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Research Article

Exact Solutions of Generalized Modified Boussinesq, Kuramoto-Sivashinsky, and Camassa-Holm Equations via Double Reduction Theory

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We find exact solutions of the Generalized Modified Boussinesq (GMB) equation, the Kuromoto-Sivashinsky (KS) equation the and, Camassa-Holm (CH) equation by utilizing the double reduction theory related to conserved vectors. The fourth order GMB equation involves the arbitrary function and mixed derivative terms in highest derivative. The partial Noether's approach yields seven conserved vectors for GMB equation and one conserved for vector KS equation. Due to presence of mixed derivative term the conserved vectors for GMB equation derived by the Noether like theorem do not satisfy the divergence relationship. The extra terms that constitute the trivial part of conserved vectors are adjusted and the resulting conserved vectors satisfy the divergence property. The double reduction theory yields two independent solutions and one reduction for GMB equation and one solution for KS equation. For CH equation two independent solutions are obtained elsewhere by double reduction theory with the help of conserved Vectors.

1. Introduction

Nonlinear differential equations have many significant implications for the mathematical models and have been of great interest in the last few decades. In the literature various techniques are used to construct exact and numerical solutions of differential equations [1–6]. Different approximations and numerical methods can be implemented to reduce nonlinear partial differential equations (PDEs) to ordinary differential equations (ODEs) but the problem occurs in the convergence of the solutions. Symmetry analysis and conservation laws play a vital role in the analysis of differential equations and have many important applications in numerical methods, linearization, and integrability. For variational type differential equations standard Lagrangian always exists and conservation laws can be computed from the well-known Noether's formula [7]. The standard Lagrangian does not exist for nonvariational differential equations. Kara and Mahomed [8] introduced the notion of partial Lagrangians for such differential equations. A Noether like theorem was invoked for the derivation of conservation laws associated with a partial Lagrangian. This technique is referred as to Noether's like approach or partial Lagrangian approach. There are some approaches for the construction of conservation laws which do not use the knowledge of standard or pariah Lagrangians. The different methods to compute the conservation laws and the comparison of approaches were studied in [9] (also see references therein).

The relationship between Noether symmetries and conservation laws is well known and is useful to reduce the numbers of variables and order of differential equations. In [10], a conserved vector is associated with the Lie-Bäcklund symmetries. Sjöberg and Mahomed [11, 12] have generalized association of a conserved vector to nonlocal symmetries. The association of Lie-Backlund symmetries or nonlocal symmetries with a conserved vector led to the development of the double reduction theory for the nonvariational type PDEs or system of PDEs with two independent variables [13, 14]. Bokhari et al. [15] generalized the theory of double reduction to find the invariant solutions for a nonlinear system of PDEs with several independent variables. This theory is helpful in finding invariant solution of PDEs having a nontrivial conserved vector and at least one symmetry associated with it. Recently, Narain and Kara [16] redefine the variational and nonvariational approaches for a class of PDE involving mixed derivative terms. Due to the presence of mixed derivative term a conserved vector computed by Noether's theorem does not satisfy the divergence relationship. A number of extra terms contributing to the trivial part of conserved vector arise and need to be adjusted to satisfy the divergence relationship.

The objective of this paper is to find the exact solutions of GMB, KS, and CH equations using the double reduction theory. The conservation laws of GMB and KS equations are constructed via partial lagrangian approach. For GMB equation the derived conserved vector failed to satisfy the divergence property. The extra terms which constitute the trivial part of the conserved vectors are adjusted to satisfy the divergence relationship. After construction of conservation laws the theory of double reduction is applied to compute the solutions of GMB and KS equations. The conserved vectors of Camassa-Holm equation are derived in [17]. Two exact solutions are computed for the Camassa-Holm equation by utilization of double reduction theory.

This paper is arranged in the following manner. In Section 2 , basic definitions, fundamental operators, and theorem for double reduction theory are invoked. In Section 3, the conservation laws and exact solutions for GMB equation are presented. The conservation laws and solution for KS equation are derived in Section 4. Section 5 deals with the exact solutions of CH equation using double reduction theory. Concluding remarks are summarized in Section 6.

2. Fundamental Operators

Assume an nth-order system of m partial differential equations of p independent variables $x = (x^1, x^2, x^3, \ldots, x^p)$ and q dependent variables $u = (u^1, u^2, \ldots, u^q)$:

$$E^\alpha = \left(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(n)}\right), \quad \alpha = 1, 2, 3, \ldots, m, \tag{1}$$

where $u_{(1)}, u_{(2)}, \ldots, u_{(n)}$ symbolize the set of all first, second, ..., nth-order partial derivatives; that is, $u_i^\alpha = D_i(u^\alpha)$, $u_{ij}^\alpha = D_j D_i(u^\alpha), \ldots$, in which D_i, the total differential operator corresponding to x^i, is given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \quad i = 1, 2, \ldots, p. \tag{2}$$

The summation convention is used whenever appropriate.

