Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 823803, 9 pages
http://dx.doi.org/10.1155/2013/823803

Research Article

# $q$-Szász-Mirakyan-Kantorovich Operators of Functions of Two Variables in Polynomial Weighted Spaces

**Mediha Örkcü**

*Department of Mathematics, Faculty of Sciences, Gazi University, Teknikokullar, 06500 Ankara, Turkey*

Correspondence should be addressed to Mediha Örkcü; medihaakcay@gazi.edu.tr

Received 20 June 2013; Accepted 10 November 2013

Academic Editor: Sergei V. Pereverzyev



The present paper deals with approximation properties of $q$-Szász-Mirakyan-Kantorovich operators. We construct new bivariate generalization by $q_R$-integral and these operators' approximation properties in polynomial weighted spaces are investigated. Also, we obtain Voronovskaya-type theorem for the proposed operators in polynomial weighted spaces of functions of two variables.

## 1. Introduction

In the past two decades, $q$-calculus has gained popularity in the construction of linear approximation processes. Lupaş [1] and Phillips [2] defined generalizations of the Bernstein operators called $q$-Bernstein operators. Then, as Phillips has done for Bernstein operators, the authors introduced modifications of the other important operators based on the $q$-integers, for example, $q$-Meyer-König operators [3, 4], $q$-Bleimann, Butzer, and Hahn operators [5, 6], $q$-Szász-Mirakyan operators [7–9], $q$-Baskakov operators [10, 11].

On the other hand, Stancu [12] first introduced new linear positive operators in two- and several dimensional variables. Recently, Barbosu [13] introduced a Stancu-type generalization of two-dimensional Bernstein operators based on $q$-integers and called them bivariate $q$-Bernstein operators. Doğru and Gupta [14] constructed a bivariate generalization of the Meyer-König and Zeller operators based on the $q$-integers. Agratini [15] presented two-dimensional extension of some univariate positive approximation processes expressed by series.

All the above mentioned new operators motivate us for current work. In this paper, we firstly extend the $q$-Szász-Mirakyan-Kantorovich operators to the case of bivariate functions. Then these operators' approximation properties in polynomial weighted spaces are investigated. Also we obtain Voronovskaya-type theorem for the proposed operators in polynomial weighted spaces of functions of two variables.

Now we recall some definitions about $q$-integers. For any nonnegative integer $r$, the $q$-integer of the number $r$ is defined by

$$[r]_q = \begin{cases} 1+q+\cdots+q^{r-1} & \text{if } q \neq 1 \\ r & \text{if } q = 1, \end{cases} \qquad [0]_q = 1, \tag{1}$$

where $q$ is a positive real number. The $q$-factorial is defined as

$$[r]_q! = \begin{cases} [1]_q[2]_q\cdots[r]_q & \text{if } r = 1,2,\ldots \\ 1 & \text{if } r = 0, \end{cases} \qquad [0]_q! = 1. \tag{2}$$

Two $q$-analogues of the exponential function $e^x$ are given as

$$\begin{aligned} E_q(x) &= \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{x^n}{[n]_q!}, \qquad x \in \mathbb{R}, \\ \varepsilon_q(x) &= \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}, \qquad |x| < \frac{1}{1-q}. \end{aligned} \tag{3}$$

The following relation between $q$-exponential functions $E_q(x)$ and $\varepsilon_q(x)$ holds:

$$E_q(x)\,\varepsilon_q(-x) = 1, \qquad |x| < \frac{1}{1-q}. \tag{4}$$

The $q$-derivative of a function $f(x)$, denoted by $D_q f$, is defined by

$$\left(D_q f\right)(x) = \frac{f(qx) - f(x)}{(q-1)x}, \quad x \neq 0,$$
$$\left(D_q f\right)(0) = \lim_{x \to 0} \left(D_q f\right)(x). \tag{5}$$

Also, it is known that $D_q E(ax) = aE(qax)$.

