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Research Article

Multiplicity of Positive Solutions for a p-q-Laplacian Type Equation with Critical Nonlinearities

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We study the effect of the coefficient $f(x)$ of the critical nonlinearity on the number of positive solutions for a p-q-Laplacian equation. Under suitable assumptions for $f(x)$ and $g(x)$, we should prove that for sufficiently small $\lambda > 0$, there exist at least k positive solutions of the following p-q-Laplacian equation, $-\Delta_p u - \Delta_q u = f(x)\left|u\right|^{p^*-2}u + \lambda g(x)\left|u\right|^{r-2}u$ in Ω, $u = 0$ on $\partial\Omega$, where $\Omega \subset \mathbf{R}^N$ is a bounded smooth domain, $N > p$, $1 < q < N(p-1)/(N-1) < p \le \max\{p, p^* - q/(p-1)\} < r < p^*$, $p^* = Np/(N-p)$ is the critical Sobolev exponent, and $\Delta_s u = \operatorname{div}(|\nabla u|^{s-2}\nabla u)$ is the s-Laplacian of u.

1. Introduction

This paper is concerned with the multiplicity of positive solutions to the following p-q-Laplacian equation with critical nonlinearities:

$$-\Delta_p u - \Delta_q u = f(x)|u|^{p^*-2}u + \lambda g(x)|u|^{r-2}u \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega, \tag{E_λ}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded smooth domain with smooth boundary $\partial\Omega$, $1 < q < p < N$, $\lambda > 0$, and $\Delta_s u = \operatorname{div}(|\nabla u|^{s-2}\nabla u)$ is the s-Laplacian of u, and assume that

($H1$) $1 < q < N(p-1)/(N-1) < p \le \max\{p, p^* - q/(p-1)\} < r < p^* = Np/(N-p)$;

($H2$) f and g are positive continuous functions in $\overline{\Omega}$;

($H3$) There exist k points $a^1, a^2, \ldots, a^k$ in Ω such that $f(a^i)$ are strict local maxima satisfying

$$f\left(a^i\right) = \max_{x\in\overline{\Omega}} f(x) = 1 \quad \text{for } 1 \le i \le k, \tag{1}$$

and for some $\beta > N/(p-1)$, $f(x) = f(a^i) + O(|x-a^i|^\beta)$ as $x \to a^i$ uniformly in i.

Problem (E_λ) comes, for example, from a general reaction-diffusion system

$$u_t = \operatorname{div}[H(u)\nabla u] + c(x,u), \tag{2}$$

where $H(u) = |\nabla u|^{p-2} + |\nabla u|^{q-2}$. This system has a wide range of applications in physics and related science such as biophysics, plasma physics, and chemical reaction design. In such applications, the function u describes a concentration, the first term on the right-hand side of (2) corresponds to the diffusion with a diffusion coefficient $H(u)$, whereas the second one is the reaction and relates to sources and loss processes. Typically, in chemical and biological applications, the reaction term $c(x,u)$ has a polynomial form with respect to the concentration u.

The stationary solution of (2) was studied by many authors; that is, many works are considered the solutions of the following problem:

$$-\operatorname{div}[H(u)\nabla u] = c(x,u). \tag{3}$$

See [1–5] for different $c(x,u)$. In the present paper we are concerned with problem (E_λ) in a bounded domain with $c(x,u) = f(x)|u|^{p^*-2}u + \lambda g(x)|u|^{r-2}u$ in (3). Recently, in [6], the authors obtain the existence of $\operatorname{cat}_\Omega(\Omega)$ positive solutions of problem (E_λ) for $N > p^2$ and $f(x) \equiv g(x) \equiv 1$ when condition $(H1)$ holds, where $\mathrm{cat}_\Omega(\Omega)$ denotes the Lusternik-Schnirelmann category of Ω in itself.

Specially, if $p = q$, (E_λ) can be reduced to the following elliptic problems:

$$-\Delta_p u = f(x)|u|^{p^*-2}u + \lambda g(x)|u|^{r-2}u \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega. \tag{4}$$

After the well-known results of Brézis and Nirenberg [7], who studied (4) in the case of $p = r = 2$ and $f(x) \equiv g(x) \equiv 1$, a lot of problems involving the critical growth in bounded and unbounded domains have been considered; see, for example, [8–10] and reference therein. In particular, the first multiplicity result for (4) has been achieved by Rey in [11] in the semilinear case. Precisely Rey proved that if $N \geq 5$, $p = r = 2$, and $f(x) \equiv g(x) \equiv 1$, for λ small enough, problem (4) has at least $\mathrm{cat}_\Omega(\Omega)$ solutions. Furthermore, Alves and Ding in [12] obtained the existence of $\mathrm{cat}_\Omega(\Omega)$ positive solutions to problem (4) with $p \geq 2$, $r \in [p, p^*)$, and $f(x) \equiv g(x) \equiv 1$. Finally, we mention that [13] studied (4) when $1 < r < p < N$ and f, g are sign-changing and verified the existence of two positive solutions for $\lambda \in (0, \lambda_0)$ for some positive constant λ_0.

The main purpose of this paper is to analyze the effect of the coefficient $f(x)$ of the critical nonlinearity to prove the multiplicity of positive solutions of problem (E_λ) for small $\lambda > 0$. By the similar argument in [14], we can construct the k compact Palais-Smale sequences that are suitably localized in correspondence of k maximum points of f. Under some assumptions $(H1)$–$(H3)$, we could show that there are at least k positive solutions of problem (E_λ) for sufficiently small $\lambda > 0$.

This paper is organized as follows. First of all, we study the argument of the Nehari manifold $\mathcal{M}_\lambda$. Next, we prove the existence of a positive solution $u_0 \in \mathcal{M}_\lambda$. Finally, we show that the condition $(H3)$ affects the number of positive solution of (E_λ); that is, there are at least k critical points $u_i \in \mathcal{M}_\lambda$ of J_λ such that $J_\lambda(u_i) = \alpha_\lambda^i$ ((PS)-value) for $1 \leq i \leq k$.

The main results of this paper are given as follows.