Definition 1. The nth-order system (1) can be expressed as

$$E^\alpha = E_0^\alpha + E_1^\alpha = 0, \quad \alpha = 1, 2, \ldots, m. \tag{3}$$

Suppose there exists a function $L = L(x, u, u_{(1)}, \ldots, u_{(n)})$ such that (1) can be expressed as

$$\frac{\delta L}{\delta u^\alpha} = F_\beta^\alpha \, E_1^\beta, \quad \alpha = 1, 2, \ldots, m. \tag{4}$$

In (4) F_β^α is the invertible matrix and $\delta/\delta u^\alpha$ is the Euler operator defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 \cdots i_s}}. \tag{5}$$

If $E_1^\beta \neq 0$ in (4), then L is said to be a partial Lagrangian of (1); otherwise, it is known as a standard Lagrangian.

Definition 2. A Lie-Bäcklund or generalized operator is

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \zeta_{i_1 i_2}^\alpha \frac{\partial}{\partial u_{i_1 i_2}^\alpha} + \cdots, \tag{6}$$

where $\xi^i, \eta^\alpha \in \mathscr{A}$ (space of differential function) and the additional coefficients can be determined from the formula

$$\begin{aligned} \zeta_i^\alpha &= D_i\left(W^\alpha\right) + \xi^j u_{ij}^\alpha, \\ \zeta_{i_1 \cdots i_s}^\alpha &= D_{i_1} \cdots D_{i_s}\left(W^\alpha\right) + \xi^j u_{j i_1 \cdots i_s}^\alpha, \quad s > 1, \end{aligned} \tag{7}$$

where W^α is the Lie characteristic function described by

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha. \tag{8}$$

Definition 3. A vector $T = (T^1, T^2, \ldots, T^n)$, $T^i \in \mathscr{A}$, $i = 1, 2, \ldots, n$, is called a conserved vector if $D_i T^i = 0$ holds for all solutions of (1).

Definition 4. A Lie-Bäcklund operator X defined in (6) is said to be a Noether symmetry generator corresponding to a Lagrangian $L = L(x, u, u_{(1)}, \ldots, u_{(n)})$ of (1) if it satisfies

$$X(L) + L D_i\left(\xi^i\right) = D_i B_i, \tag{9}$$

where D_i is the total derivative operator defined in (2) and B_i are the guage terms.

The generator X in (6) is said to be a partial Noether operator corresponding to a partial Lagrangian $L = L(x, u, u_{(1)}, \ldots, u_{(n)})$ of (1) if

$$X(L) + L D_i\left(\xi^i\right) = W^\alpha \frac{\delta L}{\delta u^\alpha} + D_i B_i. \tag{10}$$

Definition 5. If X in (6) is a Noether symmetry with respect to a Lagrangian or partial Noether operator with respect to a partial Lagrangian, then the conserved vector can be constructed from

$$T^i = B^i - \left[\xi^i + W^\alpha \frac{\delta}{\delta u_i^\alpha} + \sum_{s\geq 1} D_{i i_1} \cdots D_{i_s}\left(W^\alpha\right) \frac{\delta}{\delta u^\alpha_{i i_1 \cdots i_s}}\right] L, \tag{11}$$

in which

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s\geq 1}(-1)^s D_{j_1}\cdots D_{j_s}\frac{\partial}{\partial u^\alpha_{ij_1\dots j_s}}, \qquad i = 1,2,\dots,n,\ \alpha = 1,2,\dots,m. \quad (12)$$

Definition 6. Assume that X is a symmetry of the system (1) and T is a conserved vector of (1). If X and T satisfy

$$X\left(T^i\right) + T^i D_j\left(\xi^j\right) - T^j D_j\left(\xi^i\right) = 0, \quad i = 1,2,\dots,n. \quad (13)$$

Then X is associated with T.

The following theorem illustrates the construction of new conservation laws from symmetries and known conservation laws [18, 19].

Theorem 7. *Let X be any Lie-Bäcklund symmetry of* (1) *and T^i, $i = 1, 2, \dots, n$ comprise the components of a conserved vector of* (1)*; then*

$$T^{*i} = X\left(T^i\right) + T^i D_j\left(\xi^j\right) - T^j D_j\left(\xi^i\right) \quad (14)$$

forms a conserved vector of (1).

We now describe some results which are used in our work (see [13, 14]).