The $q$-integral of the function $f$ over the interval $[0, b]$ is defined by

$$\int_0^b f(t)\, d_q t = b(1-q) \sum_{j=0}^{\infty} f\left(bq^j\right) q^j, \quad 0 < q < 1. \tag{6}$$

If $f$ is integrable over $[0, b]$, then

$$\lim_{q \to 1^-} \int_0^b f(t)\, d_q t = \int_0^b f(t)\, dt. \tag{7}$$

Generally accepted definition for $q$-integral over an interval $[a, b]$ is

$$\int_a^b f(t)\, d_q t = \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t. \tag{8}$$

In order to generalize and spread the existing inequalities, Marinković et al. considered new type of the $q$-integral. So, the problems which ensue from the general definition of $q$-integral were overcome. The Riemann-type $q$-integral [16] in the interval $[a, b]$ was defined as

$$\int_a^b f(t)\, d_q^R t = (1-q)(b-a) \sum_{j=0}^{\infty} f\left(a + (b-a) q^j\right) q^j, \quad 0 < q < 1. \tag{9}$$

This definition includes only point inside the interval of the integration.

Details of $q$-integers can be found in [17].

## 2. Construction of the Bivariate Operators

For $q_1, q_2 \in (0,1)$ and $(m,n) \in \mathbb{N} \times \mathbb{N}$, we now define new operators that we call the $q$-Szász-Mirakyan-Kantorovich operators of functions of two variables as follows:

$$\begin{aligned} S_{m,n}^{q_1,q_2}(f; x, y) &= \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{[m]_{q_1}^{k+1} x^k}{[k]_{q_1}!} \frac{[n]_{q_2}^{l+1} y^l}{[l]_{q_2}!} q_1^{k(k-1)-1} q_2^{l(l-1)-1} \\ &\times E_{q_1}\left(-[m]_{q_1} q_1^k x\right) E_{q_2}\left(-[n]_{q_2} q_2^l y\right) \\ &\times \int_{[l]_{q_2}/q_2^{k-1}[n]_{q_2}}^{[l+1]_{q_2}/q_2^{k-1}[n]_{q_2}} \int_{[k]_{q_1}/q_1^{k-1}[m]_{q_1}}^{[k+1]_{q_1}/q_1^{k-1}[m]_{q_1}} f(t,s)\, d_{q_1}^R t\, d_{q_2}^R s, \end{aligned} \tag{10}$$

where

$$\begin{aligned} \int_c^d \int_a^b f(t,s)\, d_{q_1}^R t\, d_{q_2}^R s &= \left(1-q_1\right)\left(1-q_2\right)(b-a)(c-d) \\ &\times \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} f\left(a + (b-a) q_1^i, c + (c-d) q_2^j\right) q_1^i q_2^j, \end{aligned} \tag{11}$$

and $f$ is a $q_R$-integrable function, so the series in (11) converges. It is clear that the operators given in (10) are linear and positive. For the operator $S_{m,n}^{q_1,q_2}$, if $f$ is a $q_R$-integrable function and $f(x,y) = f_1(x) f_2(y)$, $(x,y) \in \mathbb{R}_+^2$, then

$$S_{m,n}^{q_1,q_2}\left(f(t,s); x, y\right) = S_m^{q_1}\left(f_1(t), x\right) S_n^{q_2}\left(f_2(s), y\right). \tag{12}$$

Now, in order to obtain approximation properties of proposed operators, we give some auxiliary results. For a fixed $x \in \mathbb{R}_+$, by the $q$-Taylor theorem [18], we write

$$g(t) = \sum_{k=0}^{\infty} \frac{(t-x)_q^k}{[k]_q!} D_q^k g(x), \tag{13}$$

where

$$(t-x)_q^k = \prod_{s=0}^{k-1} \left(t - q^s x\right) = \sum_{s=0}^{k} \begin{bmatrix} k \\ s \end{bmatrix}_q q^{s(s-1)/2} t^{k-s} (-x)^s. \tag{14}$$

Choosing $t = 0$ and taking into account

$$(-x)_q^k = (-1)^k x^k q^{k(k-1)/2},$$
$$D_q^k E_q\left(-[n]_q x\right) = \left(-[n]_q\right)^k q^{k(k-1)/2} E_q\left(-[n]_q q^k x\right), \tag{15}$$

we get for $g(x) = E_q(-[n]_q x)$ that

$$\begin{aligned} 1 = g(0) &= \sum_{k=0}^{\infty} \frac{(-x)_q^k}{[k]_q!} D_q^k g(x) \\ &= \sum_{k=0}^{\infty} \frac{\left([n]_q x\right)^k}{[k]_q!} q^{k(k-1)} E_q\left(-[n]_q q^k x\right). \end{aligned} \tag{16}$$