Theorem 1. *Suppose that (H1)–(H3) hold; then problem (E_λ) has a positive solution u_0 in $W_0^{1,p}(\Omega)$ for all $\lambda > 0$.*

Theorem 2. *Suppose that (H1)–(H3) hold; then there exists a $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0)$, problem (E_λ) admits at least k positive solutions in $W_0^{1,p}(\Omega)$.*

2. Preliminaries

In what follows, we denote by $\|\cdot\|_p$, $|\cdot|_p$ the norms on $W_0^{1,p}(\Omega)$ and $L^p(\Omega)$, respectively; that is,

$$\|u\|_p = \left(\int_\Omega |\nabla u|^p dx\right)^{1/p}, \qquad |u|_p = \left(\int_\Omega |u|^p dx\right)^{1/p}. \tag{5}$$

We denote the dual space of $W_0^{1,p}(\Omega)$ by $W'(\Omega)$. Set also

$$D^{1,p}(\mathbf{R}^N) := \left\{u \in L^{p^*}(\mathbf{R}^N) : \frac{\partial u}{\partial x_i} \in L^p(\mathbf{R}^N) \text{ for } i = 1, 2, \ldots, N\right\} \tag{6}$$

equipped with the norm

$$\|u\|_* = \left(\int_{\mathbf{R}^N} |\nabla u|^p dx\right)^{1/p}. \tag{7}$$

We will denote by S the best Sobolev constant as follows:

$$S = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\|u\|_p^p}{|u|_{p^*}^p}. \tag{8}$$

It is well known that S is independent of Ω and is never achieved except when $\Omega = \mathbf{R}^N$ (see [15]). Throughout this paper, we denote the Lebesgue measure of Ω by $|\Omega|$ and denote a ball centered at $a \in \mathbf{R}^N$ with radius r by $B_r(a)$ and also denote positive constants (possibly different) by C, C_i. $O(\varepsilon^t)$ denotes $|O(\varepsilon^t)|/\varepsilon^t \leq C$, $o(\varepsilon^t)$ denotes $|o(\varepsilon^t)|/\varepsilon^t \to 0$ as $\varepsilon \to 0$, and $o_n(1)$ denotes $o_n(1) \to 0$ as $n \to \infty$.

Associated with (E_λ), we consider the energy functional J_λ in $W_0^{1,p}(\Omega)$, for each $u \in W_0^{1,p}(\Omega)$,

$$J_\lambda(u) = \frac{1}{p}\|u\|_p^p + \frac{1}{q}\|u\|_q^q - \frac{1}{p^*}\int_\Omega f(x)|u|^{p^*}dx - \frac{1}{r}\int_\Omega \lambda g(x)|u|^r dx. \tag{9}$$

It is well known that J_λ is of C^1 in $W_0^{1,p}(\Omega)$ and the solutions of (E_λ) are the critical points of the energy functional J_λ (see [16]).

We define the Nehari manifold

$$\mathcal{M}_\lambda := \left\{u \in W_0^{1,p}(\Omega) \setminus \{0\} : \left\langle J_\lambda'(u), u\right\rangle = 0\right\}, \tag{10}$$

where

$$\left\langle J_\lambda'(u), u\right\rangle = \|u\|_p^p + \|u\|_q^q - \int_\Omega f(x)|u|^{p^*}dx - \int_\Omega \lambda g(x)|u|^r dx = 0. \tag{11}$$

The Nehari manifold $\mathcal{M}_\lambda$ contains all nontrivial solutions of (E_λ).

Note that J_λ is not bounded from below in $W_0^{1,p}(\Omega)$. From the following lemma, we have that J_λ is bounded from below on the Nehari manifold $\mathcal{M}_\lambda$.

Lemma 3. *Suppose that $1 < q < p < r < p^*$ and (H2) hold. Then for any $\lambda > 0$, one has that J_λ is bounded from below on $\mathcal{M}_\lambda$. Moreover, $J_\lambda(u) > 0$ for all $u \in \mathcal{M}_\lambda$.*

Proof. For $u \in \mathscr{M}_\lambda$, (10) leads to

$$J_\lambda(u) = \left(\frac{1}{p} - \frac{1}{r}\right)\|u\|_p^p + \left(\frac{1}{q} - \frac{1}{r}\right)\|u\|_q^q + \left(\frac{1}{r} - \frac{1}{p^*}\right)\int_\Omega f(x)|u|^{p^*}dx > 0. \tag{12}$$

□

Define

$$\alpha_\lambda := \inf_{u\in\mathscr{M}_\lambda} J_\lambda(u). \tag{13}$$

Now we show that J_λ possesses the mountain-pass (MP, in short) geometry.

Lemma 4. *Suppose $1 < q < p < r < p^*$ and (H2) holds. Then for any $\lambda > 0$, one has that*

(i) *there exist positive numbers R and d_0 such that $J_\lambda(u) \geq d_0$ for $\|u\|_p = R$;*

(ii) *there exists $\overline{u} \in W_0^{1,p}(\Omega)$ such that $\|\overline{u}\|_p > R$ and $J_\lambda(\overline{u}) < 0$.*

Proof. (i) By (8), the Hölder inequality, and the Sobolev embedding theorem, we have that

$$\begin{aligned} J_\lambda(u) &\geq \frac{1}{p}\|u\|_p^p - \frac{1}{p^*}\int_\Omega f(x)|u|^{p^*}dx - \frac{1}{r}\int_\Omega \lambda g(x)|u|^r dx \\ &\geq \frac{1}{p}\|u\|_p^p - \frac{1}{p^*}S^{-p^*/p}\|u\|_p^{p^*} - \frac{1}{r}\lambda|g|_\infty|\Omega|^{(p^*-r)/p^*}S^{-r/p}\|u\|_p^r. \end{aligned} \tag{14}$$

Hence, there exist positive R and d_0 such that $J_\lambda(u) \geq d_0$ for $\|u\| = R$.