Consider a scalar partial differential equation $E = 0$ with $\alpha = 2$, $(x^1, x^2) = (t, x)$ such that it has conserved vector (T^t, T^x) and admits symmetry generator X associated with conserved vectors. In terms of canonical variables r, s with the symmetry $X = \partial/\partial s$ the conservation laws can be expressed as $D_r T^r + D_s T^s = 0$. The vectors T^r and T^s in terms of (t, x) are

$$T^r = \frac{T^t D_t(r) + T^x D_x(r)}{D_t(r)D_x(s) - D_x(r)D_t(s)}, \quad (15)$$

$$T^s = \frac{T^t D_t(s) + T^x D_x(s)}{D_t(r)D_x(s) - D_x(r)D_t(s)}. \quad (16)$$

Theorem 8. *A PDE of order n with two independent variables, which admits a symmetry X that is associated with a conserved vector T, is reduced to an ODE of order $n-1$; namely, $T^r = k$, where T^r is defined in* (15) *for solutions invariant under X.*

3. Conservation Laws and Exact Solutions of GMB Equation

The generalized modified Boussinesq (GMB) equation is

$$u_{tt} - \delta u_{ttxx} - \left(f(u)\right)_{xx} = 0, \quad (17)$$

where δ is a constant and $f(u)$ is an arbitrary function. GMB equation describes nonlinear model of wave propagation of elastic rods and also arises in nonlinear lattice waves, iron sound waves, and vibrations in a nonlinear string and is thus important to study.

Equation (17) admits a partial Lagrangian

$$L = -\frac{u_t^2}{2} - \frac{\delta}{2}u_{tx}^2 + \frac{1}{2}f_u u_x^2, \quad (18)$$

and the corresponding partial Euler-Lagrange equation is

$$\frac{\delta L}{\delta u} = \frac{1}{2}f_{uu}u_x^2. \quad (19)$$

Substituting these values in (10) and comparing the coefficients of like monomials of u we obtain

$$\begin{gathered}
\tau_u = 0, \qquad \tau_x = 0,\\
\xi_u = 0, \qquad \xi_t = 0,\\
\eta_{uu} = 0, \qquad \eta_{tu} = 0, \qquad \eta_{tx} = 0,\\
\eta_{xu} - \tau_{tx} = 0,\\
\tau f_{uu} - \tau_u f_u = 0, \qquad \tau_t - \xi_x - 2\eta_u = 0,\\
f_u\left(2\eta_u + \tau_t - \xi_x\right) = 0,\\
B^1_u = -\eta_t,\\
B^2_u = f_u\eta_x,\\
B^1_t + B^2_x = 0.
\end{gathered} \quad (20)$$

In order to solve the system (20), we discuss the following cases.

Case 1. $f_{uu} = 0$.

In this case we obtain $f(u) = au + b$, $\tau = d_2$, $\xi = d_4$, $\eta = d_1 t + d_3$, $B^1 = -ud_1$, and $B^2 = 0$.

Substituting the above values of τ, ξ, η, B^1, and B^2 in (11) we obtain

$$\begin{aligned}
T^t = {} & -ud_1 + d_2\left(\frac{u_t^2}{2} + \frac{\delta}{2}u_{tx}^2 - \frac{1}{2}au_x^2\right)\\
& + \left(d_1 t + d_3 - d_2u_t - d_4u_x\right)\left(u_t - \delta u_{txx}\right)\\
& - \left(d_2u_{tx} + d_4u_{xx}\right)\left(\delta u_{tx}\right),\\
T^x = {} & d_4\left(\frac{u_t^2}{2} + \frac{\delta}{2}u_{tx}^2 - \frac{1}{2}au_x^2\right)\\
& - \left(d_1 t + d_3 - d_2u_t - d_4u_x\right)\left(au_x + \delta u_{ttx}\right)\\
& + \left(d_1 - d_2u_{tt} - d_4u_{tx}\right)\left(\delta u_{tx}\right),
\end{aligned} \quad (21)$$

where $d_1, \ldots, d_4$ are arbitrary constants. The choice of constants one by one equal to one and the rest to zero yields the following conserved vectors:

$$\begin{gathered}
T_1^t = -u + tu_t - \delta t u_{txx}, \\
T_1^x = -atu_x - \delta t u_{ttx} + \delta u_{tx}, \\
T_2^t = -\frac{u_t^2}{2} - \frac{\delta}{2}u_{tx}^2 - \frac{1}{2}au_x^2 + \delta u_t u_{txx}, \\
T_2^x = au_t u_x + \delta u_t u_{ttx} - \delta u_{tt} u_{tx}, \\
T_3^t = u_t - \delta u_{txx}, \qquad T_3^x = -au_x - \delta u_{ttx}, \\
T_4^t = -u_t u_x + \delta u_x u_{txx} - \delta u_{tx} u_{xx}, \\
T_4^x = \frac{u_t^2}{2} - \frac{\delta}{2}u_{tx}^2 + \frac{1}{2}au_x^2 + \delta u_x u_{ttx}.
\end{gathered} \tag{22}$$