Similarly, choosing $t = 0$ and taking into account

$$(-x)_q^k = (-1)^k x^k q^{k(k-1)/2},$$
$$D_q^k E_q\left(-[n]_q q x\right) = \left(-[n]_q q\right)^k q^{k(k-1)/2} E_q\left(-[n]_q q^{k+1} x\right), \tag{17}$$

we obtain for $g(x) = E_q(-[n]_q q x)$ that

$$\begin{aligned} 1 = g(0) &= \sum_{k=0}^{\infty} \frac{(-x)_q^k}{[k]_q!} D_q^k g(x) \\ &= \sum_{k=0}^{\infty} \frac{\left([n]_q x\right)^k}{[k]_q!} q^{k^2} E_q\left(-[n]_q q^{k+1} x\right). \end{aligned} \tag{18}$$

Also, using

$$(-x)_q^k = (-1)^k x^k q^{k(k-1)/2},$$
$$D_q^k E_q\left(-[n]_q q^2 x\right) = \left(-[n]_q\right)^k q^{2k} q^{k(k-1)/2} E_q\left(-[n]_q q^{k+2} x\right), \tag{19}$$

we have for $g(x) = E_q(-[n]_q q^2 x)$ that

$$1 = g(0) = x \sum_{k=0}^{\infty} \frac{(-x)_q^k}{[k]_q!} D_q^k g(x) = \sum_{k=0}^{\infty} \frac{\left([n]_q x\right)^k}{[k]_q!} q^{k^2} E_q\left(-[n]_q q^{k+2} x\right). \tag{20}$$

**Lemma 1.** *Let $q_1, q_2 \in (0,1)$ and $(m,n) \in \mathbb{N} \times \mathbb{N}$. One has*

$$S_{m,n}^{q_1,q_2}(1; x, y) = 1,$$
$$S_{m,n}^{q_1,q_2}(t; x) = x + \frac{q_1}{[2]_{q_1}[m]_{q_1}},$$
$$S_{m,n}^{q_1,q_2}(s; y) = y + \frac{q_2}{[2]_{q_2}[n]_{q_2}},$$
$$S_{m,n}^{q_1,q_2}\left(t^2; x\right) = x^2 + \frac{1}{[m]_{q_1}}\left(1 + \frac{2q_1}{[2]_{q_1}}\right)x + \frac{q_1^2}{[3]_{q_1}[m]_{q_1}^2},$$
$$S_{m,n}^{q_1,q_2}\left(s^2; y\right) = y^2 + \frac{1}{[n]_{q_2}}\left(1 + \frac{2q_2}{[2]_{q_2}}\right)y + \frac{q_2^2}{[3]_{q_2}[n]_{q_2}^2}. \tag{21}$$

*Proof.* Using $[k+1]_q = [k]_q + q^k$ and from $\int_{[l]_{q_2}/q_2^{k-1}[n]_{q_2}}^{[l+1]_{q_2}/q_2^{k-1}[n]_{q_2}} \int_{[k]_{q_1}/q_1^{k-1}[m]_{q_1}}^{[k+1]_{q_1}/q_1^{k-1}[m]_{q_1}} d_{q_1}^R t d_{q_2}^R s = q_1 q_2/[m]_{q_1}[n]_{q_2}$, we can write

$$S_{m,n}^{q_1,q_2}(1; x, y) = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \frac{[m]_{q_1}^k x^k}{[k]_{q_1}!} \frac{[n]_{q_2}^l y^l}{[l]_{q_2}!} q_1^{k(k-1)} q_2^{l(l-1)} \times E_{q_1}\left(-[m]_{q_1} q_1^k x\right) E_{q_2}\left(-[n]_{q_2} q_2^l y\right). \tag{22}$$

$S_{m,n}^{q_1,q_2}(1; x, y) = 1$ is obtained from the above identity (16).

Now, taking $f(t,s) = t$ and since $\int_{[l]_{q_2}/q_2^{k-1}[n]_{q_2}}^{[l+1]_{q_2}/q_2^{k-1}[n]_{q_2}} \int_{[k]_{q_1}/q_1^{k-1}[m]_{q_1}}^{[k+1]_{q_1}/q_1^{k-1}[m]_{q_1}} t d_{q_1}^R t d_{q_2}^R s = (q_1 q_2/[m]_{q_1}[n]_{q_2})(([k]_{q_1}/q_1^{k-1}[m]_{q_1}) + (q_1/[2]_{q_1}[m]_{q_1}))$, we get from the linearity of $S_{m,n}^{q_1,q_2}$ that