(ii) For any $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, from

$$J_\lambda(tu) = \frac{t^p}{p}\|u\|_p^p + \frac{t^q}{q}\|u\|_q^q - \frac{t^{p^*}}{p^*}\int_\Omega f(x)|u|^{p^*}dx - \frac{t^r}{r}\int_\Omega \lambda g(x)|u|^r dx, \tag{15}$$

we have $\lim_{t\to\infty}J_\lambda(tu) = -\infty$. For fixed some $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, there exist $\overline{t} > 0$ such that $\|\overline{t}u\|_p > R$ and $J_\lambda(\overline{t}u) < 0$. Let $\overline{u} = \overline{t}u$. □

Define

$$\phi_\lambda(u) := \left\langle J_\lambda'(u), u\right\rangle. \tag{16}$$

Then for $u \in \mathscr{M}_\lambda$,

$$\begin{aligned} \left\langle \phi_\lambda'(u), u\right\rangle &= p\|u\|_p^p + q\|u\|_q^q - p^*\int_\Omega f(x)|u|^{p^*}dx - r\int_\Omega \lambda g(x)|u|^r dx \\ &= (p^* - r)\int_\Omega \lambda g(x)|u|^r dx - (p^* - p)\|u\|_p^p - (p^* - q)\|u\|_q^q \\ &= (p - r)\|u\|_p^p + (q - r)\|u\|_q^q + (r - p^*)\int_\Omega f(x)|u|^{p^*}dx < 0. \end{aligned} \tag{17}$$

Lemma 5. *Suppose that $1 < q < p < r < p^*$ and (H2) holds. If $u_0 \in \mathscr{M}_\lambda$ satisfies*

$$J_\lambda(u_0) = \min_{u\in\mathscr{M}_\lambda} J_\lambda(u) = \alpha_\lambda, \tag{18}$$

then u_0 is a solution of (E_λ).

Proof. By (17), $\langle\phi_\lambda'(u), u\rangle < 0$ for $u \in \mathscr{M}_\lambda$. Since $J_\lambda(u_0) = \min_{u\in\mathscr{M}_\lambda}J_\lambda(u)$, by the Lagrange multiplier theorem, there is $\tau \in \mathbf{R}$ such that $J_\lambda'(u_0) = \tau\phi_\lambda'(u_0)$ in $W'(\Omega)$. This implies that

$$0 = \left\langle J_\lambda'(u_0), u_0\right\rangle = \tau\left\langle \phi_\lambda'(u_0), u_0\right\rangle. \tag{19}$$

It then follows that $\tau = 0$ and $J_\lambda'(u_0) = 0$ in $W'(\Omega)$. Thus, u_0 is a nontrivial solution of (E_λ) and $J_\lambda(u_0) = \alpha_\lambda$. □

Lemma 6. *Suppose that $1 < q < p < r < p^*$ and (H2) holds. For each $u \in W_0^{1,p}(\Omega)\setminus\{0\}$, there exists a unique positive number t_u such that $t_u u \in \mathscr{M}_\lambda$ and $J_\lambda(t_u u) = \sup_{t\geq 0}J_\lambda(tu)$ for any $\lambda > 0$.*

Proof. For fixed $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, consider

$$h(t) = J_\lambda(tu) = \frac{t^p}{p}\|u\|_p^p + \frac{t^q}{q}\|u\|_q^q - \frac{t^{p^*}}{p^*}\int_\Omega f(x)|u|^{p^*}dx - \frac{t^r}{r}\int_\Omega \lambda g(x)|u|^r dx. \tag{20}$$

Since $h(0) = 0$, $\lim_{t\to\infty}h(t) = -\infty$, by Lemma 4(i), then it is easy to see that there exists a unique positive number t_u such that $\sup_{t\geq 0}h(t)$ is achieved at t_u. This means that $h'(t_u) = 0$; that is, $t_u u \in \mathscr{M}_\lambda$. □

We will denote by $\widetilde{\alpha}_\lambda$ the MP level:

$$\widetilde{\alpha}_\lambda := \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}}\ \sup_{t\geq 0} J_\lambda(tu). \tag{21}$$

Then we have the following result.

Lemma 7. *Suppose that $1 < q < p < r < p^*$ and (H2) holds, then $\alpha_\lambda = \widetilde{\alpha}_\lambda$ for any $\lambda > 0$.*

Proof. By Lemma 6, we have

$$\begin{aligned}\widetilde{\alpha}_\lambda &= \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \sup_{t\geq 0} J_\lambda(tu) = \inf_{t_u u\in \mathcal{M}_\lambda} J_\lambda(t_u u)\\ &\geq \inf_{u\in\mathcal{M}_\lambda} J_\lambda(u) = \alpha_\lambda.\end{aligned} \tag{22}$$

On the other hand, for $u \in \mathcal{M}_\lambda$, by Lemma 6, we have $t_u = 1$ and $J_\lambda(u) = \sup_{t\geq 0} J_\lambda(tu)$. Hence,

$$\begin{aligned}\alpha_\lambda &= \inf_{u\in\mathcal{M}_\lambda} J_\lambda(u) = \inf_{u\in\mathcal{M}_\lambda}\sup_{t\geq 0} J_\lambda(tu)\\ &\geq \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \sup_{t\geq 0} J_\lambda(tu) = \widetilde{\alpha}_\lambda.\end{aligned} \tag{23}$$

Now the desired result follows from (22) and (23). □

Remark 8. By Lemma 7 and the definition, it is apparent that $\alpha_{\lambda_1} \leq \alpha_{\lambda_2}$ if $\lambda_1 \geq \lambda_2$; that is, α_λ is nonincreasing in λ. Moreover, by Lemma 4(i), for any $\lambda_0 > 0$, there exists a $d = d(\lambda_0)$, related to the MP geometry, such that

$$0 < d \leq \alpha_\lambda \leq \alpha_0 \quad \forall \lambda \in [0, \lambda_0]. \tag{24}$$

Here α_0 is the MP level associated to the functional

$$J_0(u) = \frac{1}{p}\|u\|_p^p + \frac{1}{q}\|u\|_q^q - \frac{1}{p^*}\int_\Omega f(x)|u|^{p^*}dx. \tag{25}$$

3. $(\mathrm{PS})_c$-Condition in $W_0^{1,p}(\Omega)$ for J_λ

First, we define the Palais-Smale (denote by (PS)) sequence, (PS)-value, and (PS)-conditions in $W_0^{1,p}(\Omega)$ for J_λ.

Definition 9. (i) For $c \in \mathbf{R}$, a sequence $\{u_n\}$ is a $(\mathrm{PS})_c$-sequence in $W_0^{1,p}(\Omega)$ for J_λ if $J_\lambda(u_n) = c + o_n(1)$ and $J'_\lambda(u_n) = o_n(1)$ strongly in $W'(\Omega)$ as $n \to \infty$.

(ii) $c \in \mathbf{R}$ is a (PS)-value in $W_0^{1,p}(\Omega)$ for J_λ if there exists a $(\mathrm{PS})_c$-sequence in $W_0^{1,p}(\Omega)$ for J_λ.

