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Research Article

Existence and Uniqueness Theorems for Impulsive Fractional Differential Equations with the Two-Point and Integral Boundary Conditions

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We study a boundary value problem for the system of nonlinear impulsive fractional differential equations of order α $(0 < \alpha \leq 1)$ involving the two-point and integral boundary conditions. Some new results on existence and uniqueness of a solution are established by using fixed point theorems. Some illustrative examples are also presented. We extend previous results even in the integer case $\alpha = 1$.

1. Introduction

For the last decades, fractional calculus has received a great attention because fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various processes of science and engineering. Indeed, we can find numerous applications in viscoelasticity [1–3], dynamical processes in self-similar structures [4], biosciences [5], signal processing [6], system control theory [7], electrochemistry [8], and diffusion processes [9].

On the other hand, the study of dynamical systems with impulsive effects has been an object of intensive investigations in physics, biology, engineering, and so forth. The interest in the study of them is that the impulsive differential systems can be used to model processes which are subject to abrupt changes and which cannot be described by the classical differential problems (e.g., see [10–13] and references therein). Cauchy problems, boundary value problems, and nonlocal problems for impulsive fractional differential equations have been attractive to many researchers; one can see [10–22] and references therein.

Fečkan et al. [22] investigated the existence and uniqueness of solutions for

$$\begin{aligned} {}^{c}D_{0+}^{\alpha}x(t) &= f(t, x(t)), \quad t \in J' := J \setminus \{t_1, \ldots, t_p\}, \\ &\qquad J := [0, T], \\ x(t_i^+) - x(t_i^-) &= a_i, \quad i = 1, 2, \ldots, p, \\ x(0) &= x_0, \quad a_k \in \mathbf{R}, \end{aligned} \tag{1}$$

where ${}^{c}D_{0+}^{\alpha}$ denotes the Caputo fractional derivative of order $\alpha \in (0, 1)$ and $f : J \times \mathbf{R} \to \mathbf{R}$ is a given continuous function.

In [21], Guo and Jiang discussed the existence of solutions for the following nonlinear fractional differential equations with boundary value conditions:

$$\begin{aligned} {}^{c}D_{0+}^{\alpha}x(t) &= f(t, x(t)), \quad t \in J' := J \setminus \{t_1, \ldots, t_p\}, \\ &\qquad J := [0, T], \\ x(t_i^+) - x(t_i^-) &= I_i(x(t_i^-)), \quad i = 1, 2, \ldots, p, \\ ax(0) + bx(T) &= c, \end{aligned} \tag{2}$$

where ${}^cD^{\alpha}_{0+}$ is the Caputo fractional derivative of order $\alpha \in (0,1)$ with the lower limit zero, $f : J \times \mathbf{R} \to \mathbf{R}$ is jointly continuous, t_k satisfy $0 = t < t_1 < \cdots < t_p < t_{p+1} = T$, $x(t_k^+) = \lim_{\varepsilon\to 0^+} x(t_k+\varepsilon)$ and $x(t_k^-) = \lim_{\varepsilon\to 0^+} x(t_k-\varepsilon)$ represent the right and left limits of $x(t)$ at $t = t_k$, $I_k \in C(\mathbf{R},\mathbf{R})$, and a, b, c are real constants with $a + b \neq 0$.

Ashyralyev and Sharifov [20] considered nonfractional n-dimensional analogues of the problem (2) with two-point and integral boundary conditions.

Motivated by the papers above, in this paper, we study impulsive fractional differential equations with the two-point and integral boundary conditions in the following form:

$$\begin{gathered} {}^cD^{\alpha}_{0+}x(t) = f(t, x(t)), \quad t \in J', \\ x(t_j^+) - x(t_j) = I_j(x(t_j)), \quad j = 1, 2, \dots, p, \\ Ax(0) + Bx(T) = \int_0^T g(s, x(s))\,ds, \end{gathered} \tag{3}$$

where $A, B \in \mathbf{R}^{n\times n}$ are given matrices and $\det(A + B) \neq 0$. Here $f, g : [0,T] \times \mathbf{R}^n \to \mathbf{R}^n$ and $I_i : \mathbf{R}^n \to \mathbf{R}^n$ are given functions.

The rest of the paper is organized as follows. In Section 2, we give some notations, recall some concepts, and introduce a concept of a piecewise continuous solution for our problem. In Section 3, we give two main results: the first result based on the Banach contraction principle and the second result based on the Schaefer fixed point theorem. Some examples are given in Section 4 to demonstrate the application of our main results.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper. By $C(J,\mathbf{R}^n)$ we denote the Banach space of all continuous functions from J to $\mathbf{R}^n$ with the norm

$$\|x\|_C = \max\{|x(t)| : t \in J\}, \tag{4}$$

where $|\cdot|$ is the norm in space $\mathbf{R}^n$. We also introduce the Banach space

$$\begin{aligned} PC(J,\mathbf{R}^n) = \{x : J \longrightarrow \mathbf{R}^n : x(t) \in C((t_i, t_{i+1}], \mathbf{R}^n), \\ i = 0, 1, 2 \dots, p,\ x(t_i^-) \text{ and } x(t_i^+) \\ \text{exist } i = 1, \dots, p, \text{ and } x(t_i^-) = x(t_i)\}, \end{aligned} \tag{5}$$

with the norm

$$\|x\|_{PC} := \sup\{|x(t)| : t \in J\}. \tag{6}$$

If $A \in \mathbf{R}^{n\times n}$, then $\|A\|$ is the norm of A.