The divergence of (22) becomes

$$\begin{gathered}
D_t T_1^t + D_x T_1^x = -t\delta u_{ttxx}, \\
D_t T_2^t + D_x T_2^x = \delta u_t u_{ttxx} - \delta u_{ttx} u_{tx}, \\
D_t T_3^t + D_x T_3^x = -\delta u_{ttxx}, \\
D_t T_4^t + D_x T_4^x = \delta u_x u_{ttxx} - \delta u_{tx} u_{txx}.
\end{gathered} \tag{23}$$

The conserved vectors in (22) fail to satisfy the divergence properties. Narain and Kara [16] prove that T^i can be adjusted to $\widetilde{T}^i$ such that $D_i \widetilde{T}^i = 0$.

Following the same line we find that the modified conserved vectors $\widetilde{T}_j^i$ $(i = 1, 2,\ j = 1, 2, 3, 4)$ are

$$\begin{gathered}
\widetilde{T}_1^t = -u + tu_t - t\delta u_{txx}, \\
\widetilde{T}_1^x = -atu_x + \delta u_{tx}, \\
\widetilde{T}_2^t = -\frac{u_t^2}{2} - \frac{\delta}{2}u_{tx}^2 - \frac{1}{2}au_x^2 + \delta u_t u_{txx}, \\
\widetilde{T}_2^x = au_t u_x - \delta u_{tt} u_{tx}, \\
\widetilde{T}_3^t = u_t, \qquad \widetilde{T}_3^x = -au_x - \delta u_{ttx}, \\
\widetilde{T}_4^t = -u_t u_x + \delta u_x u_{txx}, \\
\widetilde{T}_4^x = \frac{u_t^2}{2} - \frac{\delta}{2}u_{tx}^2 + \frac{1}{2}au_x^2.
\end{gathered} \tag{24}$$

Case 2. $f_{uu} \neq 0$.

In this case after some simple but lengthy manipulations result in $\tau = 0,\ \xi = 0,\ \eta = a_1 x + a_2 + a_3 t,\ B^1 = -ua_3$ and $B^2 = (u)_1$. Substitutig τ, ξ, η, B^1, and B^2 in (11), we arrive at

$$\begin{gathered}
T^t = -ua_3 + (a_1 x + a_2 + a_3 t)(u_t - \delta u_{txx}) + \delta a_1 u_{tx}, \\
T^x = a_1 f(u) - (a_1 x + a_2 + a_3 t)(f_u u_x + \delta u_{ttx}) + \delta a_3 u_{tx}.
\end{gathered} \tag{25}$$

The choice of constants gives rise to

$$\begin{gathered}
T_5^t = xu_t - \delta x u_{txx} + \delta u_{tx}, \\
T_5^x = f(u) - xf_u u_x - \delta x u_{ttx}, \\
T_6^t = u_t - \delta u_{txx}, \qquad T_6^x = -f_u u_x - \delta u_{ttx}, \\
T_7^t = -u + tu_t - \delta t u_{txx}, \\
T_7^x = -tf_u u_x - \delta t u_{ttx} + \delta u_{tx}.
\end{gathered} \tag{26}$$

The conserved vectors in (26) do not satisfy the relation $D_i T^i = 0$; that is,

$$\begin{gathered}
D_t T_5^t + D_x T_5^x = -\delta x u_{ttxx}, \\
D_t T_6^t + D_x T_6^x = -\delta u_{ttxx}, \\
D_t T_7^t + D_x T_7^x = -\delta t u_{ttxx}.
\end{gathered} \tag{27}$$

The new conserved vectors $\widetilde{T}^i$ after adjustment result in

$$\begin{gathered}
\widetilde{T}_5^t = xu_t + \delta u_{tx}, \\
\widetilde{T}_5^x = f(u) - xf_u u_x - \delta x u_{ttx}, \\
\widetilde{T}_6^t = u_t, \qquad \widetilde{T}_6^x = -f_u u_x - \delta u_{ttx}, \\
\widetilde{T}_7^t = -u + tu_t - \delta t u_{txx}, \\
\widetilde{T}_7^x = -tf_u u_x + \delta u_{tx}.
\end{gathered} \tag{28}$$

Now, we apply the double reduction theory to associate the Lie point symmetries with the conservation laws. Equation (17) admits the following Lie point symmetries when $f_{uu} = 0$; that is, $f(u) = au + b$. Consider

$$\begin{gathered}
X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial t}, \\
X_3 = u\frac{\partial}{\partial u}, \qquad X_4 = f(t, x)\frac{\partial}{\partial u},
\end{gathered} \tag{29}$$

where $f(t, x)$ satisfies (17).

