1 = \alpha e^{-\gamma\tau}u_1(x,t-\tau) - \beta {u_1}^2(x,t) - \frac{cu_1(x,t)u_2(x,t)}{u_1(x,t)+mu_2(x,t)}, \\
\frac{\partial u_2}{\partial t} - D_2\Delta u_2 = u_2(x,t)\left(-d + \frac{fu_1(x,t)}{u_1(x,t)+mu_2(x,t)}\right), \quad x\in\Omega,\ t>0, \\
\frac{\partial u_1}{\partial n} = 0, \quad \frac{\partial u_2}{\partial n} = 0, \quad x\in\partial\Omega,\ t>0, \\
u_1(x,t) = \varphi_1(x,t), \quad u_2(x,t) = \varphi_2(x,0), \quad x\in\bar{\Omega},\ t\in[-\tau,0].
\end{gathered}
\tag{1.7}
$$

Note that the quantities $u_2(x,t)$ and $v(x,t)$ of the system (1.5) are independent of the quantity $u_1(x,t)$, so we may only consider the subsystem (1.7) to be easy to get the properties of the system (1.5).

Before proceeding further, let us nondimensionalize the system (1.7) with the following scaling: $U_1 = \beta u_1$, $U_2 = m\beta u_2$, $T = t$, by rewriting U_1, U_2, T to u_1, u_2, t, respectively.

We obtain the following nondimensionless system:

$$\begin{aligned}
&\frac{\partial u_1}{\partial t} - D_1\Delta u_1 = au_1(x,t-\tau) - {u_1}^2(x,t) - \frac{bu_1(x,t)u_2(x,t)}{u_1(x,t)+u_2(x,t)},\\
&\frac{\partial u_2}{\partial t} - D_2\Delta u_2 = u_2(x,t)\left(-d + \frac{fu_1(x,t)}{u_1(x,t)+u_2(x,t)}\right), \quad x\in\Omega,\ t>0,\\
&\frac{\partial u_1}{\partial n} = 0, \quad \frac{\partial u_2}{\partial n} = 0, \quad x\in\partial\Omega,\ t>0,\\
&u_1(x,t) = \varphi_1(x,t), \quad u_2(x,t) = \varphi_2(x,0), \quad x\in\bar{\Omega},\ t\in[-\tau,0],
\end{aligned} \tag{1.8}$$

where $a = \alpha e^{-\gamma\tau}$, $b = c/m$.

The remaining part of this paper is organized as follows. The existence and uniqueness of the solutions of system (1.8) will be proved in Section 2. In Section 3, we obtain conditions for local asymptotic stability of the nonnegative equilibria of system (1.8). In Section 4, we analyze the global asymptotic stability and obtain conditions for global asymptotic stability of the nonnegative equilibria of system (1.8).

2. Existence and uniqueness of the solutions

In order to solve the problem and prove theorems, we devote to some preliminaries. We rewrite system (1.8) to

$$\begin{aligned}
&\frac{\partial u_1}{\partial t} - D_1\Delta u_1 = F_1\big(u_1(x,t),u_2(x,t),u_1(x,t-\tau)\big),\\
&\frac{\partial u_2}{\partial t} - D_2\Delta u_2 = F_2\big(u_1(x,t),u_2(x,t)\big), \quad x\in\Omega,\ t>0,\\
&\frac{\partial u_1}{\partial n} = 0, \quad \frac{\partial u_2}{\partial n} = 0, \quad x\in\partial\Omega,\ t>0,\\
&u_1(x,t) = \varphi_1(x,t), \quad u_2(x,0) = \varphi_2(x,0), \quad x\in\bar{\Omega},\ t\in[-\tau,0],
\end{aligned} \tag{2.1}$$

where $F_1(u_1(x,t),u_2(x,t),u_1(x,t-\tau)) = au_1(x,t-\tau) - {u_1}^2(x,t) - bu_1(x,t)u_2(x,t)/(u_1(x,t)+u_2(x,t))$, and $F_2(u_1(x,t),u_2(x,t)) = u_2(x,t)(-d+fu_1(x,t)/(u_1(x,t)+u_2(x,t)))$.

Definition 2.1. Suppose $\varphi_1(x,t)$, $\varphi_2(x,t)$, $\psi(x,t)$ be Hölder continuous, call $(\tilde{u}_1,\tilde{u}_2)$, $(\hat{u}_1,\hat{u}_2)$ to be a pair of strong upper and lower solutions, if $\tilde{u}_1$, $\hat{u}_1$, $\tilde{u}_2$, and $\hat{u}_2 \in C(\bar{\Omega}\times[0,+\infty))\cap C^{2,1}(\Omega\times[0,+\infty))$ such that $\hat{u}_1 \le \tilde{u}_1$, $\hat{u}_2 \le \tilde{u}_2$, and

$$\begin{aligned}
&\frac{\partial \tilde{u}_1}{\partial t} - D_1\Delta\tilde{u}_1 \ge a\tilde{u}_1(x,t-\tau) - \tilde{u}_1^2(x,t) - \frac{b\tilde{u}_1(x,t)\hat{u}_2(x,t)}{\tilde{u}_1(x,t)+\hat{u}_2(x,t)},\\
&\frac{\partial \hat{u}_1}{\partial t} - D_1\Delta\hat{u}_1 \le a\hat{u}_1(x,t-\tau) - \hat{u}_1^2(x,t) - \frac{b\hat{u}_1(x,t)\tilde{u}_2(x,t)}{\hat{u}_1(x,t)+\tilde{u}_2(x,t)},\\
&\frac{\partial \tilde{u}_2}{\partial t} - D_2\Delta\tilde{u}_2 \ge -d\tilde{u}_2(x,t) + \frac{f\tilde{u}_1(x,t)\tilde{u}_2(x,t)}{\tilde{u}_1(x,t)+\tilde{u}_2(x,t)},
\end{aligned}$$

$$\begin{gathered}
\frac{\partial \hat{u}_2}{\partial t} - D_2\Delta \hat{u}_2 \le -d\hat{u}_2(x,t) + \frac{f\hat{u}_1(x,t)\hat{u}_2(x,t)}{\hat{u}_1(x,t)+\hat{u}_2(x,t)}, \quad x \in \Omega,\ t>0, \\
\frac{\partial \hat{u}_1}{\partial n} \le 0 \le \frac{\partial \tilde{u}_1}{\partial n}, \quad \frac{\partial \hat{u}_2}{\partial n} \le 0 \le \frac{\partial \tilde{u}_2}{\partial n}, \quad (x,t) \in \partial\Omega \times [0,+\infty), \\
\hat{u}_1(x,t) \le \varphi_1(x,t) \le \tilde{u}_1(x,t), \quad (x,t) \in \bar{\Omega} \times [-\tau,0], \\
\hat{u}_2(x,0) \le \varphi_2(x,0) \le \tilde{u}_2(x,0), \quad x \in \bar{\Omega}.
\end{gathered} \tag{2.2}$$

Similar to Definition 2.1, the definition of a pair of strong upper and lower solutions of the elliptic system corresponding to system (2.1) is easy to be given.

Lemma 2.2 [14]. *Suppose that* $u_i(x,t) \in C(\bar{\Omega} \times [0,T]) \cap C^{2,1}(\Omega \times [0,T])$ *satisfy*

$$\begin{gathered}
\frac{\partial u_i}{\partial t} - D_i\Delta u_i \ge \sum_{j=1}^{2} b_{ij}u_j(x,t) + \sum_{j=1}^{2} c_{ij}u_j(x,t-\tau_i), \quad (x,t) \in \Omega \times [0,T], \\
\frac{\partial u_i}{\partial n} \ge 0,\ (x,t) \in \partial\Omega \times [0,T]; \qquad u_i(x,t) \ge 0,\ (x,t) \in \Omega \times [-\tau,0],
\end{gathered} \tag{2.3}$$

where $b_{ij}(x,t), c_{ij}(x,t) \in C(\bar{\Omega} \times [0,T])$, $b_{ij} \ge 0$ *for* $(i \ne j)$, *and* $c_{ij} \ge 0$ *for* $i,j=1,2$, *and* $\tau_2 = 0$. *Then*

$$u_i(x,t) \ge 0, \quad (x,t) \in \bar{\Omega} \times [0,T]. \tag{2.4}$$

From Lemma 2.2 we easily get the following lemma.