Let us recall the following known definitions and results. For more details see [15, 16].

Definition 1. If $g \in C[a,b]$ and $\alpha > 0$, then the Riemann-Liouville fractional integral is defined by

$$I^{\alpha}_{a+}g(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{g(s)}{(t-s)^{1-\alpha}}ds, \tag{7}$$

where $\Gamma(\cdot)$ is the Gamma function defined for any complex number z as

$$\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}dt. \tag{8}$$

Definition 2. The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $g : [a,b] \to \mathbf{R}$ is defined by

$${}^cD^{\alpha}_{a+}g(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{g^{(n)}(s)}{(t-s)^{\alpha-n+1}}ds, \tag{9}$$

where $n = [\alpha] + 1$ (the notation $[\alpha]$ stands for the largest integer not greater than α).

Remark 3. Under natural conditions on $g(t)$, the Caputo fractional derivative becomes the conventional integer order derivative of the function $g(t)$ as $\alpha \to n$.

Remark 4. Let $\alpha, \beta > 0$ and $n = [\alpha] + 1$; then the following relations hold:

$$\begin{gathered} {}^cD^{\alpha}_{0+}t^{\beta} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}t^{\beta-1}, \quad \beta > n, \\ {}^cD^{\alpha}_{0+}t^{k} = 0, \quad k = 0, 1, 2, \dots, n-1. \end{gathered} \tag{10}$$

Lemma 5. *For $\alpha > 0$, $g(t) \in C[0,T] \bigcap L_1[0,T]$, the homogeneous fractional differential equation,*

$${}^cD^{\alpha}_{0+}g(t) = 0, \tag{11}$$

has a solution

$$g(t) = c_0 + c_1t + c_2t^2 + \cdots + c_{n-1}t^{n-1}, \tag{12}$$

where $c_i \in \mathbf{R}$, $i = 0, 1, \dots, n-1$, and $n = [\alpha] + 1$.

Lemma 6. *Assume that $g(t) \in C[0,T] \bigcap L_1[0,T]$, with derivative of order n that belongs to $C[0,T] \bigcap L_1[0,T]$; then*

$$I^{\alpha}_{0+}\,{}^cD^{\alpha}_{0+}\,g(t) = g(t) + c_0 + c_1t + c_2t^2 + \cdots + c_{n-1}t^{n-1}, \tag{13}$$

where $c_i \in \mathbf{R}$, $i = 0, 1, \dots, n-1$, and $n = [\alpha] + 1$.

Lemma 7. *Let $p, q \geq 0$, $f \in L_1[0,T]$. Then*

$$I^p_{0+}I^q_{0+}f(t) = I^{p+q}_{0+}f(t) = I^q_{0+}I^p_{0+}f(t) \tag{14}$$

is satisfied almost everywhere on $[0,T]$. Moreover, if $f \in C[0,T]$, then (14) is true for all $t \in [0,T]$.

Lemma 8. *If $\alpha > 0$, $f \in C([0,T])$, then ${}^cD^{\alpha}_{0+}I^{\alpha}_{0+}f(t) = f(t)$ for all $t \in [0,T]$.*

We define a solution problem (3) as follows.

Definition 9. A function $x \in PC(J, \mathbf{R}^n)$ is said to be a solution of problem (3) if ${}^cD_{0+}^{\alpha}x(t) = f(t, x(t))$, for $t \in [0,T]$, $t \neq t_i$, $i = 1,2,\ldots,p$, and for each $i = 1,2,\ldots,p$, $x(t_i^+)-x(t_i) = I_i(x(t_i))$, $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = T$, and the boundary conditions $Ax(0) + Bx(T) = \int_0^T g(s,x(s))ds$ are satisfied.

We have the following result which is useful in what follows.