(iii) J_λ satisfies the $(\mathrm{PS})_c$-condition in $W_0^{1,p}(\Omega)$ if every $(\mathrm{PS})_c$-sequence $\{u_n\}$ in $W_0^{1,p}(\Omega)$ for J_λ contains a convergent subsequence.

Applying Ekeland's variational principle and using the same argument as in Cao and Zhou [17] or Tarantello [18], we have the following lemma.

Lemma 10. *Suppose that $1 < q < p < r < p^*$ and (H2) holds. Then for any $\lambda > 0$, there exists a $(PS)_{\alpha_\lambda}$-sequence $\{u_n\}$ in $\mathcal{M}_\lambda$ for J_λ.*

To prove the existence of positive solutions, we claim that J_λ satisfies the $(\mathrm{PS})_c$-condition in $W_0^{1,p}(\Omega)$ for $c \in (0, (1/N)S^{N/p})$.

Lemma 11. *Suppose that $1 < q < p < r < p^*$, $|f|_\infty = 1$, and (H2) holds. Then for any $\lambda > 0$, J_λ satisfies the $(PS)_c$-condition in $W_0^{1,p}(\Omega)$ for all $c \in (0, (1/N)S^{N/p})$.*

Proof. Let $\{u_n\} \subset W_0^{1,p}(\Omega)$ be a $(\mathrm{PS})_c$-sequence for J_λ which satisfies

$$J_\lambda(u_n) = c + o_n(1), \quad J'_\lambda(u_n) = o_n(1) \quad \text{in } W'(\Omega). \tag{26}$$

Then

$$\begin{aligned}c + s_n + \frac{t_n\|u_n\|}{p} &\geq J_\lambda(u_n) - \frac{1}{r}\left\langle J'_\lambda(u_n), u_n\right\rangle\\ &= \left(\frac{1}{p} - \frac{1}{r}\right)\|u_n\|_p^p + \left(\frac{1}{q} - \frac{1}{r}\right)\|u_n\|_q^q\\ &\quad + \left(\frac{1}{r} - \frac{1}{p^*}\right)\int_\Omega f(x)|u_n|^{p^*}dx\\ &\geq \frac{r-p}{rp}\|u_n\|_p^p,\end{aligned} \tag{27}$$

where $s_n = o_n(1)$, $t_n = o_n(1)$, as $n \to \infty$. It follows that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Thus, there exist a subsequence still denoted by $\{u_n\}$ and $u \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned}&u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega),\\ &u_n \longrightarrow u \quad \text{strongly in } L^s(\Omega)\ \forall 1 \leq s < p^*,\\ &u_n \longrightarrow u \quad \text{a.e. in } \Omega.\end{aligned} \tag{28}$$

Furthermore, we have that $J'_\lambda(u) = 0$ in $W'(\Omega)$. By g being continuous on $\overline{\Omega}$, we get

$$\lambda\int_\Omega g(x)|u_n|^r dx = \lambda\int_\Omega g(x)|u|^r dx + o_n(1). \tag{29}$$

Let $v_n = u_n - u$. Then by f being positive continuous on $\overline{\Omega}$ and Brézis-Lieb lemma (see [19]), we obtain

$$\begin{aligned}\|v_n\|_p^p &= \|u_n\|_p^p - \|u\|_p^p + o_n(1),\\ \|v_n\|_q^q &= \|u_n\|_q^q - \|u\|_q^q + o_n(1),\\ \int_\Omega f(x)|v_n|^{p^*}dx &= \int_\Omega f(x)|u_n|^{p^*}dx - \int_\Omega f(x)|u|^{p^*}dx + o_n(1).\end{aligned} \tag{30}$$

From (26)–(30), we can deduce that

$$\begin{aligned}&\frac{1}{p}\|v_n\|_p^p + \frac{1}{q}\|v_n\|_q^q - \frac{1}{p^*}\int_\Omega f(x)|v_n|^{p^*}dx\\ &\quad = c - J_\lambda(u) + o_n(1),\end{aligned} \tag{31}$$

$$\|v_n\|_p^p + \|v_n\|_q^q = \int_\Omega f(x)|v_n|^{p^*}dx + o_n(1). \tag{32}$$

Without loss of generality, we may assume that

$$\|v_n\|_p^p = a + o_n(1), \qquad \|v_n\|_q^q = b + o_n(1). \tag{33}$$

So (32) and $|f|_\infty = 1$ imply that

$$\int_\Omega |v_n|^{p^*} dx \geq \int_\Omega f(x) |v_n|^{p^*} dx = a + b + o_n(1). \quad (34)$$

By the Sobolev inequality and (33) and (34), we have $\|v_n\|_p^p \geq S(\int_\Omega |v_n|^{p^*} dx)^{p/p^*}$ and

$$a \geq S(a+b)^{p/p^*} \geq S a^{p/p^*}. \quad (35)$$

If $a > 0$, then (35) implies that $a \geq S^{N/p}$, combined with (31), (33)–(35) and Lemma 3, $1 < q < p < p^*$, as $n \to \infty$; we get

$$c = \frac{a}{p} + \frac{b}{q} - \frac{a+b}{p^*} + J_\lambda(u) \geq \frac{1}{N} a \geq \frac{1}{N} S^{N/p}, \quad (36)$$

which is a contradiction. So, we have $a = 0$; J_λ satisfies the $(\mathrm{PS})_c$-condition in $W_0^{1,p}(\Omega)$ for all $c \in (0, (1/N)S^{N/p})$. □

4. Existence of k Positive Solutions

In this section, we first give some preliminary notations and useful lemmas.

Choose $r_0 > 0$ small enough such that $\overline{B_{r_0}(a^i)} \subset \Omega$ and $\overline{B_{r_0}(a^i)} \cap \overline{B_{r_0}(a^j)} = \varnothing$ for $i \neq j$, $i, j = 1, 2, \ldots, k$.

Define

$$Q_i(u) = \frac{\int_\Omega \phi_i(x) |u|^{p^*} dx}{\int_\Omega |u|^{p^*} dx},$$
$$\phi_i(x) = \min\left\{1, \left|x - a^i\right|\right\}, \quad 1 \leq i \leq k. \quad (37)$$

Then we have the following separation result.