Lemma 2.3. *For any given* $T>0$, *if* $u(x,t)$ *and* $v(x,t)$ *belong to* $C(\bar{\Omega} \times [0,T]) \cap C^{2,1}(\Omega \times [0,T])$ *and satisfy the relations*

$$\begin{aligned}
&\frac{\partial u}{\partial t} - D\Delta u - \left(au(x,t-\tau) - \beta u^2(x,t)\right) \\
&\qquad \ge \frac{\partial v}{\partial t} - D\Delta v - \left(av(x,t-\tau) - \beta v^2(x,t)\right), \quad x \in \Omega,\ t \in [0,T], \\
&\frac{\partial u}{\partial n} \ge \frac{\partial v}{\partial n}, \quad x \in \partial\Omega,\ t \in [0,T]; \qquad u(x,t) = \varphi(x,t) \ge v(x,t), \quad x \in \bar{\Omega},\ t \in [-\tau,0].
\end{aligned} \tag{2.5}$$

Then $u(x,t) \ge v(x,t)$.

Proof. Let $\omega(x,t) = u(x,t) - v(x,t)$, then

$$\begin{aligned}
\frac{\partial \omega}{\partial t} - D\Delta\omega &\ge \left(au(x,t-\tau) - \beta u^2(x,t)\right) - \left(av(x,t-\tau) - \beta v^2(x,t)\right) \\
&= a\omega(x,t-\tau) - \beta\omega(x,t)\left(u(x,t)+v(x,t)\right).
\end{aligned} \tag{2.6}$$

Let $c_{11} = a$, $b_{11} = -\beta(u(x,t)+v(x,t))$. Since $c_{11} = a = \alpha e^{-\gamma\tau} > 0$, by Lemma 2.2 we have $\omega(x,t) \geq 0$, that is,

$$u(x,t) \geq v(x,t). \tag{2.7}$$

□

THEOREM 2.4. *Let $u_1(x,t)$ and $u_2(x,t)$ be the solutions of system (2.1) in $C(\bar{\Omega}\times[0,T]) \cap C^{2,1}(\Omega\times[0,T])$, and if $f > d$, then*

$$\begin{aligned} 0 \leq u_1(x,t) &\leq \max\{\|\varphi_1\|_\infty, a\} \stackrel{\text{def}}{=} M_1, \\ 0 \leq u_2(x,t) &\leq \max\left\{\|\varphi_2\|_\infty, \frac{M_1(f-d)}{d}\right\} \stackrel{\text{def}}{=} M_2. \end{aligned} \tag{2.8}$$

Proof. Let $0 \leq \sigma \leq T$. In order to investigate system (2.1), we firstly consider the following system:

$$\begin{aligned} &\frac{\partial\psi_1}{\partial t} - D_1\Delta\psi_1 = a\psi_1(x,t-\tau) + \psi_1(x,t)(-\psi_1(x,t)), \quad x\in\Omega,\ t\in[0,T], \\ &\frac{\partial\psi_2}{\partial t} - D_2\Delta\psi_2 = \psi_2(x,t)\left(-d + \frac{f\psi_1(x,t)}{\psi_1(x,t)+\psi_2(x,t)}\right), \quad x\in\Omega,\ t\in[0,T], \\ &\frac{\partial\psi_1}{\partial n} \geq 0, \quad \frac{\partial\psi_2}{\partial n} \geq 0, \quad x\in\partial\Omega,\ t\in[0,T], \\ &\psi_1(x,t) \geq 0, \quad \psi_2(x,0) \geq 0, \quad x\in\bar{\Omega},\ t\in[-\tau,0]. \end{aligned} \tag{2.9}$$

Since $a = \alpha e^{-\gamma\tau} \neq 0$ and $b_{12} \equiv 0$, by Lemma 2.2 we have

$$u_i(x,t) \geq 0, \quad (x,t)\in\bar{\Omega}\times[0,\sigma]. \tag{2.10}$$

Note that $\psi_1(x,t)$ is bounded in $\bar{\Omega}\times[0,\sigma]$ for any $\sigma(0<\sigma\leq T)$. If $\max_{\Omega\times[0,\sigma]}\psi_1(x,t) \geq \|\varphi_1\|_\infty$, due to $\psi_1(x,t)$ satisfying the homogeneous Neumann boundary condition, there exists $(x_0,t_0)\in\Omega\times[0,\sigma]$ such that

$$\psi_1(x_0,t_0) = \max_{\Omega\times[0,\sigma]}\psi_1(x,t) \geq \|\varphi_1\|_\infty. \tag{2.11}$$

Therefore, from the first equation of system (2.9) at the point (x_0,t_0), we have

$$\left(a\psi_1(x,t-\tau) - {\psi_1}^2(x,t)\right)\big|_{(x_0,t_0)} \geq 0. \tag{2.12}$$

That is

$$\psi_1(x_0,t_0) \leq a. \tag{2.13}$$

Hence, we obtain

$$0 \leq \psi_1(x,t) \leq \max\{\|\varphi_1\|_\infty, a\}, \quad (x,t)\in\Omega\times[0,\sigma]. \tag{2.14}$$

Taking the same argument in $[\sigma,2\sigma]$, $[2\sigma,3\sigma]$,...,$[(n-1)\sigma,\ n\sigma(=T)]$, we have

$$0 \le \psi_1(x,t) \le M_1, \quad (x,t) \in \Omega \times [0,T]. \tag{2.15}$$

Similarly, there exists $(x'_0,t'_0) \in \Omega \times [0,T]$ such that

$$\left(\psi_2(x,t)\left(-d+\frac{f\psi_1(x,t)}{\psi_1(x,t)+\psi_2(x,t)}\right)\right)\Big|_{(x'_0,t'_0)} \ge 0. \tag{2.16}$$

Hence, if $f > d$, then

$$0 \le \psi_2(x,t) \le \frac{M_1(f-d)}{d}. \tag{2.17}$$

By Lemma 2.3, we have

$$u_i(x,t) \le \psi_i(x,t), \quad i = 1,2. \tag{2.18}$$

So we have

$$\begin{gathered} 0 \le u_1(x,t) \le \max\{||\varphi_1||_\infty, a\}, \\ 0 \le u_2(x,t) \le \max\left\{||\varphi_2||_\infty, \frac{M_1(f-d)}{d}\right\}. \end{gathered} \tag{2.19}$$

□

3. Local asymptotic stability of the equilibria

In this section, we discuss local asymptotic stability of the nonnegative equilibria by linearization method and analyzing the so-called characteristic equation of the equilibrium. It is obvious that system (2.1) only has three nonnegative equilibria: the equilibrium $E_1(0,0)$, the equilibrium $E_2(a,0)$, and the positive equilibrium $E_3(c_1^*,c_2^*)$ when $f > d$ and $a/b > 1 - d/f$, where

$$c_1^* = \frac{(a-b)f+bd}{f}, \qquad c_2^* = \frac{(f-d)c_1^*}{d}. \tag{3.1}$$

We will point out that $E_1(0,0)$ cannot be linearized though it is defined for system (2.1), so the local stability of $E_1(0,0)$ will be studied in another paper.