Theorem 10. *Let f, $g \in C(J, \mathbf{R}^n)$. Then the function x is a solution of the boundary value problem for impulsive differential equation*

$$\begin{gathered} {}^cD_{0+}^{\alpha}x(t) = f(t), \quad t \in J', \\ x(t_j^+) - x(t_j) = I_j(x(t_j)), \quad j = 1,2,\ldots,p, \\ Ax(0) + Bx(T) = \int_0^T g(s)\,ds \end{gathered} \tag{15}$$

if and only if

$$\begin{aligned} x(t) &= (A+B)^{-1}\int_0^T g(s)\,ds \\ &\quad + \sum_{0<t_j<T} K(t_k,t_j) I_j(x(t_j^-)) \\ &\quad + \frac{1}{\Gamma(\alpha)}\sum_{0<t_j\le T} K(t_k,t_j)\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-1}f(s)\,ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}f(s)\,ds, \quad t_k < t \le t_{k+1}, \end{aligned} \tag{16}$$

where

$$K(t,\tau) = \begin{cases} 0, & t = 0, \\ (A+B)^{-1}A, & 0 < \tau \le t, \\ -(A+B)^{-1}B, & t < \tau \le T. \end{cases} \tag{17}$$

Proof. Assume that x is a solution of the boundary value problem (15); then we have

$$x(t) = x(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\,ds, \quad 0 \le t \le t_1. \tag{18}$$

If $t_1 < t \le t_2$, then

$$\begin{gathered} {}^cD_{0+}^{\alpha}x(t) = f(t), \quad t_1 < t \le t_2, \\ x(t_1^+) - x(t_1) = I_1(x(t_1)). \end{gathered} \tag{19}$$

Integrating the expression (19) from t_1 to t, one can obtain

$$x(t) = x(t_1^+) + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}f(s)\,ds. \tag{20}$$

It follows that

$$\begin{aligned} x(t) &= x(t_1) + I_1(x(t_1)) + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}f(s)\,ds \\ &= x(0) + I_1(x(t_1)) + \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}f(s)\,ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}f(s)\,ds. \end{aligned} \tag{21}$$

Thus if $t \in (t_k, t_{k+1}]$, we get

$$\begin{aligned} x(t) &= x(t_k) + I_k(x(t_k)) + \frac{1}{\Gamma(\alpha)}\int_{t_k}^t (t-s)^{\alpha-1}f(s)\,ds \\ &= x(0) + \sum_{0<t_k<t} I_k(x(t_k)) \\ &\quad + \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}f(s)\,ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_{t_k}^t (t-s)^{\alpha-1}f(s)\,ds, \end{aligned} \tag{22}$$

where $x(0)$ is still an arbitrary constant vector. For determining $x(0)$ we use the boundary value condition $Ax(0) + Bx(T) = \int_0^T g(s)ds$:

$$\begin{aligned} \int_0^T g(s)\,ds &= Ax(0) + Bx(T) \\ &= (A+B)x(0) + B\sum_{0<t_k<T} I_k(x(t_k)) \\ &\quad + \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<T} B\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}f(s)\,ds \\ &\quad + \frac{1}{\Gamma(\alpha)}B\int_{t_k}^T (T-s)^{\alpha-1}f(s)\,ds. \end{aligned} \tag{23}$$

Hence, we obtain

$$\begin{aligned} x(0) &= (A+B)^{-1}\int_0^T g(s)\,ds - (A+B)^{-1}B\sum_{0<t_j<T} I_j(x(t_j)) \\ &\quad - \frac{1}{\Gamma(\alpha)}(A+B)^{-1}B\sum_{0<t_j\le T}\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-1}f(s)\,ds, \end{aligned} \tag{24}$$

and consequently for all $t \in (t_k, t_{k+1}]$

$$\begin{aligned} x(t) &= (A+B)^{-1}\int_0^T g(s)\,ds - (A+B)^{-1}B\sum_{0<t_j<T} I_j(x(t_j)) \\ &\quad - \frac{1}{\Gamma(\alpha)}(A+B)^{-1}B\sum_{0<t_j\le T}\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-1}f(s)\,ds \\ &\quad + \sum_{0<t_j<t} I_j(x(t_j)) \end{aligned}$$

$$+\frac{1}{\Gamma(\alpha)}\sum_{0<t_j<t}\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-1}f(s)\,ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}f(s)\,ds$$
$$=(A+B)^{-1}\int_0^T g(s)\,ds-(A+B)^{-1}B\sum_{t_k<t_j<T}I_j(x(t_j))$$
$$+(A+B)^{-1}A\sum_{0<t_j<t}I_j(x(t_j))$$
$$-\frac{1}{\Gamma(\alpha)}(A+B)^{-1}B\sum_{t_k<t_j\le T}\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-1}f(s)\,ds$$
$$+\frac{1}{\Gamma(\alpha)}(A+B)^{-1}A\sum_{0<t_j<t}\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-1}f(s)\,ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}f(s)\,ds$$
$$=(A+B)^{-1}\int_0^T g(s)\,ds+\sum_{0<t_j<T}K(t_k,t_j)I_j(x(t_j^-))$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{0<t_j\le T}K(t_k,t_j)\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-1}f(s)\,ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}f(s)\,ds. \tag{25}$$