Lemma 12. *If $Q_i(u) \leq r_0/3$ and $Q_j(u) \leq r_0/3$ for $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, then $i = j$.*

Proof. For any $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ satisfying $Q_i(u) \leq r_0/3$ $(1 \leq i \leq k)$, we get

$$\begin{aligned} \frac{r_0}{3} \int_\Omega |u|^{p^*} dx &\geq \int_\Omega \phi_i(x) |u|^{p^*} dx \\ &\geq \int_{\Omega \setminus B_{r_0}(a^i)} \phi_i(x) |u|^{p^*} dx \\ &\geq r_0 \int_{\Omega \setminus B_{r_0}(a^i)} |u|^{p^*} dx, \end{aligned} \quad (38)$$

which implies that

$$\int_\Omega |u|^{p^*} dx \geq 3 \int_{\Omega \setminus B_{r_0}(a^i)} |u|^{p^*} dx, \quad 1 \leq i \leq k. \quad (39)$$

Hence, from (39), we obtain

$$\begin{aligned} 2\int_\Omega |u|^{p^*} dx &\geq 3\left(\int_{\Omega \setminus B_{r_0}(a^i)} |u|^{p^*} dx + \int_{\Omega \setminus B_{r_0}(a^j)} |u|^{p^*} dx\right) \\ &\geq 3 \int_\Omega |u|^{p^*} dx \quad \text{if } i \neq j, \end{aligned} \quad (40)$$

which is a contradiction. □

For $i = 1, 2, \ldots, k$, we set

$$\begin{aligned} \mathcal{N}_\lambda^i &:= \left\{u \in \mathcal{M}_\lambda : Q_i(u) < \frac{r_0}{3}\right\} \\ \partial\mathcal{N}_\lambda^i &:= \left\{u \in \mathcal{M}_\lambda : Q_i(u) = \frac{r_0}{3}\right\}, \end{aligned} \quad (41)$$

and define

$$\alpha_\lambda^i := \inf_{\mathcal{N}_\lambda^i} J_\lambda(u), \qquad \tilde{\alpha}_\lambda^i := \inf_{\partial\mathcal{N}_\lambda^i} J_\lambda(u). \quad (42)$$

Now let us assume that $(H1)$–$(H3)$ hold. From conditions $(H2)$ and $(H3)$, we can choose a $\rho \in (0, r_0/2)$ small enough and there exist some positive constants γ_1, γ_2 such that for $1 \leq i \leq k$, we have

$$\begin{aligned} &\overline{\bigcup_{1\leq i\leq k} B_{2\rho}(a^i)} \subset \Omega, \\ &\left|f(x) - f\left(a^i\right)\right| \leq \gamma_1 \left|x - a^i\right|^\beta \quad \forall x \in \overline{B_{2\rho}(a^i)}, \\ &g(x) \geq \gamma_2 \quad \forall x \in \overline{\bigcup_{1\leq i\leq k} B_{2\rho}(a^i)}, \end{aligned} \quad (43)$$

for some $\beta > N/(p-1)$. For $i \in \{1, 2, \ldots, k\}$ and $\varepsilon > 0$, we define

$$\begin{aligned} u_\varepsilon^{a^i}(x) &= \frac{\eta_i(x)}{\left(\varepsilon + \left|x - a^i\right|^{p/(p-1)}\right)^{(N-p)/p}}, \\ v_\varepsilon^{a^i}(x) &= \varepsilon^{(N-p)/p^2} u_\varepsilon^{a^i}(x), \end{aligned} \quad (44)$$

where $\eta_i \in C_0^\infty(B_{2\rho}(a^i))$ is a function such that $0 \leq \eta_i(x) \leq 1$ and $\eta_i(x) \equiv 1$ on $B_\rho(a^i)$. Then we obtain the following estimates (see [20]):

$$\int_\Omega \left|u_\varepsilon^{a^i}\right|^t dx = \begin{cases} K_1 \varepsilon^{(N(p-1)-t(N-p))/p} + O(1), & t > \dfrac{N(p-1)}{p}, \\ K_1 |\ln \varepsilon| + O(1), & t = \dfrac{N(p-1)}{p}, \\ O(1), & t < \dfrac{N(p-1)}{p}, \end{cases} \quad (45)$$

$$\int_\Omega \left|\nabla u_\varepsilon^{a^i}\right|^t dx = \begin{cases} K_2 \varepsilon^{(t+N(p-1)-tN)/p} + O(1), & t > \dfrac{N(p-1)}{p}, \\ K_2 |\ln \varepsilon| + O(1), & t = \dfrac{N(p-1)}{p}, \\ O(1), & t < \dfrac{N(p-1)}{p}. \end{cases} \quad (46)$$

From (43)–(46) [13, Lemma 4.2] and conditions $(H2)$-$(H3)$, we can deduce the following estimates:

$$\int_\Omega \left|\nabla v_\varepsilon^{a^i}\right|^p dx = K_2 + O\left(\varepsilon^{(N-p)/p}\right),$$
$$\int_\Omega \left|\nabla v_\varepsilon^{a^i}\right|^q dx = K_2 + O\left(\varepsilon^{q(N-p)/p^2}\right), \tag{47}$$

$$\int_\Omega f(x)\left|v_\varepsilon^{a^i}\right|^{p^*} dx = K_3^{p^*/p} + O\left(\varepsilon^{N/p}\right), \tag{48}$$

$$\int_\Omega \left|v_\varepsilon^{a^i}\right|^r dx = K_1 \varepsilon^{((p-1)/p)(N-r((N-p)/p))} + O\left(\varepsilon^{r(N-p)/p^2}\right), \tag{49}$$

where K_1, K_2, and K_3 are positive constants independent of ε, and $S = K_2/K_3$ is the best Sobolev constant given in (8).

Next, we will investigate the effect of the coefficient $f(x)$ to find some Palais-Smale sequences which are used to prove Theorem 2.

Lemma 13. *If (H1)–(H3) hold, then for any $i \in \{1, 2, \ldots, k\}$ and any $\lambda > 0$, there exists a $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ one has*

$$\sup_{t\ge 0} J_\lambda\left(t v_\varepsilon^{a^i}\right) < \frac{1}{N} S^{N/p} \quad \textit{uniformly in } i. \tag{50}$$

In particular, $0 < \alpha_\lambda \le \alpha_\lambda^i < (1/N) S^{N/p}$ for all $\lambda > 0$.