Let $\mu_1 < \mu_2 < \mu_3 < \cdots < \mu_n < \cdots$ be the eigenvalues of the operator $-\Delta$ on Ω with the homogeneous Neumann boundary condition, and let $E(\mu_i)$ be the eigenfunction space corresponding to μ_i in $C^1(\Omega)$. It is well known that $\mu_1 = 0$ and the corresponding eigenfunction $\phi_1(x) > 0$. Let $\{\phi_{ij} \mid j = 1,2,\dots,\ \mathrm{dimE}(\mu_i)\}$ be an orthogonal basis of $E(\mu_i)$, $X = \{\mathbf{u} = (u_1,u_2) \mid \mathbf{u} \in [C^1(\Omega)]^2\}$ and $X_{ij} = \{\mathbf{c}\phi_{ij} \mid \mathbf{c} \in \mathbb{R}^2\}$, thus $X = \bigoplus_{i=1}^{\infty} X_i$, $X_i = \bigoplus_{j=1}^{\mathrm{dimE}(\mu_i)} X_{ij}$.

Let $u_1(x,t) = u_1^*(x,t) + c_1^*$, $u_2(x,t) = u_2^*(x,t) + c_2^*$, where c_1^* and c_2^* are both not zero. We still make $u_1(x,t)$, $u_2(x,t)$ corresponding to $u_1^*(x,t)$, $u_2^*(x,t)$, so the linearized equation of the system (2.1) at (c_1^*, c_2^*) is

$$\begin{gathered}\frac{\partial u_1}{\partial t} - D_1\Delta u_1 = au_1(x,t-\tau) - 2c_1^* u_1(x,t) - \frac{b(c_2^*)^2}{(c_1^*+c_2^*)^2}u_1(x,t) - \frac{b(c_1^*)^2}{(c_1^*+c_2^*)^2}u_2(x,t),\\ \frac{\partial u_2}{\partial t} - D_2\Delta u_2 = \frac{f(c_2^*)^2}{(c_1^*+c_2^*)^2}u_1(x,t) - du_2(x,t) + \frac{f(c_1^*)^2}{(c_1^*+c_2^*)^2}u_2(x,t), \quad x\in\Omega,\ t>0,\\ \frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} = 0, \quad x\in\partial\Omega,\ t>0,\\ u_1(x,t) = \varphi_1(x,t) - c_1^*, \quad u_2(x,t) = \varphi_2(x,0) - c_2^*, \quad x\in\Omega,\ t\in[-\tau,0].\end{gathered} \tag{3.2}$$

From [5], we know that the characteristic equation for the system (3.2) is equivalent to

$$\begin{vmatrix} \lambda + \mu_k D_1 - ae^{-\lambda\tau} + 2c_1^* + \dfrac{b(c_2^*)^2}{(c_1^*+c_2^*)^2} & \dfrac{b(c_1^*)^2}{(c_1^*+c_2^*)^2} \\ -\dfrac{f(c_2^*)^2}{(c_1^*+c_2^*)^2} & \lambda + \mu_k D_2 + d - \dfrac{f(c_1^*)^2}{(c_1^*+c_2^*)^2} \end{vmatrix} = 0. \tag{3.3}$$

That is

$$\begin{aligned}&\left(\lambda + \mu_k D_1 - ae^{-\lambda\tau} + 2c_1^* + \frac{b(c_2^*)^2}{(c_1^*+c_2^*)^2}\right)\left(\lambda + \mu_k D_2 + d - \frac{f(c_1^*)^2}{(c_1^*+c_2^*)^2}\right)\\ &\quad + \left(\frac{b(c_1^*)^2}{(c_1^*+c_2^*)^2}\right)\left(\frac{f(c_2^*)^2}{(c_1^*+c_2^*)^2}\right) = 0.\end{aligned} \tag{3.4}$$

3.1. Local asymptotic stability of the equilibrium $E_2(a,0)$. From (3.4), it follows that at the equilibrium $E_2(a,0)$,

$$(\lambda + \mu_k D_1 - ae^{-\lambda\tau} + 2a)(\lambda + \mu_k D_2 + d - f) = 0. \tag{3.5}$$

From the first factor of (3.5), we see

$$\lambda + \mu_k D_1 + 2a = ae^{-\lambda\tau}. \tag{3.6}$$

Therefore,

$$|\lambda + \mu_k D_1 + 2a| = |ae^{-\lambda\tau}|. \tag{3.7}$$

Now we will determine that all roots of (3.7) satisfy $\mathrm{Re}\lambda < 0$. Suppose that there exists λ_0 such that $\mathrm{Re}\lambda_0 \geq 0$. From (3.7), we deduce that

$$|\lambda_0 + \mu_k D_1 + 2a| \leq |a|\,|e^{-\tau\mathrm{Re}\lambda_0}| \leq |a|. \tag{3.8}$$

This implies that λ_0 is in the circle in the complex plane centered at $(-(\mu_k D_1 + 2a), 0)$ and of radius a. However, as for given μ_k and D_1, it follows for ever that $\mu_k D_1 + 2a > a$, therefore,

$$\operatorname{Re}\lambda < 0. \tag{3.9}$$

By the second factor of (3.5), we have

$$\lambda = -\mu_k D_2 - d + f \leq f - d. \tag{3.10}$$

If $f > d$, by taking $k = 1(\mu_1 = 0)$, from (3.10), we obtain that there at least exists a root λ_0 of (3.5) such that $\operatorname{Re}\lambda_0 > 0$. Therefore, $E_2(a,0)$ is unstable if the condition $f > d$ holds.

If $f < d$, then $f - d < 0$, by (3.10), we have $\operatorname{Re}\lambda < 0$. Therefore, if $f < d$, then $E_2(a,0)$ is locally asymptotically stable.

3.2. Local asymptotic stability of the equilibrium $E_3(c_1^*, c_2^*)$. Let $\lambda = x + iy$, using (3.4), a direct calculation yields

$$\begin{aligned}
&\Bigg(x + iy + \mu_k D_1 - ae^{-x\tau}\big(\cos(-y\tau) + i\sin(-y\tau)\big)\\
&\quad + 2c_1^* + \frac{b(c_2^*)^2}{(c_1^* + c_2^*)^2}\Bigg)\left(x + iy + \mu_k D_2 + d - \frac{f(c_1^*)^2}{(c_1^* + c_2^*)^2}\right)\\
&\quad + \left(\frac{b(c_1^*)^2}{(c_1^* + c_2^*)^2}\right)\left(\frac{f(c_2^*)^2}{(c_1^* + c_2^*)^2}\right) = 0,
\end{aligned} \tag{3.11}$$

where $c_1^* = ((a-b)f + bd)/f$, $c_2^* = ((f-d)c_1^*)/d$.