Conversely, assume that x satisfies (16). If $t\in[0,t_1]$, then, using the fact that ${}^cD^{\alpha}_{0^+}$ is the left inverse of I^{α}_{0+}, we get ${}^cD^{\alpha}_{0+}x(t)=f(t)$, $t_0<t\le t_1$. If $t\in(t_k,t_{k+1}]$, $k=1,2,\ldots,p$, then, using the fact that the Caputo derivative of a constant is equal to zero, we obtain ${}^cD^{\alpha}_{0+}x(t)=f(t)$, $t_k<t\le t_{k+1}$, and $x(t_k^+)-x(t_k)=I_k(x(t_k))$. The lemma is proved. □

Theorem 11 (see [18]). *Let X be a Banach space and $W\subset PC(J,X)$. If the following conditions are satisfied,*

(1) *W is uniformly bounded subset of $PC(J,X)$,*

(2) *W is equicontinuous in (t_k,t_{k+1}), $k=0,1,2,\ldots,p$, where $t_0=0$, $t_{p+1}=T$,*

(3) *$W(t)=\{u(t):u\in W,t\in J'\}$, $W(t_k^+)=\{u(t_k^+):u\in W\}$, and $W(t_k^-)=\{u(t_k^-):u\in W\}$ are relatively compact subsets of X,*

then W is a relatively compact subset of $PC(J,X)$.

3. Main Results

Our first result is based on Banach fixed point theorem. Before stating and proving the main results, we introduce the following hypotheses.

(H1) $f,g:J\times\mathbf{R}^n\to\mathbf{R}^n$ are continuous functions.

(H2) There are constants $L_f>0$ and $L_g>0$ such that

$$|f(t,x)-f(t,y)|\le L_f|x-y|,\qquad |g(t,x)-g(t,y)|\le L_g|x-y| \tag{26}$$

for each $t\in[0,T]$ and all $x,y\in\mathbf{R}^n$.

(H3) There exist constants $l_i>0$, $i=1,2,\ldots,p$ such that

$$|I_i(x)-I_i(y)|\le l_i|x-y| \tag{27}$$

for all $x,y\in\mathbf{R}^n$.

For brevity, let

$$L_{AB}:=\max\left(\left\|(A+B)^{-1}A\right\|,\ \left\|(A+B)^{-1}B\right\|\right). \tag{28}$$

Theorem 12. *Assume that (H1)–(H3) hold. If*

$$L_g\left\|(A+B)^{-1}\right\|T+L_{AB}\sum_{j=1}^{p}l_j+\frac{1}{\Gamma(\alpha+1)}L_fL_{AB}\sum_{j=1}^{p+1}(t_j-t_{j-1})^{\alpha}+\frac{T^{\alpha}}{\Gamma(\alpha+1)}L_f<1, \tag{29}$$

then the boundary value problem (3) *has a unique solution on J.*

Proof. The proof is based on the classical Banach fixed theorem for contractions. Let us set

$$\sup_{t\in J}|f(t,0)|=M_f,\qquad \sup_{t\in J}|g(t,0)|=M_g,$$
$$|I_k(0)|=m_k,$$
$$\delta(T):=L_g\left\|(A+B)^{-1}\right\|T+L_{AB}\sum_{j=1}^{p}l_j+\frac{1}{\Gamma(\alpha+1)}L_fL_{AB}\sum_{j=1}^{p+1}(t_j-t_{j-1})^{\alpha}+\frac{T^{\alpha}}{\Gamma(\alpha+1)}L_f<1,$$
$$\gamma:=\left\|(A+B)^{-1}\right\|TM_g+L_{AB}\sum_{j=1}^{p}m_j+\frac{1}{\Gamma(\alpha+1)}M_fL_{AB}\sum_{j=1}^{p+1}(t_j-t_{j-1})^{\alpha}+\frac{T^{\alpha}}{\Gamma(\alpha+1)}M_f. \tag{30}$$

It is clear that

$$\begin{aligned}&\left|f(t,x)\right| \le M_f + L_f|x|,\\&\left|g(t,x)\right| \le M_g + L_g|x|,\\&\left|I_k(x)\right| \le m_k + l_k|x|, \quad t\in J,\ x\in \mathbf{R}^n,\\&\left\|K(t_k,t_j)\right\| \le L_{AB}, \quad k,j=0,1,\ldots,p+1.\end{aligned} \tag{31}$$

Consider

$$B_r := \{x \in PC(J,\mathbf{R}^n) : \|x\|_{PC} \le r\}, \tag{32}$$

where

$$r \ge \frac{\gamma}{1-\delta(T)}. \tag{33}$$

Let Q be the following operator:

$$(Qx)(t) = \begin{cases}(A+B)^{-1}\int_0^T g(s)\,ds \\ \quad + \sum_{0<t_j<T} K(t_k,t_j) I_j(x(t_j)) \\ \quad + \frac{1}{\Gamma(\alpha)} \sum_{0<t_j\le T} K(t_k,t_j) \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-1} f(s)\,ds \\ \quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f(s)\,ds, \quad t_k<t\le t_{k+1}, \\ \qquad k=1,\ldots,p.\end{cases} \tag{34}$$