Proof. By Lemma 6, there exists a $t_\varepsilon^i > 0$ such that $t_\varepsilon^i v_\varepsilon^{a^i} \in \mathcal{M}_\lambda$. Furthermore,

$$\begin{aligned} Q_i\left(t_\varepsilon^i v_\varepsilon^{a^i}\right) &= \frac{\int_\Omega \phi_i(x)\left|v_\varepsilon^{a^i}\right|^{p^*} dx}{\int_\Omega \left|v_\varepsilon^{a^i}\right|^{p^*} dx} \\ &= \frac{\int_{\Omega_\varepsilon} \phi_i\left(a^i + \varepsilon y\right)\left|\eta_i\left(a^i + \varepsilon y\right) V(y)\right|^{p^*} dy}{\int_{\Omega_\varepsilon} \left|\eta_i\left(a^i + \varepsilon y\right) V(y)\right|^{p^*} dy} \\ &\longrightarrow \phi_i\left(a^i\right) = 0 \quad \text{as } \varepsilon \longrightarrow 0, \end{aligned} \tag{51}$$

where $\Omega_\varepsilon = \{x : \varepsilon x + a^i \in \Omega\}$ and $V(y) = 1/(1 + |y|^{p/(p-1)})^{(N-p)/p}$. Hence, there exists an $\varepsilon_1 > 0$ small enough such that for any $\varepsilon \in (0, \varepsilon_1)$, we have

$$Q_i\left(t_\varepsilon^i v_\varepsilon^{a^i}\right) < \frac{r_0}{3}, \tag{52}$$

which implies $t_\varepsilon^i v_\varepsilon^{a^i} \in \mathcal{N}_\lambda^i$ for any $\varepsilon \in (0, \varepsilon_1)$, and then

$$0 < \alpha_\lambda \le \alpha_\lambda^i \le J_\lambda\left(t_\varepsilon^i v_\varepsilon^{a^i}\right) \le \sup_{t\ge 0} J_\lambda\left(t t_\varepsilon^i v_\varepsilon^{a^i}\right) = \sup_{t\ge 0} J_\lambda\left(t v_\varepsilon^{a^i}\right). \tag{53}$$

Set

$$\begin{aligned} h(t) = J_\lambda\left(t v_\varepsilon^{a^i}\right) &= \frac{t^p}{p}\left\|v_\varepsilon^{a^i}\right\|_p^p dx + \frac{t^q}{q}\left\|v_\varepsilon^{a^i}\right\|_q^q \\ &\quad - \frac{t^{p^*}}{p^*}\int_\Omega f(x)\left|v_\varepsilon^{a^i}\right|^{p^*} dx - \frac{t^r}{r}\int_\Omega \lambda g(x)\left|v_\varepsilon^{a^i}\right|^r dx. \end{aligned} \tag{54}$$

Since $h(0) = 0$, $\lim_{t\to+\infty} h(t) = -\infty$, then there exists a t_ε such that $\sup_{t\ge 0} J_\lambda(t v_\varepsilon^{a^i}) = J_\lambda(t_\varepsilon v_\varepsilon^{a^i})$ hold, and then t_ε satisfies

$$\begin{aligned} 0 = h'(t_\varepsilon) &= t_\varepsilon^{p-1}\left\|v_\varepsilon^{a^i}\right\|_p^p + t_\varepsilon^{q-1}\left\|v_\varepsilon^{a^i}\right\|_q^q \\ &\quad - t_\varepsilon^{p^*-1}\int_\Omega f(x)\left|v_\varepsilon^{a^i}\right|^{p^*} dx - t_\varepsilon^{r-1}\int_\Omega \lambda g(x)\left|v_\varepsilon^{a^i}\right|^r dx; \end{aligned} \tag{55}$$

then we have

$$\left\|v_\varepsilon^{a^i}\right\|_p^p + t_\varepsilon^{q-p}\left\|v_\varepsilon^{a^i}\right\|_q^q > t_\varepsilon^{p^*-p}\int_\Omega f(x)\left|v_\varepsilon^{a^i}\right|^{p^*} dx. \tag{56}$$

From (47) and (48), fixing any $\varepsilon_2 > 0$ small enough, there exists $T_1 > 0$ such that

$$t_\varepsilon \le T_1 \quad \text{for any } \varepsilon \in (0, \varepsilon_2). \tag{57}$$

Also, from (55), we obtain

$$\left\|v_\varepsilon^{a^i}\right\|_p^p < t_\varepsilon^{p^*-p}\int_\Omega f(x)\left|v_\varepsilon^{a^i}\right|^{p^*} dx + t_\varepsilon^{r-p}|g|_\infty \int_\Omega \lambda\left|v_\varepsilon^{a^i}\right|^r dx. \tag{58}$$

From (47)–(49) and (58), there exist $\varepsilon_3 > 0$ and $T_2 > 0$ such that

$$t_\varepsilon \ge T_2 \quad \text{for any } \varepsilon \in (0, \varepsilon_3). \tag{59}$$

Let $\varepsilon_4 = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\} > 0$; then

$$0 < T_2 \le t_\varepsilon \le T_1 \quad \forall \varepsilon \in (0, \varepsilon_4), \tag{60}$$

where T_1 and T_2 are independent of ε. From [13, Lemma 4.2] and conditions $(H2)$-$(H3)$, we also have

$$\begin{aligned} &\sup_{t\ge 0}\left(\frac{t^p}{p}\left\|v_\varepsilon^{a^i}\right\|_p^p - \frac{t^{p^*}}{p^*}\int_\Omega f(x)\left|v_\varepsilon^{a^i}\right|^{p^*} dx\right) \\ &\quad = \frac{1}{N} S^{N/p} + O\left(\varepsilon^{(N-p)/p}\right). \end{aligned} \tag{61}$$