Throughout the section we assume $f \geq 2d$ and $af \geq 2b(f-d)$ and let

$$\begin{aligned}
M_1 &= x + \mu_k D_1 - ae^{-x\tau}\cos(-y\tau) + 2c_1^* + \frac{b(c_2^*)^2}{(c_1^* + c_2^*)^2},\\
M_2 &= y + ae^{-x\tau}\sin(y\tau),\\
N_1 &= x + \mu_k D_2 + d - \frac{f(c_1^*)^2}{(c_1^* + c_2^*)^2},\\
N_2 &= y.
\end{aligned} \tag{3.12}$$

Separating real and imaginary parts and applying (3.12) to (3.11), we obtain the equations

$$M_1 N_1 - M_2 N_2 + \left(\frac{b(c_1^*)^2}{(c_1^* + c_2^*)^2}\right)\left(\frac{f(c_2^*)^2}{(c_1^* + c_2^*)^2}\right) = 0, \tag{3.13}$$

$$M_1 N_2 + M_2 N_1 = 0. \tag{3.14}$$

Assume, for contradiction, that there exists a root λ such that $\mathrm{Re}\lambda = x \geq 0$. By (3.12), we have

$$\begin{aligned}
M_1 &= x + \mu_k D_1 - ae^{-x\tau}\cos(-y\tau) + 2c^*1 + \frac{b(c_2^*)^2}{(c_1^* + c_2^*)^2} \\
&\geq x + 0 - ae^{-x\tau}\cos(-y\tau) + 2c_1^* + \frac{b(c_2^*)^2}{(c_1^* + c_2^*)^2} \\
&\geq x - a + 2c_1^* + \frac{b(c_2^*)^2}{(c_1^* + c_2^*)^2} \\
&\geq \left(-a + c_1^* + \frac{bc_2^*}{c_1^* + c_2^*}\right) + \frac{b(c_2^*)^2}{(c_1^* + c_2^*)^2} + c_1^* - \frac{bc_2^*}{c_1^* + c_2^*} \\
&\geq \frac{b(c_2^*)^2}{(c_1^* + c_2^*)^2} + b(f-d)f - \frac{bc_2^*}{c_1^* + c_2^*} = \frac{b(c_2^*)^2}{(c_1^* + c_2^*)^2} > 0,
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
N_1 &= x + \mu_k D_2 + d - \frac{f(c_1^*)^2}{(c_1^* + c_2^*)^2} \\
&= d - \frac{fc_1^*}{c_1^* + c_2^*} + \frac{fc_1^*}{c_1^* + c_2^*} - \frac{f(c_1^*)^2}{(c_1^* + c_2^*)^2} \geq \frac{f(c_1^*)^2}{(c_1^* + c_2^*)^2} > 0.
\end{aligned} \tag{3.16}$$

Applying (3.15) and (3.16), one can obtain

$$\left(\frac{b(c_2^*)^2}{(c_1^* + c_2^*)^2}\right)\left(\frac{f(c_1^*)^2}{(c_1^* + c_2^*)^2}\right) \leq M_1 N_1. \tag{3.17}$$

Using (3.13) and (3.14), we have

$$\begin{aligned}
&\left(\left(\frac{b(c_1^*)^2}{(c_1^* + c_2^*)^2}\right)\left(\frac{f(c_2^*)^2}{(c_1^* + c_2^*)^2}\right)\right)^2 \\
&\quad = (M_2N_2)^2 + (M_1N_1)^2 + (M_1N_2)^2 + (M_2N_1)^2.
\end{aligned} \tag{3.18}$$

If $N_2 \neq 0$, by (3.15) and (3.18), we get

$$\begin{aligned}
&\left(\left(\frac{b(c_1^*)^2}{(c_1^* + c_2^*)^2}\right)\left(\frac{f(c_2^*)^2}{(c_1^* + c_2^*)^2}\right)\right)^2 \\
&\quad = (M_2N_2)^2 + (M_1N_1)^2 + (M_1N_2)^2 + (M_2N_1)^2 > (M_1N_1)^2,
\end{aligned} \tag{3.19}$$

it is a contradiction to (3.17). If $N_2 = 0$, from (3.12), we deduce $M_2 = 0$, again using (3.13), we have

$$\begin{aligned}&\left(x+\mu_k D_1 - ae^{-x\tau} + 2c^*{}_1 + \frac{b(c^*{}_2)^2}{(c_1^*+c_2^*)^2}\right)\left(x+\mu_k D_2 + d - \frac{f(c_1^*)^2}{(c_1^*+c_2^*)^2}\right)\\&\quad+\left(\frac{b(c_1^*)^2}{(c_1^*+c_2^*)^2}\right)\left(\frac{f(c_2^*)^2}{(c_1^*+c_2^*)^2}\right)=0,\end{aligned} \tag{3.20}$$

that is,

$$\begin{aligned}&\left(x+\mu_k D_2 + \frac{fc_1^*c_2^*}{(c_1^*+c_2^*)^2}\right)\left(x+\mu_k D_1 + a - ae^{-x\tau} + c_1^* - \frac{bc_1^*c_2^*}{(c_1^*+c_2^*)^2}\right)\\&\quad+\left(\frac{b(c_1^*)^2}{(c_1^*+c_2^*)^2}\right)\left(\frac{f(c_2^*)^2}{(c_1^*+c_2^*)^2}\right)=0.\end{aligned} \tag{3.21}$$

It is obvious that $x = -\mu_k D_2 - fc_1^*c_2^*/(c_1^*+c_2^*)^2$ does not satisfy (3.21), so we have

$$\begin{aligned}&\left(x+\mu_k D_2 + \frac{fc_1^*c_2^*}{(c_1^*+c_2^*)^2}\right)\\&\quad\times\left(x+\mu_k D_1 + a - ae^{-x\tau} + c_1^* - \frac{bc_1^*c_2^*}{(c_1^*+c_2^*)^2}\right.\\&\qquad\left.+\frac{\left(b(c_1^*)^2/(c_1^*+c_2^*)^2\right)\left(f(c_2^*)^2/(c_1^*+c_2^*)^2\right)}{x+\mu_k D_2 + fc_1^*c_2^*/(c_1^*+c^*{}_2)^2}\right)=0.\end{aligned} \tag{3.22}$$

So all roots of (3.22) are given by (3.23), that is,

$$\begin{aligned}x &= -\mu_k D_1 - a - c_1^* + ae^{-x\tau} + \frac{bc_1^*c_2^*}{(c_1^*+c_2^*)^2} - \frac{\left(b(c_1^*)^2/(c_1^*+c_2^*)^2\right)\left(f(c_2^*)^2/(c_1^*+c_2^*)^2\right)}{x+\mu_k D_2 + fc_1^*c_2^*/(c_1^*+c_2^*)^2}\\&\le -c_1^* + \frac{bc_1^*c_2^*}{(c_1^*+c_2^*)^2} - \frac{\left(b(c_1^*)^2/(c_1^*+c_2^*)^2\right)\left(f(c_2^*)^2/(c_1^*+c_2^*)^2\right)}{x+\mu_k D_2 + fc_1^*c_2^*/(c_1^*+c_2^*)^2}\\&\le -c_1^* + \frac{bc_1^*c_2^*}{(c_1^*+c_2^*)^2}\left(\frac{x+\mu_k D_2}{x+\mu_k D_2 + fc_1^*c_2^*/(c_1^*+c_2^*)^2}\right)\\&< -c_1^* + \frac{bc_1^*c_2^*}{(c_1^*+c_2^*)^2} < c_1^*\left(-1+\frac{b}{c_1^*+c_2^*}\right) \le c_1^*\left(-1+\frac{b}{2c_1^*}\right) \le 0,\end{aligned} \tag{3.23}$$

it is a contradiction to $\operatorname{Re}\lambda = x \ge 0$. So we have that $\operatorname{Re}\lambda < 0$ if $f \ge 2d$ and $af \ge 2b(f-d)$, that is, the positive equilibrium $E_3(c_1^*, c_2^*)$ is locally asymptotically stable.

From the above discussion, we can conclude the following.