We show that Q maps B_r into B_r. It is clear that Q is well defined on $PC(J,\mathbf{R}^n)$. Moreover for $x\in B_r$ and $t\in(t_k,t_{k+1}]$, $k=0,\ldots,p$, we have

$$\begin{aligned}|(Qx)(t)| \le{}& \left\|(A+B)^{-1}\right\| \int_0^T |g(s,x(s))|\,ds \\ &+ \sum_{0<t_j<T} \left\|K(t_k,t_j)\right\| \left|I_j(x(t_j))\right| \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_j\le T} \left\|K(t_k,t_j)\right\| \\ &\times \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-1} |f(s,x(s))|\,ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |f(s,x(s))|\,ds \\ \le{}& \left\|(A+B)^{-1}\right\| T (M_g+L_g r) + L_{AB} \sum_{j=1}^{p} (m_j+l_j r) \\ &+ \frac{1}{\alpha\Gamma(\alpha)} L_{AB} (M_f+L_f r) \sum_{j=1}^{p+1} (t_j-t_{j-1})^{\alpha} \\ &+ \frac{T^{\alpha}}{\alpha\Gamma(\alpha)} (M_f+L_f r) = \gamma + \delta(T) r \le r.\end{aligned} \tag{35}$$

Consequently Q maps $PC(J,\mathbf{R}^n)$ into itself.

Next we will show that Q is a contraction. Let $x,y\in PC(J,\mathbf{R}^n)$. Then, for each $t\in(t_k,t_{k+1}]$, $k=0,\ldots,p$, we have

$$\begin{aligned}&\left|(Qx)(t)-(Qy)(t)\right| \\ &\le \left\|(A+B)^{-1}\right\| \int_0^T \left|g(s,x(s))-g(s,y(s))\right| ds \\ &\quad + \sum_{0<t_j<T} \left\|K(t_k,t_j)\right\| \left|I_j(x(t_j))-I_j(y(t_j))\right| \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{0<t_j\le T} \left\|K(t_k,t_j)\right\| \\ &\quad \times \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-1} \left|f(s,x(s))-f(s,y(s))\right| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} \left|f(s,x(s))-f(s,y(s))\right| ds \\ &\le L_g \left\|(A+B)^{-1}\right\| \int_0^T |x(s)-y(s)|\,ds \\ &\quad + L_{AB} \sum_{j=1}^{p} \left|I_j(x(t_j))-I_j(y(t_j))\right| \\ &\quad + \frac{L_f L_{AB}}{\Gamma(\alpha)} \sum_{0<t_j\le T} \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-1} |x(s)-y(s)|\,ds \\ &\quad + \frac{L_f}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |x(s)-y(s)|\,ds \\ &\le \Bigg(L_g \left\|(A+B)^{-1}\right\| T + L_{AB} \sum_{j=1}^{p} l_j \\ &\qquad + \frac{1}{\alpha\Gamma(\alpha)} L_f L_{AB} \sum_{j=1}^{p+1} (t_j-t_{j-1})^{\alpha} + \frac{T^{\alpha}}{\alpha\Gamma(\alpha)} L_f \Bigg) \\ &\quad \times \|x-y\|_{PC}.\end{aligned} \tag{36}$$

Thus, Q is a contraction mapping on $PC(J,\mathbf{R}^n)$ due to condition (29) and the operator Q has a unique fixed point on $PC(J,\mathbf{R}^n)$ which is a unique solution to (3). □

The second result is based on the Schaefer fixed point theorem. We introduce the following assumptions.

(H4) There exist constants $N_f > 0$, $N_g > 0$ such that $|f(t,x)| \le N_f$, $|g(t,x)| \le N_g$ for each $t\in J$ and all $x\in\mathbf{R}^n$.

(H5) $I_k \in C(\mathbf{R}^n,\mathbf{R}^n)$.

Theorem 13. *Assume that (H1), (H4), and (H5) hold. Then the boundary value problem* (3) *has at least one solution on* J.

Proof. We will use Schaefer's fixed point theorem to prove that Q defined by (34) has a fixed point. The proof will be given in several steps.

Step 1. Operator Q is continuous.

Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $PC(J,\mathbf{R}^n)$. Then, for each $k = 0,1,2,\dots,p$ and for all $t \in (t_k,t_{k+1}]$, we have

$$\begin{aligned} &\left|(Qx)(t) - (Qx_n)(t)\right| \\ &\quad\le \left\|(A+B)^{-1}\right\| \int_0^T \left|g(s,x(s)) - g(s,x_n(s))\right| ds \\ &\qquad + \sum_{0<t_j<T} \left\|K(t_k,t_j)\right\| \left|I_j(x(t_j)) - I_j(x_n(t_j))\right| \\ &\qquad + \frac{1}{\Gamma(\alpha)} \sum_{0<t_j\le T} \left\|K(t_k,t_j)\right\| \\ &\qquad \times \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-1} \left|f(s,x(s)) - f(s,x_n(s))\right| ds \\ &\qquad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left|f(s,x(s)) - f(s,x_n(s))\right| ds. \end{aligned} \tag{37}$$

Since f, g, and I_k, $k = 0,1,2,\dots,p$, are continuous functions, we have

$$\|Qx_n - Qx\|_{PC} \longrightarrow 0 \tag{38}$$

as $n \to \infty$.