By (43), (46)–(49), (60), and (61), for $\varepsilon \in (0, \varepsilon_4)$, we obtain

$$\begin{aligned} h(t_\varepsilon) &\le \sup_{t\ge 0}\left(\frac{t^p}{p}\left\|v_\varepsilon^{a^i}\right\|_p^p - \frac{t^{p^*}}{p^*}\int_\Omega f(x)\left|v_\varepsilon^{a^i}\right|^{p^*}dx\right) \\ &\quad + \frac{t_\varepsilon^q}{q}\left\|v_\varepsilon^{a^i}\right\|_q^q - \frac{t^r}{r}\int_\Omega \lambda g(x)\left|v_\varepsilon^{a^i}\right|^r dx \\ &\le \frac{1}{N}S^{N/p} + O\left(\varepsilon^{(N-p)/p}\right) + \frac{T_1^q}{q}\left\|v_\varepsilon^{a^i}\right\|_q^q \\ &\quad - \frac{T_2^r}{r}\lambda\gamma_2\int_\Omega\left|v_\varepsilon^{a^i}\right|^r dx \\ &\le \frac{1}{N}S^{N/p} + C_1\varepsilon^{(N-p)/p} \\ &\quad + C_2\varepsilon^{q(N-p)/p^2} - C_3\varepsilon^{(p/(p-1))(N-r((N-p)/p))}, \end{aligned} \tag{62}$$

where C_1, C_2, C_3 are positive constants independent of ε. Since $1 < q < N(p-1)/(N-1) < p \le \max\{p, p^* - q/(p-1)\} < r < p^*$, we obtain that

$$\frac{N-p}{p} > \frac{q(N-p)}{p^2} > \frac{p}{p-1}\left(N - r\frac{N-p}{p}\right); \tag{63}$$

then there exists an $\varepsilon_0 \in (0, \varepsilon_3)$ such that $h(t_\varepsilon) = \sup_{t\ge0} J_\lambda(t v_\varepsilon^{a^i}) < (1/N)S^{N/p}$ uniformly in i for all $\varepsilon \in (0, \varepsilon_0)$. Moreover, from (53), we have $0 < \alpha_\lambda \le \alpha_\lambda^i < (1/N)S^{N/p}$ for all $1 \le i \le k$ and $\lambda > 0$. This completes the proof. □

Proof of Theorem 1. From Lemmas 5, 10, 11, and 13, we get for all $\lambda > 0$ that there exists a u_0 such that $J'_\lambda(u_0) = 0$ and $J_\lambda(u_0) = \alpha_\lambda$. Set $u_+ = \max\{u, 0\}$. Replace the terms $\int_\Omega f(x)|u|^{p^*}dx$ and $\int_\Omega g(x)|u|^r dx$ of the functional J_λ by $\int_\Omega f(x)u_+^{p^*}dx$ and $\int_\Omega g(x)u_+^r dx$, respectively. It then follows that u_0 is a nonnegative solution of (E_λ). Applying the maximum principle, (E_λ) admits at least one positive solution u_0 in $W_0^{1,p}(\Omega)$. □

By studying the argument as in [21, Theorem III 3.1] and [22], we obtain the following lemma.

Lemma 14. *Let $\{u_n\} \subset W_0^{1,p}(\Omega)$ be a nonnegative function sequence with $|u_n|_{p^*} = 1$ and $\|u_n\|_p^p \to S$. Then there exists a sequence $(y_n, \sigma_n) \in \Omega \times \mathbf{R}^+$ such that*

$$v_n(x) := \sigma_n^{(N-p)/p} u_n(\sigma_n x + y_n) \tag{64}$$

contains a convergent subsequence denoted again by $\{v_n\}$ such that

$$v_n \longrightarrow v \quad \text{in } D^{1,p}\left(\mathbf{R}^N\right), \tag{65}$$

where $v(x) > 0$ in $\mathbf{R}^N$. Moreover, we have $\sigma_n \to 0$, $(1/\sigma_n)\operatorname{dist}(y_n, \partial\Omega) \to \infty$, and $y_n \to y \in \overline{\Omega}$ as $n \to \infty$.

Lemma 15. *Suppose that (H2) and (H3) hold. Then for any $i \in \{1, 2, \ldots, k\}$, there exists $\tilde{\lambda}_i > 0$ such that*

$$\tilde{\alpha}_\lambda^i > \frac{1}{N}S^{N/p} \quad \forall \lambda \in \left(0, \tilde{\lambda}_i\right). \tag{66}$$

Proof. Fix $i \in \{1, 2, \ldots, k\}$. Assume the contrary; that is, there then exists a sequence $\{\lambda_n\}$ with $\lambda_n \to 0^+$ as $n \to \infty$ such that $\tilde{\alpha}_{\lambda_n}^i \to c \le (1/N)S^{N/p}$. Consequently, there exists a sequence $\{u_n\} \subset \partial\mathcal{N}_{\lambda_n}^i$ such that, as $n \to \infty$,

$$\begin{aligned} \|u_n\|_p^p + \|u_n\|_q^q &= \int_\Omega f(x)|u_n|^{p^*}dx \\ &\quad + \lambda_n\int_\Omega g(x)|u_n|^r dx + o_n(1), \end{aligned} \tag{67}$$

and by Remark 8, we have that there exists a $d > 0$ such that

$$0 < d \le \liminf_{n\to\infty}\alpha_{\lambda_n} \le \lim_{n\to\infty} J_{\lambda_n}(u_n) = c \le \frac{1}{N}S^{N/p}, \tag{68}$$

where d is independent of λ_n for all n. It then follows easily that $\{u_n\}$ is uniformly bounded in $W_0^{1,p}(\Omega)$, and since $g(x)$ is continuous on $\overline{\Omega}$, we obtain

$$\lambda_n\int_\Omega g(x)|u_n|^r dx = o_n(1) \quad \text{as } n \longrightarrow \infty. \tag{69}$$

From (67)–(69), we may assume that there exist $a \ge 0$ and $b \ge 0$ such that

$$\|u_n\|_p^p = a + o_n(1), \qquad \|u_n\|_q^q = b + o_n(1). \tag{70}$$

So (70) and $|f|_\infty = 1$ imply that

$$|u_n|_{p^*}^{p^*} \ge \int_\Omega f(x)|u_n|^{p^*}dx = a + b + o_n(1). \tag{71}$$

By (70), (71), and the Sobolev inequality, we have

$$\begin{aligned} a &= \lim_{n\to\infty}\|u_n\|_p^p \ge S\lim_{n\to\infty}|u_n|_{p^*}^p \\ &\ge S\lim_{n\to\infty}\left(\int_\Omega f(x)|u_n|^{p^*}dx\right)^{p/p^*} \\ &\ge S(a+b)^{p/p^*} \ge Sa^{p/p^*}, \end{aligned} \tag{72}$$

which implies $a = 0$ or $a \ge S^{N/p}$. If $a = 0$, then by (72) we have that $b = 0$. From $a = b = 0$, we can deduce that $c = 0$ which is a contradiction. Hence,

$$a \ge S^{N/p}. \tag{73}$$

On the other hand, by $J_{\lambda_n}(u_n) = c + o_n(1)$, $c \le (1/N)S^{N/p}$, and (69)–(71), we get

$$\frac{a}{p} + \frac{b}{q} - \frac{a+b}{p^*} = \lim_{n\to\infty} J_{\lambda_n}(u_n) = c \le \frac{1}{N}S^{N/p}. \tag{74}$$