The symmetries X_1 and X_2 in (29) are associated with the conservation laws of GMB equation given below when $f_{uu} = 0$:

$$\begin{gathered}
T_1 = \left(-\frac{u_t^2}{2} - \frac{\delta}{2}u_{tx}^2 - \frac{1}{2}au_x^2 + \delta u_t u_{txx}, au_t u_x - \delta u_{tt} u_{tx}\right), \\
T_2 = (u_t, -au_x - \delta u_{ttx}).
\end{gathered} \tag{30}$$

The combination of these symmetries $X = X_1 + \alpha X_2$ yields the generator, X in canonical form $X = \partial/\partial q$ if

$$\frac{dx}{1} = \frac{dt}{\alpha} = \frac{du}{0} = \frac{dr}{0} = \frac{ds}{1} \tag{31}$$

or

$$s = x, \qquad r = \alpha x - t, \qquad u = u(r). \tag{32}$$

Using relation (15), we determine T_1^r and T_2^r corresponding to T_1 and T_2 given in (30):

$$T_1^r = -\frac{u_r^2}{2} - \frac{\delta\alpha^2}{2}u_{rr}^2 + \frac{a}{2}\alpha^2 u_r^2 + \delta\alpha^2 u_r u_{rrr}, \tag{33}$$

$$T_2^r = -u_r + \delta\alpha^2 u_{rrr} + a\alpha^2 u_r. \tag{34}$$

Since $T^1 = (T_1^r, T_1^s)$ is associated with X, so $T_1^r = k$ in (33) implies

$$-\frac{u_r^2}{2} - \frac{\delta\alpha^2}{2}u_{rr}^2 + \frac{a}{2}\alpha^2 u_r^2 + \delta\alpha^2 u_r u_{rrr} = k. \tag{35}$$

The solution of (35) is

$$u(r) = c_1 + mr + c_2 \sin\left(\frac{\sqrt{\alpha^2 a - 1}r}{\alpha\sqrt{\delta}}\right) + c_3 \cos\left(\frac{\sqrt{\alpha^2 a - 1}r}{\alpha\sqrt{\delta}}\right), \tag{36}$$

where

$$m = \Big((\alpha^2 a - 1) \times \delta\big(c_2^2 + c_3^2 + 2\delta k\alpha^2 - 2\alpha^2 a c_2^2 + \alpha^4 a^2 c_3^2 - 2\alpha^2 a c_3^2 + \alpha^4 a^2 c_2^2\big)\Big)^{1/2} \times \left((\alpha^2 a - 1)\delta\alpha\right)^{-1} \tag{37}$$

and c_1, c_2, and c_3 are arbitrary constants. Hence

$$u(t,x) = c_1 + m(\alpha x - t) + c_2 \sin\left(\frac{\sqrt{\alpha^2 a - 1}(\alpha x - t)}{\alpha\sqrt{\delta}}\right) + c_3 \cos\left(\frac{\sqrt{\alpha^2 a - 1}(\alpha x - t)}{\alpha\sqrt{\delta}}\right) \tag{38}$$

is a solution of (17) invariant under X.

Similarly from (34), one can easily find that

$$u(t,x) = \left(\left(c_1\alpha\sqrt{\delta}\sin\left(\frac{\sqrt{\alpha^2 a - 1}(\alpha x - t)}{\alpha\sqrt{\delta}}\right) - c_2\alpha\sqrt{\delta}\cos\left(\frac{\sqrt{\alpha^2 a - 1}(\alpha x - t)}{\alpha\sqrt{\delta}}\right)\right) \times \left(\sqrt{\alpha^2 a - 1}\right)^{-1}\right) + \frac{k(\alpha x - t)}{\alpha^2 a - 1} + c_3 \tag{39}$$

is also a solution of (17).

Equation (17) admits the trivial symmetry generators $X_1 = \partial/\partial x$ and $X_2 = \partial/\partial t$ in Case 2 when $f(u)$ is an arbitrary function. In this case the symmetry generators $X_1 = \partial/\partial x$ and $X_2 = \partial/\partial t$ are associated with the conserved vectors

$$T_3 = (u_t, -f_u u_x - \delta u_{ttx}) \tag{40}$$

of GMB equation. The similarity variables are defined in (32) and with the help of conserved vectors (40), (17) reduces to

$$\delta\alpha^2 u_{rr} - u_r + \alpha^2 f(u(r)) = kr - c_1\delta\alpha^2. \tag{41}$$

Equation (41) cannot be reduced further due to an arbitrary function $f(u)$.