Step 2. Q maps bounded sets in bounded sets in $PC(J,\mathbf{R}^n)$.

Indeed, it is enough to show that, for any $\eta > 0$, there exists a positive constant l such that, for each $x \in B_\eta = \{x \in PC(J,\mathbf{R}^n) : \|x\|_{PC} \le \eta\}$, we have $\|Q(x)\|_{PC} \le l$. By (H4), (H5) we have, for each $k = 1,2,\dots,p$ and for all $t \in (t_k,t_{k+1}]$,

$$\begin{aligned} &|(Qx)(t)| \\ &\quad\le \left\|(A+B)^{-1}\right\| \int_0^T |g(s,x(s))| ds \\ &\qquad + \sum_{0<t_j<T} \left\|K(t_k,t_j)\right\| \left|I_j(x(t_j))\right| \\ &\qquad + \frac{1}{\Gamma(\alpha)} \sum_{0<t_j\le T} \left\|K(t_k,t_j)\right\| \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-1} |f(s,x(s))| ds \\ &\qquad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} |f(s,x(s))| ds \\ &\quad\le \left\|(A+B)^{-1}\right\| TN_g + L_{AB} \sum_{j=1}^{p} \left|I_j(x(t_j))\right| \\ &\qquad + \frac{1}{\alpha\Gamma(\alpha)} L_{AB} N_f \sum_{j=1}^{p+1} (t_j - t_{j-1})^\alpha + \frac{T^\alpha}{\alpha\Gamma(\alpha)} N_f := l. \end{aligned} \tag{39}$$

Thus

$$\|Qx\|_{PC} \le l. \tag{40}$$

Step 3. Q maps bounded sets into equicontinuous sets of $PC(J,\mathbf{R}^n)$.

Let $\tau_1, \tau_2 \in (t_k,t_{k+1}]$, $\tau_1 < \tau_2$, B_η be a bounded set of $PC(J,\mathbf{R}^n)$ as in Step 2, and let $x \in B_\eta$. Then

$$\begin{aligned} &\left|(Qx)(\tau_2) - (Qx)(\tau_1)\right| \\ &\quad\le \left| \frac{1}{\Gamma(\alpha)} \int_{t_i}^{\tau_1} \left[(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}\right] f(s,x(s)) ds \right. \\ &\qquad \left. + \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1} f(s,x(s)) ds \right| \\ &\quad\le \frac{N_f}{\Gamma(\alpha)} \int_{t_i}^{\tau_1} \left[(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}\right] ds \\ &\qquad + \frac{N_f}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1} ds \\ &\quad\le \frac{N_f}{\Gamma(\alpha+1)} \left[2(\tau_2 - \tau_1)^\alpha + \tau_2^\alpha - \tau_1^\alpha\right]. \end{aligned} \tag{41}$$

As $\tau_1 \to \tau_2$, the right-hand side of the above inequality tends to zero.

As a consequence of Steps 1 to 3 together with the Arzela-Ascoli theorem (Theorem 11 with $X = \mathbf{R}^n$), we can conclude that the operator $Q : PC(J,\mathbf{R}^n) \to PC(J,\mathbf{R}^n)$ is completely continuous.

Step 4. One has a priori bounds.

Now it remains to show that the set

$$\Delta = \left\{x \in PC(J,\mathbf{R}^n) : x = \lambda Q(x),\ \text{for some } 0 < \lambda < 1\right\} \tag{42}$$

is bounded.

Let then $x = \lambda Q(x)$ for some $0 < \lambda < 1$. Thus, for each $t \in (t_k,t_{k+1}]$, we have

$$\begin{aligned} &|x(t)| = |\lambda (Qx)(t)| \\ &\quad\le \left\|(A+B)^{-1}\right\| \int_0^T |g(s,x(s))| ds \\ &\qquad + \sum_{0<t_j<T} \left\|K(t_k,t_j)\right\| \left|I_j(x(t_j))\right| \\ &\qquad + \frac{1}{\Gamma(\alpha)} \sum_{0<t_j\le T} \left\|K(t_k,t_j)\right\| \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-1} |f(s,x(s))| ds \\ &\qquad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} |f(s,x(s))| ds \\ &\quad\le \left\|(A+B)^{-1}\right\| TN_g + L_{AB} \sum_{j=1}^{p} \left|I_j(x(t_j))\right| \\ &\qquad + \frac{1}{\alpha\Gamma(\alpha)} L_{AB} N_f \sum_{j=1}^{p+1} (t_j - t_{j-1})^\alpha + \frac{T^\alpha}{\alpha\Gamma(\alpha)} N_f. \end{aligned} \tag{43}$$