This implies that

$$\frac{a}{N} \le \left(\frac{a}{p} - \frac{a}{p^*}\right) + \left(\frac{b}{q} - \frac{b}{p^*}\right) = c \le \frac{1}{N}S^{N/p}. \tag{75}$$

Hence, together with (73), we get $a = S^{N/p}$ and $b = 0$, and then, from (71) and (72), we also have

$$\lim_{n\to\infty}|u_n|_{p^*}^{p^*} = \lim_{n\to\infty}\int_\Omega f(x)|u_n|^{p^*}dx = a = S^{N/p}. \tag{76}$$

Set $w_n = u_n/|u_n|_{p^*}$; then we have

$$|w_n|_{p^*} = 1, \qquad \lim_{n\to\infty} \|w_n\|_p^p = \lim_{n\to\infty} \frac{\|u_n\|_p^p}{|u_n|_{p^*}^p} = S. \tag{77}$$

Using Lemma 14, there exists a sequence $(y_n, \sigma_n) \in \Omega \times \mathbf{R}^+$ such that the sequence

$$v_n(x) := \sigma_n^{(N-p)/p} w_n(\sigma_n x + y_n) \tag{78}$$

converges strongly to $v \in D^{1,p}(\mathbf{R}^N)$, $\sigma_n \to 0$, $y_n \to y \in \overline{\Omega}$, and $(1/\sigma_n)\operatorname{dist}(y_n, \partial\Omega) \to \infty$ as $n \to \infty$.

Let $\Omega_n = \{x : \sigma_n x + y_n \in \Omega\}$. Since $\sigma_n \to 0$, $y_n \to y \in \overline{\Omega}$, and $(1/\sigma_n)\operatorname{dist}(y_n, \partial\Omega) \to \infty$ as $n \to \infty$, then $\Omega_n \to \mathbf{R}^N$ as $n \to \infty$. Observe that $Q_i(w_n) = Q_i(u_n) = r_0/3$. By the Lebesgue dominated convergence theorem, we have

$$\begin{aligned} \frac{r_0}{3} &= \lim_{n\to\infty} Q_i(w_n) = \lim_{n\to\infty} \frac{\int_\Omega \phi_i(x)|w_n|^{p^*}dx}{\int_\Omega |w_n|^{p^*}dx} \\ &= \lim_{n\to\infty} \frac{\int_\Omega \phi_i(x)|v_n((x-y_n)/\sigma_n)|^{p^*}dx}{\int_\Omega |v_n((x-y_n)/\sigma_n)|^{p^*}dx} \\ &= \lim_{n\to\infty} \frac{\int_{\Omega_n} \phi_i(\sigma_n x + y_n)|v_n(x)|^{p^*}dx}{\int_{\Omega_n} |v_n(x)|^{p^*}dx} = \phi_i(y), \end{aligned} \tag{79}$$

which implies that $y \neq a^i$ by the definition of $\phi_i(x)$. On the other hand, by the Lebesgue dominated convergence theorem again and (76), we get

$$\begin{aligned} 1 &= \lim_{n\to\infty} \int_\Omega f(x)|w_n(x)|^{p^*}dx \\ &= \lim_{n\to\infty} \int_{\Omega_n} f(\sigma_n x + y_n)|v_n(x)|^{p^*}dx = f(y), \end{aligned} \tag{80}$$

which is impossible, because $f(x)$ is not a constant function by condition $(H3)$. □

According to Lemma 13, we have

$$0 < \alpha_\lambda \le \alpha_\lambda^i < \frac{1}{N} S^{N/p} \quad \forall \lambda > 0. \tag{81}$$

According to Lemma 15, for each $i \in \{1, 2, \ldots, k\}$, there exists $\tilde{\lambda}_i > 0$ such that

$$\tilde{\alpha}_\lambda^i > \frac{1}{N} S^{N/p} \quad \forall \lambda \in (0, \tilde{\lambda}_i). \tag{82}$$

Let $\lambda_0 = \min_{1\le i\le k} \tilde{\lambda}_i > 0$. Then for each $i \in \{1, 2, \ldots, k\}$, by (81) and (82), we obtain that

$$\alpha_\lambda^i < \tilde{\alpha}_\lambda^i \quad \forall \lambda \in (0, \lambda_0). \tag{83}$$

Hence

$$\alpha_\lambda^i = \inf_{u \in \mathcal{N}_\lambda^i \cup \partial\mathcal{N}_\lambda^i} J_\lambda(u) \quad \forall \lambda \in (0, \lambda_0). \tag{84}$$

Applying Ekeland's variational principle and using the standard computation, we have the following lemma.

Lemma 16. *If $\lambda \in (0, \lambda_0)$, then for each $i \in \{1, 2, \ldots, k\}$, there exists a $(PS)_{\alpha_\lambda^i}$-sequence $\{u_n^i\} \subset \mathcal{N}_\lambda^i$ in $W_0^{1,p}(\Omega)$ for J_λ.*

Proof. See Cao and Zhou [17] or Tarantello [18]. □

Proof of Theorem 2. By Lemma 16, for all $\lambda \in (0, \lambda_0)$, there exists a $(\mathrm{PS})_{\alpha_\lambda^i}$-sequence $\{u_n^i\} \subset \mathcal{N}_\lambda^i$ in $W_0^{1,p}(\Omega)$ for J_λ where $1 \le i \le k$. From (81), we have

$$\alpha_\lambda^i \in \left(0, \frac{1}{N} S^{N/p}\right). \tag{85}$$

Note that J_λ satisfies the $(\mathrm{PS})_c$-condition for $c \in (0, (1/N)S^{N/p})$. Hence, we obtain that J_λ at least k critical points in $\mathcal{M}_\lambda$ for all $\lambda \in (0, \lambda_0)$. Set $u_+ = \max\{u, 0\}$. Replace the terms $\int_\Omega f(x)|u|^{p^*}dx$ and $\int_\Omega g(x)|u|^r dx$ of the functional J_λ by $\int_\Omega f(x)u_+^{p^*}dx$ and $\int_\Omega g(x)u_+^r dx$, respectively. It then follows that (E_λ) has k nonnegative solutions. Applying the maximum principle, (E_λ) admits at least k positive solutions. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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