4. Conservation Laws and Exact Solution of the Kuramoto-Sivashinsky Equation

The Kuramoto-Sivashinsky (KS) equation plays a dominant role in stability of flame fronts, reaction diffusion, and other physical phenomena. The KS equation is

$$u_t + auu_x + bu_{xx} + ku_{xxxx} = 0, \tag{42}$$

where a, b, and k are constants.

Equation (42) does not admit a standard Lagrangian but the partial Lagrangian is

$$L = \frac{ku_{xx}^2}{2} - \frac{b}{2}u_x^2, \tag{43}$$

and the corresponding partial Euler-Lagrange equation is

$$\frac{\delta L}{\delta u} = -u_t - auu_x. \tag{44}$$

Substituting these values in (10) and after some lengthy manipulation results in $\tau = 0$, $\xi = 0$, $\eta = c$, $B^1 = cu$, and $B^2 = acu$. Setting $c = 1$, we obtain $\tau = 0$, $\xi = 0$, $\eta = 1$, $B^1 = u$, and $B^2 = au$.

Equation (11) with the substitution of τ, ξ, η, B^1, and B^2 gives rise to

$$T^t = u, \qquad T^x = a\frac{u^2}{2} + bu_x + ku_{xxx}, \tag{45}$$

which satisfies

$$D_t T^t + D_x T^x = 0. \tag{46}$$

We reduce (42) using the double reduction theory, that is, association of symmetries with the conserved vectors.

Equation (42) possesses three Lie point symmetry generators:

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial t}, \qquad X_3 = \frac{\partial}{\partial u} + at\frac{\partial}{\partial x}. \tag{47}$$

Using (13) one can easily verify that X_1, X_2 in (47) are associated with the conserved vector

$$T = \left(u, a\frac{u^2}{2} + bu_x + ku_{xxx}\right) \tag{48}$$

of KS equation. We set $X = \alpha X_1 + X_2$; then, the canonical coordinates of X are $s = t$, $r = \alpha t - x$, and $u = u(r)$. Using (15) and with the use of the above canonical coordinates, we find that

$$T^r = \alpha u - a\frac{u^2}{2} + bu_r + ku_{rrr}. \tag{49}$$

Replacing $T^r = m$ (arbitrary constant), (49) becomes

$$\alpha u - a\frac{u^2}{2} + bu_r + ku_{rrr} = m. \tag{50}$$

Equation (50) admits the symmetry generator $X = \partial/\partial r$ and the similarity variables are $v = u$, $w = u_r$. After using the similarity variables, (50) reduces to

$$k = \alpha v - a\frac{v^2}{2} + bw + kw^2w'' + kww'^2, \quad w' = \frac{dw}{dv}. \tag{51}$$

The solution of (51) gives rise to

$$\begin{aligned} w(v) = \text{Root Of}\Bigg[& -\ln\left(2\alpha v - av^2 - 2c\right) \\ & +4\Bigg[\int \frac{1}{\left(-1 + 2bv + 8k\alpha^2 v^3\right)b} \\ & \times \Bigg[k\alpha^2 v\Bigg[-2bv \\ & + \text{Root Of}\Bigg[\int \frac{b^3}{k\alpha^2v^3 + b^3v - b^3}dv \\ & + \int\Bigg[-\frac{2b}{-1 + 2bv + 8k\alpha^2v^3}\Bigg]dv \\ & +c_1\Bigg]\Bigg]\Bigg]dv\Bigg] + c_2\Bigg] \\ & \times\left(-2\alpha v + av^2 + 2c\right). \end{aligned} \tag{52}$$

Equation (52) in terms of $w = u_r$ and $v = u$ yields

$$\begin{aligned} u_r = \text{Root Of}\Bigg[& -\ln\left(2\alpha u - au^2 - 2c\right) \\ & +4\Bigg[\int \frac{1}{\left(-1 + 2bu + 8k\alpha^2 u^3\right)b} \\ & \times \Bigg[k\alpha^2 u\Bigg[-2bu \\ & + \text{Root Of}\Bigg[\int \frac{b^3}{k\alpha^2u^3 + b^3u - b^3}du \\ & + \int\Bigg[-\frac{2b}{-1 + 2bu + 8k\alpha^2u^3}\Bigg]du \\ & +c_1\Bigg]\Bigg]\Bigg]du\Bigg] + c_2\Bigg] \\ & \times\left(-2\alpha u + au^2 + 2c\right), \end{aligned} \tag{53}$$