Thus

$$\|x\|_{PC} \leq \left\|(A+B)^{-1}\right\| TN_g + L_{AB}\sum_{j=1}^{p}\left|I_j\left(x\left(t_j\right)\right)\right| + \frac{1}{\alpha\Gamma(\alpha)}L_{AB}N_f\sum_{j=1}^{p+1}\left(t_j - t_{j-1}\right)^{\alpha} + \frac{T^{\alpha}}{\alpha\Gamma(\alpha)}N_f. \tag{44}$$

This shows that the set Δ is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that Q has a fixed point which is a solution of the problem (3). □

4. Examples

In this section, we give some examples to illustrate our main results.

Example 1. Consider

$$\begin{aligned} &{}^{c}D_{0+}^{\alpha}x_1(t) = \cos\left(\frac{1}{10}x_2(t)\right), \quad t \in (0,2)\setminus\{1\},\\ &{}^{c}D_{0+}^{\alpha}x_2(t) = \frac{e^{-t}}{9+e^{t}}\frac{\left|x_1(t)\right|}{1+\left|x_1(t)\right|}, \quad t \in (0,2)\setminus\{1\},\\ &x_1(0) + \frac{1}{2}x_2(1) = 0, \quad x_2(0) = 1,\\ &\Delta x_1(1) = \frac{1}{10}x_2(1), \qquad \Delta x_2(1) = \frac{1}{10}x_1(1) + 5. \end{aligned} \tag{45}$$

Consider boundary value problem (3) with $f_1(t,x_1,x_2) = \cos((1/10)x_2(t))$, $f_2(t,x_1,x_2) = (e^{-t}/(9+e^{t}))\cdot(|x_1|/(1+|x_1|))$, and $T = 1$, $p = 1$, $I_1(x_1,x_2) = \left[\begin{smallmatrix}(1/10)x_2\\(1/10)x_1+5\end{smallmatrix}\right]$.

Evidently,

$$\begin{aligned} &A = I = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad B = \begin{pmatrix}0 & 0.5\\ 0 & 0\end{pmatrix},\\ &\left\|(A+B)^{-1}A\right\| = \left\|\begin{pmatrix}1 & -\frac{1}{2}\\ 0 & 1\end{pmatrix}\right\| = \frac{3}{2},\\ &\left\|(A+B)^{-1}B\right\| = \left\|\begin{pmatrix}0 & \frac{1}{2}\\ 0 & 0\end{pmatrix}\right\| = \frac{1}{2},\\ &L_{AB} = \max\left(\frac{3}{2},\frac{1}{2}\right) = \frac{3}{2}, \end{aligned} \tag{46}$$

and conditions (H1)–(H3) hold. We will show that condition (29) is satisfied for, say, $\alpha = 0,2$. Indeed,

$$\begin{aligned} &L_{AB}l_1 + \frac{2^{\alpha+1}}{\alpha\Gamma(\alpha)}L_fL_{AB} + \frac{2^{\alpha}}{\alpha\Gamma(\alpha)}L_f\\ &= \frac{3}{2}\times\frac{1}{10} + \frac{2^{\alpha}}{\Gamma(\alpha+1)}\times\frac{3}{10} + \frac{2^{\alpha}}{\Gamma(\alpha+1)}\times\frac{1}{10}\\ &< \frac{3}{2}\times\frac{1}{10} + \frac{5}{4}\times\frac{2}{5} = \frac{13}{20} < 1, \end{aligned} \tag{47}$$

where we used

$$\Gamma(\alpha+1) = \Gamma(1,2) = 0.92, \qquad \frac{2^{\alpha}}{\Gamma(\alpha+1)} < \frac{1.15}{0.92} = 1.25. \tag{48}$$

Then, by Theorem 12, boundary value problem (45) has unique solution on $[0,2]$.

Example 2. Consider

$$\begin{aligned} &{}^{c}D_{0+}^{\alpha}x_1(t) = \frac{e^{-t}}{9+e^{t}}\frac{\left|x_1(t)\right|}{1+x_2^2(t)}, \quad t \in (0,1),\ t \neq 0,5\\ &{}^{c}D_{0+}^{\alpha}x_2(t) = \sin x_1(t), \quad t \in (0,1),\ t \neq 0,5,\\ &x_1(0) = 1, \qquad x_2(0) + \frac{1}{2}x_1(1) = 0,\\ &\Delta x_1\left(\frac{1}{2}\right) = \frac{1}{1+x_2^2(1/2)},\\ &\Delta x_2\left(\frac{1}{2}\right) = \frac{1}{1+\cos^2 x_1(1/2)}. \end{aligned} \tag{49}$$