Theorem 3.1. *If $f \geq 2d$ and $af \geq 2b(f-d)$, then the positive equilibrium $E_3(c_1^*, c_2^*)$ is locally asymptotically stable.*

Theorem 3.2. *If $f > d$, then the equilibrium $E_2(a,0)$ is unstable.*

Theorem 3.3. *If $f < d$, then the equilibrium $E_2(a,0)$ is locally asymptotically stable.*

4. Global asymptotic stability of the equilibria

Note that $F_1(u_1(x,t), u_2(x,t), u_1(x,t-\tau))$ and $F_2(u_1(x,t), u_2(x,t))$, with respect to u_1, u_2, are continuous and mixed quasimonotone in $\Sigma \times \Sigma^*$, where Σ, Σ^* are fixed and bounded subsets of $\mathbb{R}^2$. Thus there exist $K_i \geq 0$ $(i = 1,2)$ such that

$$\begin{aligned} &\left| F_i(u_1, u_2, u_1(x,t-\tau)) - F_i(u'_1, u'_2, u'_1(x,t-\tau)) \right| \\ &\quad \leq K_i\left(\left| u_1(x,t) - u'_1(x,t) \right| + u_2(x,t) - u'_2(x,t) \right| \right), \end{aligned} \tag{4.1}$$

when $i = 2$, $\tau = 0$, where $(u_1, u'_1), (u_2, u'_2) \in \Sigma \times \Sigma^*$.

In order to investigate the dynamics of the system (2.1) we define two sequences of constant vectors $\{\bar{\mathbf{c}}^{(m)}\} = \{\bar{c}_1{}^{(m)}, \bar{c}_2{}^{(m)}\}_{m=1}^{\infty}$, $\{\underline{\mathbf{c}}^{(m)}\} = \{\underline{c}_1{}^{(m)}, \underline{c}_2{}^{(m)}\}_{m=1}^{\infty}$ satisfying the following relation:

$$\begin{aligned} \bar{c}_1{}^{(m)} &= \bar{c}_1{}^{(m-1)} + \frac{1}{K_1}\bar{c}_1{}^{(m-1)}\left(a - \bar{c}_1{}^{(m-1)} - \frac{b\underline{c}_2{}^{(m-1)}}{\bar{c}_1{}^{(m)} + \underline{c}_2{}^{(m-1)}} \right), \\ \underline{c}_1{}^{(m)} &= \underline{c}_1{}^{(m-1)} + \frac{1}{K_1}\underline{c}_1{}^{(m-1)}\left(a - \underline{c}_1{}^{(m-1)} - \frac{b\bar{c}_2{}^{(m-1)}}{\underline{c}_1{}^{(m-1)} + \bar{c}_2{}^{(m-1)}} \right), \\ \bar{c}_2{}^{(m)} &= \bar{c}_2{}^{(m-1)} + \frac{1}{K_2}\bar{c}_2{}^{(m-1)}\left(-d + \frac{f\bar{c}_1{}^{(m)}}{\bar{c}_1{}^{(m)} + \bar{c}_2{}^{(m)}} \right), \\ \underline{c}_2{}^{(m)} &= \underline{c}_2{}^{(m-1)} + \frac{1}{K_2}\underline{c}_2{}^{(m-1)}\left(-d + \frac{f\underline{c}_2{}^{(m-1)}}{\underline{c}_2{}^{(m-1)} + \underline{c}_2{}^{(m-1)}} \right), \\ \bar{\mathbf{c}}^{(0)} &= \tilde{\mathbf{c}}, \qquad \underline{\mathbf{c}}^{(0)} = \hat{\mathbf{c}}, \quad m = 1,2,\ldots, \end{aligned} \tag{4.2}$$

where $(\tilde{\mathbf{c}}, \hat{\mathbf{c}})$ is a pair of coupled upper and lower solutions of system (2.1). It is easy to prove the following lemma.

Lemma 4.1. *The sequences $\{\bar{\mathbf{c}}^{(m)}\}$, $\{\underline{\mathbf{c}}^{(m)}\}$ given by (4.2) with $\bar{\mathbf{c}}^{(0)} = \tilde{\mathbf{c}}$ and $\underline{\mathbf{c}}^{(0)} = \hat{\mathbf{c}}$ possess the monotone property*

$$\hat{\mathbf{c}} \leq \underline{\mathbf{c}}^{(m)} \leq \underline{\mathbf{c}}^{(m+1)} \leq \bar{\mathbf{c}}^{(m+1)} \leq \bar{\mathbf{c}}^{(m)} \leq \tilde{\mathbf{c}}, \quad m = 1,2,\ldots. \tag{4.3}$$

Proof. Since $(\tilde{\mathbf{c}},\hat{\mathbf{c}})$ is a pair of coupled upper and lower solutions of the system (2.1), so it follows from (2.2) that

$$\begin{aligned}\bar{c}_1{}^{(0)}-\bar{c}_1{}^{(1)}&=\bar{c}_1{}^{(0)}-\left(\bar{c}_1{}^{(0)}+\frac{1}{K_1}\bar{c}_1{}^{(0)}\left(a-\bar{c}_1{}^{(0)}-\frac{b\underline{c}_2{}^{(0)}}{\bar{c}_1{}^{(0)}+\underline{c}_2{}^{(0)}}\right)\right)\\&=-\frac{1}{K_1}\bar{c}_1{}^{(0)}\left(a-\bar{c}_1{}^{(0)}-\frac{b\underline{c}_2{}^{(0)}}{\bar{c}_1{}^{(0)}+\underline{c}_2{}^{(0)}}\right)\geq 0,\end{aligned}\tag{4.4}$$

$$\begin{aligned}\underline{c}_1{}^{(1)}-\underline{c}_1{}^{(0)}&=\left(\underline{c}_1{}^{(0)}+\frac{1}{K_1}\underline{c}_1{}^{(0)}\left(a-\underline{c}_1{}^{(0)}-\frac{b\bar{c}_2{}^{(0)}}{\underline{c}_1{}^{(0)}+\bar{c}_2{}^{(0)}}\right)\right)-\underline{c}_1{}^{(0)}\\&=\frac{1}{K_1}\underline{c}_1{}^{(0)}\left(a-\underline{c}_1{}^{(0)}-\frac{b\bar{c}_2{}^{(0)}}{\underline{c}_1{}^{(0)}+\bar{c}_2{}^{(0)}}\right)\geq 0.\end{aligned}\tag{4.5}$$

This gives $\bar{c}_1{}^{(0)}\geq\bar{c}_1{}^{(1)}$ and $\underline{c}_1{}^{(1)}\geq\underline{c}_1{}^{(0)}$. Similarly by (2.2) and the quasimonotone property, we have

$$\begin{aligned}K_1\left(\bar{c}_1{}^{(1)}-\underline{c}_1{}^{(1)}\right)&=K_1\left(\bar{c}_1{}^{(0)}-\underline{c}_1{}^{(0)}\right)+\bar{c}_1{}^{(0)}\left(a-\bar{c}_1{}^{(0)}-\frac{b\underline{c}_2{}^{(0)}}{\bar{c}_1{}^{(0)}+\underline{c}_2{}^{(0)}}\right)\\&\quad-\underline{c}_1{}^{(0)}\left(a-\underline{c}_1{}^{(0)}-\frac{b\bar{c}_2{}^{(0)}}{\underline{c}_1{}^{(0)}+\bar{c}_2{}^{(0)}}\right)\geq 0.\end{aligned}\tag{4.6}$$

This yields $\bar{c}_1{}^{(1)}\geq\underline{c}_1{}^{(1)}$. The above conclusions show that

$$\underline{\mathbf{c}}^{(0)}\leq\underline{\mathbf{c}}^{(1)}\leq\bar{\mathbf{c}}^{(1)}\leq\bar{\mathbf{c}}^{(0)}.\tag{4.7}$$