which in turn results in

$$\begin{aligned} r - \Bigg[\int 1 \times \Bigg[\text{Root Of}\Bigg[& -\ln\left(2\alpha u - au^2 - 2c\right) \\ & + 4\Bigg[\int \frac{1}{\left(-1 + 2bu + 8k\alpha^2 u^3\right)b} \\ & \times \Bigg[k\alpha^2 u\Bigg[-2bu \\ & + \text{Root Of}\Bigg[\int \frac{b^3}{k\alpha^2u^3 + b^3u - b^3}du \\ & + \int\Bigg[-\frac{2b}{-1 + 2bu + 8k\alpha^2u^3}\Bigg]du \\ & +c_1\Bigg]\Bigg]\Bigg]du\Bigg] + c_2\Bigg] \\ & \times\left(-2\alpha u + au^2 + 2c\right)\Bigg]^{-1}du\Bigg] + c_3, r = \alpha t - x, \end{aligned} \tag{54}$$

which is a solution of (42).

5. Exact Solutions of the Camassa-Holm Equation

The Camassa Holm (CH) equation is described by Camassa and Holm as a bi-Hamiltonian model for waves in shallow water. It is prominent for turbulent flows and waves in a hyperelastic rod. The CH equation is

$$u_t + 2\omega u_x + 3uu_x - \alpha^2\left(u_{xxt} + 2u_xu_{xx} + uu_{xxx}\right) = 0, \tag{55}$$

where α and ω are constants.

Conservation laws of CH equation are derived in [17]:

$$\begin{aligned} T_1^t &= u_t - \alpha^2 u_{xx}, \\ T_1^x &= 3\frac{u^2}{2} + 2\omega u - \alpha^2 uu_{xx} - \frac{\alpha^2}{2}u_x^2, \\ T_2^t &= \frac{u^2}{2} - \alpha^2 uu_{xx} - \frac{\alpha^2}{2}u_x^2, \\ T_2^x &= \omega u^2 + u^3 - \alpha^2u^2u_{xx} + \alpha^2u_tu_x. \end{aligned} \tag{56}$$

Equation (55) admits the following symmetries:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \\ X_3 &= t\frac{\partial}{\partial t} - \omega t\frac{\partial}{\partial x} - (u + \omega)\frac{\partial}{\partial u}. \end{aligned} \tag{57}$$

From (13) it can be easily shown that the conserved flow (T^1, T^2) for CH equation are associated with only X_1, X_2. We define the combination of these symmetries $X = X_1 + \alpha X_2$. The generator X has the canonical form $X = \partial/\partial q$ if

$$\frac{dt}{1} = \frac{dx}{\alpha} = \frac{du}{0} = \frac{dr}{0} = \frac{ds}{1} = \frac{dw}{0}, \tag{58}$$

or

$$s = t, \qquad r = \alpha t - x, \qquad u = u(r). \tag{59}$$

Equation (15) with the use of similarity variables defined above reduces (56) to

$$T_1^r = (\alpha - 2\omega) u + \alpha^2 (u - \alpha) u_{rr} - 3\frac{u^2}{2} + \alpha^2 \frac{u_r^2}{2}, \tag{60}$$

$$T_2^r = \alpha \frac{u^2}{2} - \alpha^3 u u_{rr} + \alpha^2 u^2 u_{rr} + \alpha^3 \frac{u_r^2}{2} - \omega u^2 - u^3. \tag{61}$$

Setting $T_1^r = k$ in (60), we obtain

$$(\alpha - 2\omega) u + \alpha^2 (u - \alpha) u_{rr} - 3\frac{u^2}{2} + \alpha^2 \frac{u_r^2}{2} = k. \tag{62}$$

The solution of (62) yields

$$\pm \int \frac{(u - \alpha)\alpha}{(u - \alpha)(2u^2\omega - u^2\alpha + u^3 + 2ku + c_1\alpha^2)} du = r + c_2, \qquad r = \alpha t - x, \tag{63}$$

which is a solution of (55).

Similarly from (61), applying the same procedure, we have

$$\pm \int \frac{(u - \alpha)\alpha}{(u - \alpha)(2\omega u^2 - \alpha u^2 + u^3 - 2k + c_3\alpha^2 u)} du = \alpha t - x + c_4 \tag{64}$$

which constitutes the solution of (55).