Here $0 < \alpha \leq 1$, $f_1(t,x_1,x_2) = e^{-t}/(1+x_2^2(t))$, $f_2(t,x_1,x_2) = \sin x_1(t)$, $A = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix}0 & 0\\ 0.5 & 0\end{smallmatrix}\right)$, and $T = 1$, $p = 1$, $I_1(x_1,x_2) = \left[\begin{smallmatrix}1/(1+x_2^2)\\ 1/(1+\cos^2 x_1)\end{smallmatrix}\right]$. Clearly, all the conditions of Theorem 13 are satisfied ($N_f = 1$, $N_g = 0$), and consequently boundary value problem (49) has at least one solution.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] R. L. Bagley, "A theoretical basis for the application of fractional calculus to viscoelasticity," *Journal of Rheology*, vol. 27, no. 3, pp. 201–210, 1983.

[2] G. Sorrentinos, "Fractional derivative linear models for describing the viscoelastic dynamic behavior of polymeric beams," in *Proceedings of the IMAS Conference and Exposition on Structural Dynamics*, St. Louis, Mo, USA, 2006.

[3] G. Sorrentinos, "Analytic modeling and experimental identification of viscoelastic mechanical systems," in *Advances in Fractional Calculus*, Springer, Berlin, Germany, 2007.

[4] F. Mainardi, *Fractals and Fractional Calculus in Continuum Mechanics*, Springer, New York, NY, USA, 1997.

[5] R. L. Magin, "Fractional calculus in bioengineering," *Critical Reviews in Biomedical Engineering*, vol. 32, no. 1, pp. 1–104, 2004.

[6] M. D. Ortigueira and J. A. T. Machado, "Special issue on fractional signal processing and applications," *Signal Processing*, vol. 83, no. 11, pp. 2285–2480, 2003.

[7] B. M. Vinagre, I. Podlubny, A. Hernandez, and V. Feliu, "Some approximations of fractional order operators used in control theory and applications," *Fractional Calculus and Applied Analysis*, vol. 3, no. 3, pp. 231–248, 2000.

[8] K. B. Oldham, "Fractional differential equations in electrochemistry," *Advances in Engineering Software*, vol. 41, no. 1, pp. 9–12, 2010.

[9] R. Metzler and K. Joseph, "Boundary value problems for fractional diffusion equations," *Physica A: Statistical Mechanics and Its Applications*, vol. 278, no. 1, pp. 107–125, 2000.

[10] M. Benchohra, J. Henderson, and S. Ntouyas, *Impulsive Differential Equations and Inclusions (Contemporary Mathematics and Its Applications*, vol. 2, Hindawi Publishing Corporation, New York, NY, USA, 1st edition, 2006.

[11] D. D. Bainov and P. S. Simeonov, *Systems with Impulsive Effect*, Horwood, Chichister, UK, 1989.

[12] V. Lakshmikantham, D. D. Bainov, and P. S. Semeonov, *Theory of Impulsive Differential Equations*, Worlds Scientific, Singapore, 1989.

[13] A. M. Samoilenko and N. A. Perestyuk, *Impulsive Differential Equations*, Worlds Scientific, Singapore, 1995.

[14] R. P. Agarwal, M. Benchohra, and S. Hamani, "Boundary value problems for fractional differential equations," *Georgian Mathematical Journal*, vol. 16, no. 3, pp. 401–411, 2009.

[15] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204 of *North-Holland Mathematics Studies*, Elsevier, Amsterdam, The Netherlands, 2006.

[16] I. Podlubny, *Fractional Differential Equations*, vol. 198 of *Mathematics in Science and Engineering*, Academic Press, New York, NY, USA, 1999.

[17] Z. Bai and H. Lü, "Positive solutions for boundary value problem of nonlinear fractional differential equation," *Journal of Mathematical Analysis and Applications*, vol. 311, no. 2, pp. 495–505, 2005.

[18] W. Wei, X. Xiang, and Y. Peng, "Nonlinear impulsive integro-differential equations of mixed type and optimal controls," *Optimization*, vol. 55, no. 1-2, pp. 141–156, 2006.

[19] A. Ashyralyev and Y. A. Sharifov, "Existence and uniqueness of solutions for the system of nonlinear fractional differential equations with nonlocal and integral boundary conditions," *Abstract and Applied Analysis*, vol. 2012, Article ID 594802, 14 pages, 2012.

[20] A. Ashyralyev and Y. A. Sharifov, "Existence and uniqueness of solutions for nonlinear impulsive differential equations with two-point and integral boundary conditions," *Advances in Difference Equations*, vol. 2013, article 173, 2013.

[21] T. L. Guo and W. Jiang, "Impulsive problems for fractional differential equations with boundary value conditions," *Computers and Mathematics with Applications*, vol. 64, pp. 3281–3291, 2012.

[22] M. Fečkan, Y. Zhou, and J. Wang, "On the concept and existence of solution for impulsive fractional differential equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 7, pp. 3050–3060, 2012.