Assume, by induction, that $\underline{\mathbf{c}}^{(m-1)}\leq\underline{\mathbf{c}}^{(m)}\leq\bar{\mathbf{c}}^{(m)}\leq\bar{\mathbf{c}}^{(m-1)}$ for some $m>1$, Then by (4.6) and (2.2), we have

$$\begin{aligned}&K_1\left(\bar{c}_1{}^{(m)}-\underline{c}_1{}^{(m+1)}\right)\\&\quad=K_1\left(\bar{c}_1{}^{(m-1)}-\underline{c}_1{}^{(m)}\right)+\bar{c}_1{}^{(m-1)}\left(a-\bar{c}_1{}^{(m-1)}-\frac{b\underline{c}_2{}^{(m-1)}}{\bar{c}_1{}^{(m-1)}+\underline{c}_2{}^{(m-1)}}\right)\\&\quad-\underline{c}_1{}^{(m)}\left(a-\underline{c}_1{}^{(m)}-\frac{b\bar{c}_2{}^{(m)}}{\underline{c}_1{}^{(m)}+\bar{c}_2{}^{(m)}}\right)\geq 0.\end{aligned}\tag{4.8}$$

This yields $\bar{c}_1{}^{(m)}\geq\underline{c}_1{}^{(m+1)}$. A similar argument gives $\underline{\mathbf{c}}^{(m)}\leq\underline{\mathbf{c}}^{(m+1)}\leq\bar{\mathbf{c}}^{(m+1)}\leq\bar{\mathbf{c}}^{(m)}$. A similar argument gives $\bar{c}_2{}^{(m)}$ and $\underline{c}_2{}^{(m)}$. Therefore, the monotone property (4.3) follows by the principle of induction. □

By monotone bounds principle, we get

$$\lim_{m\to\infty}\underline{\mathbf{c}}^{(m)}=\underline{\mathbf{c}},\qquad\lim_{m\to\infty}\bar{\mathbf{c}}^{(m)}=\bar{\mathbf{c}},\tag{4.9}$$

with

$$\hat{\mathbf{c}} \le \underline{\mathbf{c}}^{(m)} \le \underline{\mathbf{c}}^{(m+1)} \le \underline{\mathbf{c}} \le \bar{\mathbf{c}} \le \bar{\mathbf{c}}^{(m+1)} \le \bar{\mathbf{c}}^{(m)} \le \tilde{\mathbf{c}}. \tag{4.10}$$

In (4.2), letting $m \to \infty$, we have

$$\begin{aligned} &\bar{c_1}\left(a - \bar{c_1} - \frac{b\underline{c}_2}{\bar{c_1} + \underline{c}_2}\right) = 0, \qquad \underline{c}_1\left(a - \underline{c}_1 - \frac{b\bar{c_2}}{\underline{c}_1 + \bar{c_2}}\right) = 0,\\ &\bar{c_2}\left(-d + \frac{f\bar{c_1}}{\bar{c_1} + \bar{c_2}}\right) = 0, \qquad \underline{c}_2\left(-d + \frac{f\underline{c}_1}{\underline{c}_1 + \underline{c}_2}\right) = 0. \end{aligned} \tag{4.11}$$

Considering [16, Theorem 2.2] as a corollary, we obtain the following.

Theorem 4.2. *Let $\underline{\mathbf{c}}$, $\bar{\mathbf{c}}$ be the limits in (4.2), then for any initial function $\varphi = (\varphi_1, \varphi_2)$ in $\langle \underline{\mathbf{c}}, \bar{\mathbf{c}} \rangle$ the solution of system (2.1) satisfies the relation*

$$\underline{\mathbf{c}} \le \mathbf{u}(x,t) \le \bar{\mathbf{c}} \quad \text{as } t \to \infty. \tag{4.12}$$

Moreover, if, in addition, $\underline{\mathbf{c}} = \bar{\mathbf{c}} (\equiv \mathbf{c}^*)$, then $\mathbf{c}^*$ is the unique global solution of the system (2.1) in $\langle \underline{\mathbf{c}}, \bar{\mathbf{c}} \rangle$, and

$$\lim_{t \to \infty} \mathbf{u}(x,t) = \mathbf{c}^*, \quad x \in \Omega. \tag{4.13}$$

Theorem 4.3. *Let $(u_1(x,t), u_2(x,t))$ be the solution of the system (2.1), if $\varphi_1(x,0) \neq 0$ and $\varphi_2(x,0) \equiv 0$, then $u_1(x,t) > 0$, $u_2(x,t) \equiv 0$, and*

$$(u_1(x,t), u_2(x,t)) \longrightarrow (a,0), \quad t \longrightarrow +\infty. \tag{4.14}$$

Proof. By the standard maximum principle for parabolic boundary-value problems with homogeneous Neumann boundary condition, we see that

$$(u_1(x,t), u_2(x,t)) \ge (0,0). \tag{4.15}$$

It is obvious that if $\varphi_1(x,0) \neq 0$, $\varphi_2(x,0) \equiv 0$, then

$$u_1(x,t) > 0, \quad u_2(x,t) \equiv 0. \tag{4.16}$$

Now, the system (2.1) becomes the scalar boundary-value problem

$$\begin{aligned} &\frac{\partial u_1}{\partial t} - D_1 \Delta u_1 = a u_1(x, t-\tau) - {u_1}^2(x,t), \quad x \in \Omega,\ t > 0,\\ &\frac{\partial u_1}{\partial n} = 0, \quad x \in \partial\Omega,\ t > 0; \quad u_1(x,t) = \varphi_1(x,t) \ge 0, \quad x \in \Omega,\ t \in [-\tau, 0]. \end{aligned} \tag{4.17}$$

By continuity and Lemma 2.3, it follows that there exist $\delta > 0$ and $t^* > 0$ satisfying

$$u_1(x,t) \ge \delta, \quad (x,t) \in [t^*, t^* + \tau]. \tag{4.18}$$

Let $\tilde{\mathbf{c}} = M$, $\hat{\mathbf{c}} = \varepsilon \geq 0$, where

$$M = \max\left\{\max_{\Omega\times[t^*,t^*+\tau]} u_1(x,t), a\right\}, \qquad \varepsilon = \min\{a,\delta\}. \tag{4.19}$$

Then

$$\begin{gathered} \varepsilon \leq a \Longrightarrow \varepsilon^2 \leq a\varepsilon \Longrightarrow a\varepsilon - \varepsilon^2 \geq 0, \\ a \leq M \Longrightarrow aM \leq M^2 \Longrightarrow aM - M^2 \leq 0. \end{gathered} \tag{4.20}$$

Therefore, $(\tilde{\mathbf{c}}, \hat{\mathbf{c}})$ is a pair of strong upper and lower solutions of the system (4.17). Applying Theorem 4.2 and (4.3), we obtain that there exist $\underline{\mathbf{c}}$, $\bar{\mathbf{c}}$ such that

$$\begin{gathered} \varepsilon \leq \hat{\mathbf{c}} \leq \underline{\mathbf{c}} \leq \bar{\mathbf{c}} \leq \tilde{\mathbf{c}} \leq M, \\ \underline{c}(a-\underline{c}) = 0, \qquad \bar{c}(a-\bar{c}) = 0, \end{gathered} \tag{4.21}$$

while the equation $c(a-c) = 0$ has a unique positive solution $c^* = a$.