6. Conclusions

Exact solutions of the GMB equation, the KS equation, and the CH equation were constructed by utilizing the conservation laws. Firstly GMB equation was considered and conservation laws were computed by partial Noether's approach. Two cases arise; namely, Case 1: $f_{uu} = 0$ and Case 2: $f_{uu} \neq 0$. In Case 1, when $f(u)$ was a linear function four conserved vectors were obtained, whereas in Case 2 for arbitrary $f(u)$ three conserved vectors were reported. The derived conserved vectors failed to satisfy divergence condition. The extra terms arising in conserved vectors were absorbed and the new forms of conserved vectors satisfying the divergence property were found. When $f(u)$ is linear only two conserved vectors (30) satisfy the symmetry conservation laws relationship. The double reduction theory was applied to these two conserved vectors and two independent solutions were constructed. The symmetry was associated with only one conserved vector (40) when $f(u)$ is arbitrary. For this case GMB equation was reduced to a second order ODE. The partial Noether approach for KS equation yielded one conserved vector which satisfies the symmetry conservation laws relation. The conserved vector reduced the KS equation to a third order ODE (50) which further reduced to a second order ODE (51) which in turn results in the exact solution (54) of KS equation. A similar procedure is carried out to obtain two exact solutions of CH equation. These solutions are new and not obtained in the literature. The derived solutions cannot be interpreted physically due to deficiency of experimental sources; however these are important for numerical simulations.

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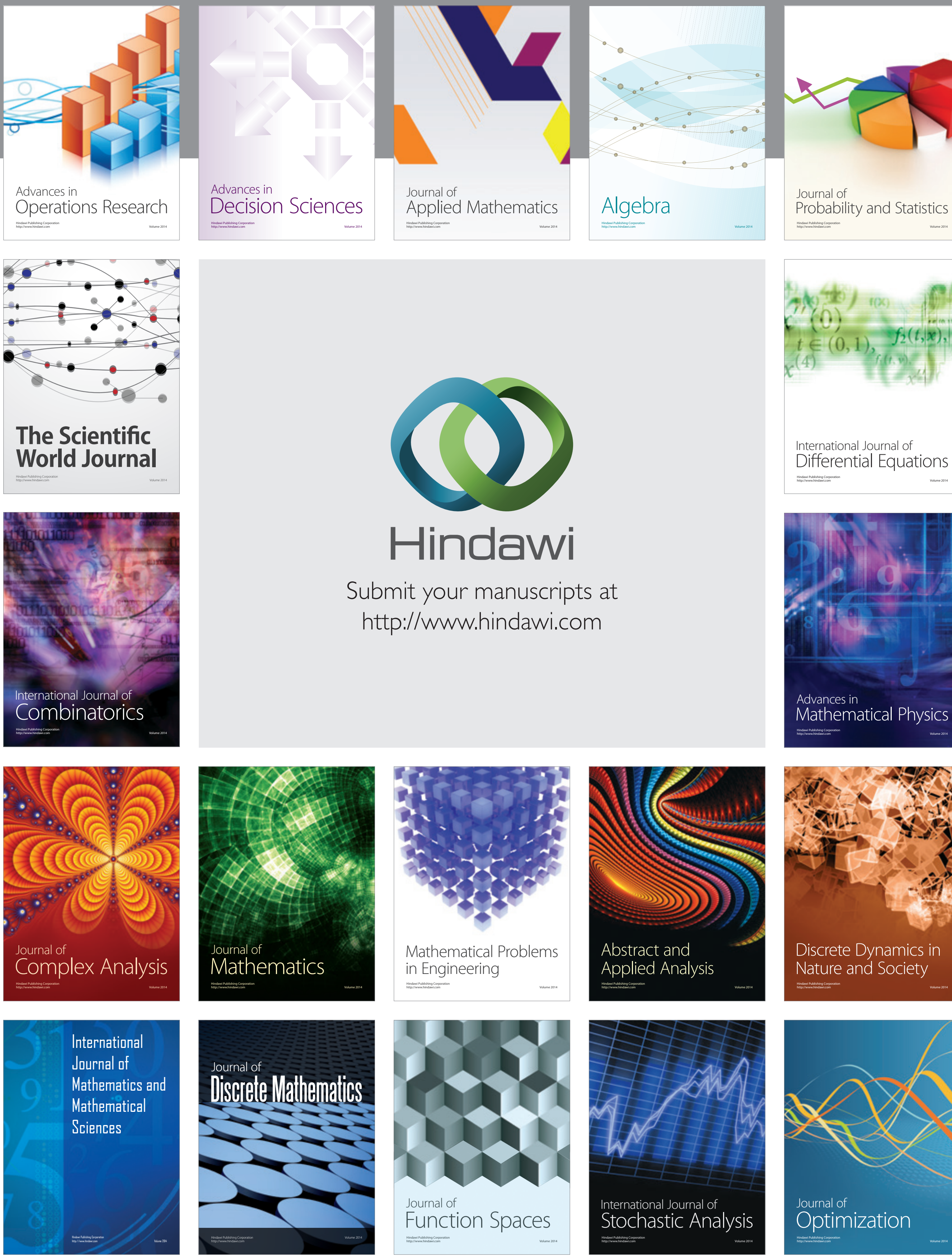
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