Therefore

$$\bar{c} = \underline{c} = c^* = a, \qquad \lim_{t\to\infty} \mathbf{u}_1(x,t) = \mathbf{c}^* = a. \tag{4.22}$$

□

Theorem 4.4. *If $f \geq 2d$ and $af \geq 2b(f-d)$, then for any nonnegative initial function $\varphi = (\varphi_1, \varphi_2)$ the system (2.1) has a unique global solution with*

$$\left(u_1(x,t), u_2(x,t)\right) \longrightarrow \left(c_1{}^*, c_2{}^*\right) \quad \text{as } t \longrightarrow +\infty. \tag{4.23}$$

Proof. Let $\omega_1(x,t)$ be the solution of the scalar boundary-value problem

$$\begin{gathered} \frac{\partial \omega_1}{\partial t} - D_1\Delta\omega_1 = a\omega_1(x,t-\tau) - \omega_1{}^2(x,t), \quad x \in \Omega,\ t > 0, \\ \frac{\partial \omega_1}{\partial n} = 0, \quad x \in \partial\Omega,\ t > 0, \\ \omega_1(x,t) = \varphi_1(x,t) \geq 0, \quad x \in \Omega,\ t \in [-\tau, 0]. \end{gathered} \tag{4.24}$$

Applying Lemma 2.3, we have

$$u_1(x,t) \leq \omega_1(x,t), \quad (x,t) \in \Omega \times [0,+\infty]. \tag{4.25}$$

By Theorem 4.3, it follows that

$$\omega_1(x,t) \longrightarrow a, \quad t \longrightarrow \infty. \tag{4.26}$$

Hence, for any $\varepsilon > 0$, there always exists $t^{**} > 0$ as $t > t^{**}$ such that

$$u_1(x,t) \leq \omega_1(x,t) \leq a + \varepsilon. \tag{4.27}$$

Let $\omega_2(x,t)$ be the solution of the scalar boundary-value problem

$$\begin{aligned}\frac{\partial \omega_2}{\partial t} - D_2\Delta\omega_2 &= -d\omega_2(x,t) + \frac{f(a+\varepsilon)\omega_2(x,t)}{(a+\varepsilon)+\omega_2(x,t)}\\ &\le -d\omega_2(x,t) + f(a+\varepsilon), \quad (x,t)\in\Omega\times[t^{**},+\infty],\\ \frac{\partial\omega_2}{\partial n} &= 0, \quad (x,t)\in\partial\Omega\times[t^{**},+\infty],\\ \omega_2(x,t) &= \varphi_1(x,t^{**}) \ge 0, \quad x\in\Omega.\end{aligned} \tag{4.28}$$

By the quasimonotone property of $F_2(u_1,u_2)$, we obtain

$$u_2(x,t) \le \omega_2(x,t), \quad (x,t)\in\Omega\times[t^{**},+\infty]. \tag{4.29}$$

Now we consider the scalar boundary-value problem

$$\begin{aligned}\frac{\partial \omega_2}{\partial t} - D_2\Delta\omega_2 &= -d\omega_2(x,t) + f(a+\varepsilon), \quad (x,t)\in\Omega\times[t^{**},+\infty],\\ \frac{\partial\omega_2}{\partial n} &= 0, \quad (x,t)\in\partial\Omega\times[t^{**},+\infty],\\ \omega_2(x,t) &= \varphi_1(x,t^{**}) \ge 0, \quad x\in\Omega.\end{aligned} \tag{4.30}$$

By [22], we have $\omega_2(x,t) \to f(a+\varepsilon)/d$ as t sufficiently large. Therefore, for the above given ε, there exists $t_0 > 0$ satisfying

$$u_2(x,t) \le \frac{f(a+\varepsilon)}{d}, \quad (x,t)\in\Omega\times[t_0,+\infty]. \tag{4.31}$$

Let

$$\begin{aligned}\tilde{c}_1 &= a+\varepsilon, \qquad \hat{c}_1 = \delta,\\ \tilde{c}_2 &= \frac{f(a+\varepsilon)}{d}, \qquad \hat{c}_2 = \delta,\end{aligned} \tag{4.32}$$

where ε and δ are sufficiently small positive constants.

By $f \ge 2d$ and $af \ge 2b(f-d)$, applying Lemmas 2.2 and 2.3, for t sufficiently large, we obtain that

$$\begin{gathered}\hat{c}_1 \le u_1(x,t) \le \tilde{c}_1, \qquad \hat{c}_2 \le u_2(x,t) \le \tilde{c}_2,\\ \tilde{c}_2 = \frac{f(a+\varepsilon)}{d} \Longrightarrow d\tilde{c}_2 = f(a+\varepsilon) \Longrightarrow d\tilde{c}_2 \ge \frac{f(a+\varepsilon)\tilde{c}_2}{\tilde{c}_1+\tilde{c}_2} \Longrightarrow -d\tilde{c}_2 + \frac{f(a+\varepsilon)\tilde{c}_2}{\tilde{c}_1+\tilde{c}_2} \le 0,\\ \tilde{c}_1 = a+\varepsilon \ge a \Longrightarrow \tilde{c}_1^{\,2} \ge a\tilde{c}_1 \Longrightarrow 0 \ge a\tilde{c}_1 - \tilde{c}_1^{\,2} \Longrightarrow a\tilde{c}_1 - \tilde{c}_1^{\,2} - \frac{b\delta\tilde{c}_1}{\delta+\tilde{c}_1} \le 0.\end{gathered} \tag{4.33}$$

Let ε and δ be sufficiently small positive constants. Applying $f \ge 2d$ and $af \ge 2b(f-d)$, it is not difficult to prove that $\hat{c}_1 = \delta_1$ and $\hat{c}_2 = \delta_2$ satisfy (2.2).

Hence, $(\tilde{c_1},\tilde{c_2})$, $(\hat{c_1},\hat{c_2})$ are a pair of coupled upper and lower solutions of the system (2.1).

By Theorem 4.2, we obtain that $\underline{\mathbf{c}}$ and $\bar{\mathbf{c}}$ satisfy

$$0<\hat{\mathbf{c}}\le\underline{\mathbf{c}}\le\bar{\mathbf{c}}\le\tilde{\mathbf{c}},$$
$$\bar{c_1}\left(a-\bar{c_1}-\frac{b\underline{c}_2}{\bar{c_1}+\underline{c}_2}\right)=0,$$
$$\underline{c}_1\left(a-\underline{c}_1-\frac{b\bar{c_2}}{\underline{c}_1+\bar{c_2}}\right)=0, \tag{4.34}$$
$$\bar{c_2}\left(-d+\frac{f\bar{c_1}}{\bar{c_1}+\bar{c_2}}\right)=0,$$
$$\underline{c}_2\left(-d+\frac{f\underline{c}_1}{\underline{c}_1+\underline{c}_2}\right)=0.$$

Since $f\ge 2d$ and $af\ge 2b(f-d)$ and the following equations have unique positive solutions:

$$c_1\left(a-c_1-\frac{bc_2}{c_1+c_2}\right)=0,\qquad c_2\left(-d+\frac{fc_1}{c_1+c_2}\right)=0, \tag{4.35}$$

therefore

$$\bar{c_1}=\underline{c}_1=c_1{}^*,\qquad \bar{c_2}=\underline{c}_2=c_2{}^*, \tag{4.36}$$

where

$$c^*{}_1=\frac{(a-b)f+bd}{f},\qquad c^*{}_2=\frac{(f-d)c^*{}_1}{d}. \tag{4.37}$$

Therefore

$$\lim_{t\to+\infty}u_1(x,t)=c_1^*,\qquad \lim_{t\to+\infty}u_2(x,t)=c_2^*. \tag{4.38}$$

□

Theorem 4.5. *If $f<d$, then for any nonnegative initial function $\varphi_i(x,t)$ $(i=1,2)$, the system (2.1) has a unique global nonnegative solution $(u_1(x,t),u_2(x,t))$ satisfying*

$$(u_1(x,t),u_2(x,t))\longrightarrow(a,0),\quad t\longrightarrow+\infty. \tag{4.39}$$

Proof. Let $c_1{}^*$ and $c_2{}^*$ be the solutions of the following system:

$$c_1{}^*\left(a-c_1{}^*-\frac{bc_2{}^*}{c_1{}^*+c_2{}^*}\right)=0,$$
$$c_2{}^*\left(-d+\frac{fc_1{}^*}{c_1{}^*+c_2{}^*}\right)=0. \tag{4.40}$$

If $f < d$, then (4.40) has only one nonzero solution and one nonnegative solution:

$$c_1{}^* = a, \qquad c_2{}^* = 0. \tag{4.41}$$

Let $\tilde{c}_2$ be sufficiently large and let $0 < \delta(\le a)$ be sufficiently small, and $\tilde{c}_1$, $\hat{c}_1$, $\hat{c}_2$ satisfy

$$\tilde{c}_1 \ge a, \qquad \hat{c}_1 = \delta, \qquad \hat{c}_2 = 0. \tag{4.42}$$

Applying Lemmas 2.2 and 2.3, if the condition $f < d$ holds, it is easy to prove

$$\hat{c}_1 \le u_1(x,t) \le \tilde{c}_1, \qquad \hat{c}_2 \le u_2(x,t) \le \tilde{c}_2, \tag{4.43}$$

and $(\tilde{c}_1,\tilde{c}_2)$, $(\hat{c}_1,\hat{c}_2)$ satisfying (2.2). Hence, $(\tilde{c}_1,\tilde{c}_2)$, $(\hat{c}_1,\hat{c}_2)$ are a pair of coupled upper and lower solutions of the system (2.1). By Theorem 4.2, we obtain $\underline{\mathbf{c}}$ and $\bar{\mathbf{c}}$ satisfying

$$\begin{gathered} 0 < \hat{\mathbf{c}} \le \underline{\mathbf{c}} \le \bar{\mathbf{c}} \le \tilde{\mathbf{c}}, \\ \bar{c}_1\left(a - \bar{c}_1 - \frac{b\underline{c}_2}{\bar{c}_1 + \underline{c}_2}\right) = 0, \\ \underline{c}_1\left(a - \underline{c}_1 - \frac{b\bar{c}_2}{\underline{c}_1 + \bar{c}_2}\right) = 0, \\ \bar{c}_2\left(-d + \frac{f\bar{c}_1}{\bar{c}_1 + \bar{c}_2}\right) = 0, \\ \underline{c}_2\left(-d + \frac{f\underline{c}_2}{\underline{c}_1 + \underline{c}_2}\right) = 0. \end{gathered} \tag{4.44}$$

We see from $\underline{c}_2{}^{(0)} = 0$ that $\underline{c}_2{}^{(m)} = 0$ for every $m = 1,2,\ldots$. This implies $\underline{c}_2 = 0$.

Using $0 < \delta \le \underline{c}_1 \le \bar{c}_1$, we have $\underline{c}_1 = a$. Since $f < d$, and

$$\begin{gathered} \bar{c}_1\left(a - \bar{c}_1 - \frac{b\underline{c}_2}{\bar{c}_1 + \underline{c}_2}\right) = 0, \\ \underline{c}_1\left(a - \underline{c}_1 - \frac{b\bar{c}_2}{\underline{c}_1 + \bar{c}_2}\right) = 0, \\ \bar{c}_2\left(-d + \frac{f\bar{c}_2}{\bar{c}_1 + \bar{c}_2}\right) = 0, \end{gathered} \tag{4.45}$$

so we have

$$\bar{c}_1 = \underline{c}_1 = a, \qquad \bar{c}_2 = \underline{c}_2 = 0. \tag{4.46}$$

By Theorem 4.2, for any nonnegative initial function $\varphi_i(x,t)$, $i = 1,2$, the system (2.1) has a unique global nonnegative solution $(u_1(x,t),u_2(x,t))$ in $\varphi_i(x,t) \in \langle\hat{\mathbf{c}},\tilde{\mathbf{c}}\rangle$ satisfying

$$(u_1(x,t),u_2(x,t)) \longrightarrow (a,0), \quad t \longrightarrow +\infty. \tag{4.47}$$

□

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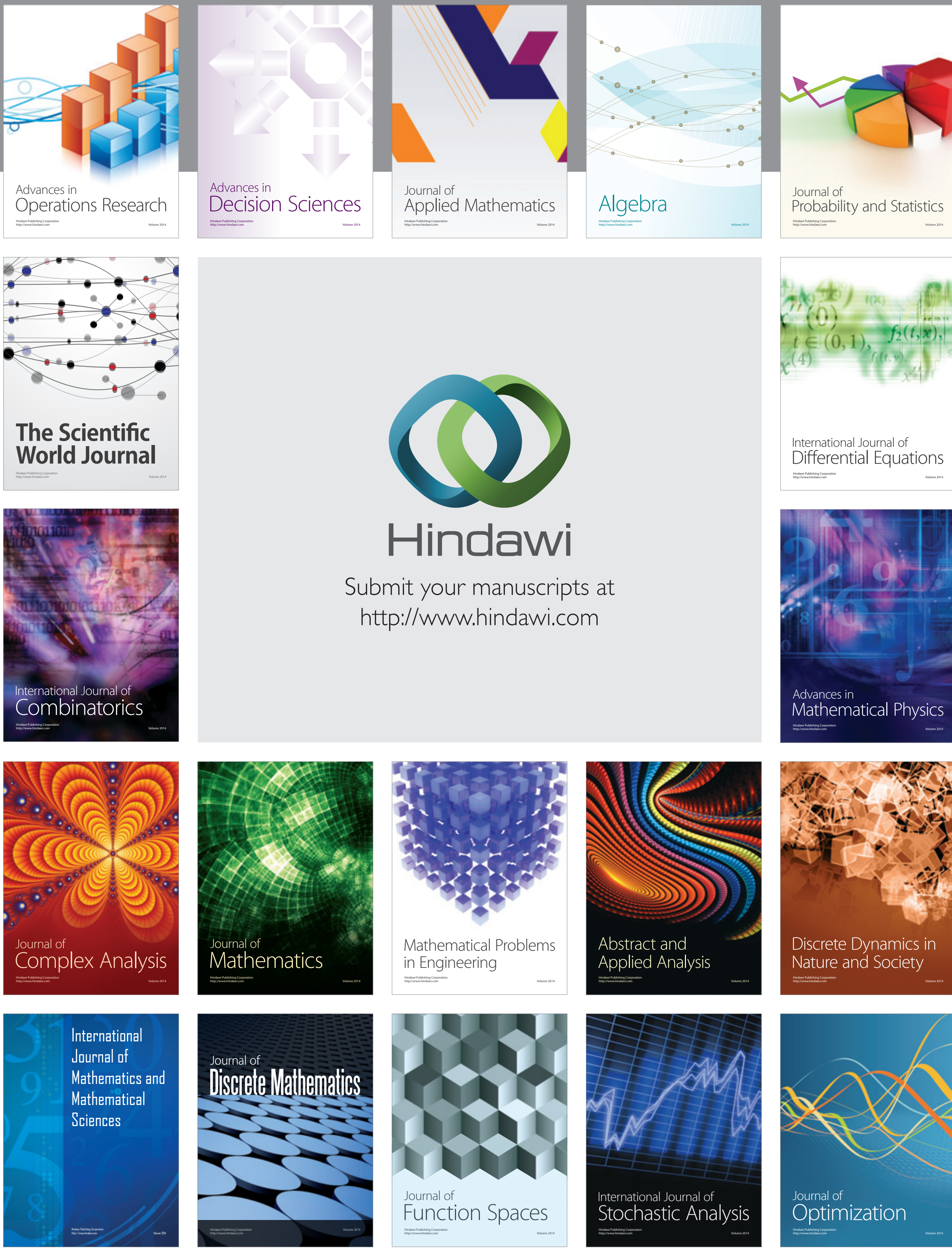
Xinyu Song: Department of Mathematics, Xinyang Normal University, Xinyang 464000, Henan, China
E-mail address: xysong88@163.com

Zhihao Ge: Faculty of Science, Xi'an Jiaotong University, Xi'an 710049, China
E-mail address: zhihaoge@gmail.com

Jingang Wu: Department of Mathematics, Xinyang Normal University, Xinyang 464000, Henan, China
E-mail address: kyczrk@mail2.xytc.edu.cn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