$$S_{m,n}^{q_1,q_2}(t; x) = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \frac{[m]_{q_1}^k x^k}{[k]_{q_1}!} \frac{[n]_{q_2}^l y^l}{[l]_{q_2}!} q_1^{k(k-1)} q_2^{l(l-1)} \times E_{q_1}\left(-[m]_{q_1} q_1^k x\right) E_{q_2}\left(-[n]_{q_2} q_2^l y\right) \frac{[k]_{q_1}}{q_1^{k-1}[m]_{q_1}}$$
$$+ \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \frac{[m]_{q_1}^k x^k}{[k]_{q_1}!} \frac{[n]_{q_2}^l y^l}{[l]_{q_2}!} q_1^{k(k-1)} q_2^{l(l-1)} \times E_{q_1}\left(-[m]_{q_1} q_1^k x\right) E_{q_2}\left(-[n]_{q_2} q_2^l y\right) \frac{q_1}{[2]_{q_1}[m]_{q_1}}. \tag{23}$$

From this, applying (16) and (18), we have

$$S_{m,n}^{q_1,q_2}(t; x) = \sum_{l=0}^{\infty}\sum_{k=1}^{\infty} \frac{[m]_{q_1}^{k-1} x^k}{[k-1]_{q_1}!} \frac{[n]_{q_2}^l y^l}{[l]_{q_2}!} \frac{q_1^{k(k-1)}}{q_1^{k-1}} q_2^{l(l-1)} \times E_{q_1}\left(-[m]_{q_1} q_1^k x\right) E_{q_2}\left(-[n]_{q_2} q_2^l y\right) + \frac{q_1}{[2]_{q_1}[m]_{q_1}}$$
$$= x \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \frac{[m]_{q_1}^k x^k}{[k]_{q_1}!} \frac{[n]_{q_2}^l y^l}{[l]_{q_2}!} q_1^{k^2} q_2^{l(l-1)} \times E_{q_1}\left(-[m]_{q_1} q_1^{k+1} x\right) E_{q_2}\left(-[n]_{q_2} q_2^l y\right) + \frac{q_1}{[2]_{q_1}[m]_{q_1}}$$
$$= x + \frac{q_1}{[2]_{q_1}[m]_{q_1}}. \tag{24}$$

Similarly, we write that

$$S_{m,n}^{q_1,q_2}(s; y) = y + \frac{q_2}{[2]_{q_2}[n]_{q_2}}. \tag{25}$$

Now, taking $f(t,s) = t^2$ and from

$$\int_{[l]_{q_2}/q_2^{k-1}[n]_{q_2}}^{[l+1]_{q_2}/q_2^{k-1}[n]_{q_2}} \int_{[k]_{q_1}/q_1^{k-1}[m]_{q_1}}^{[k+1]_{q_1}/q_1^{k-1}[m]_{q_1}} t^2 d_{q_1}^R t d_{q_2}^R s = \frac{q_1 q_2}{[m]_{q_1}[n]_{q_2}} \times \left(\frac{[k]_{q_1}^2}{q_1^{2k-2}[m]_{q_1}^2} + \frac{[k]_{q_1}}{q_1^{k-1}[m]_{q_1}} \frac{2q_1}{[2]_{q_1}[m]_{q_1}} + \frac{q_1^2}{[3]_{q_1}[m]_{q_1}^2}\right), \tag{26}$$

we have

$$S_{m,n}^{q_1,q_2}\left(t^2;x\right) = \sum_{l=0}^{\infty}\sum_{k=1}^{\infty}\frac{[m]_{q_1}^{k-1}x^k}{[k-1]_{q_1}!}\frac{[n]_{q_2}^{l}y^l}{[l]_{q_2}!}q_1^{k(k-1)}q_2^{l(l-1)} \times E_{q_1}\left(-[m]_{q_1}q_1^k x\right)E_{q_2}\left(-[n]_{q_2}q_2^l y\right)\frac{[k]_{q_1}}{q_1^{2k-2}[m]_{q_1}} + \frac{2q_1}{[2]_{q_1}[m]_{q_1}}x + \frac{q_1^2}{[3]_{q_1}[m]_{q_1}^2}. \tag{27}$$

From this, using $[k]_q = [k-1]_q + q^{k-1}$ we get

$$S_{m,n}^{q_1,q_2}\left(t^2;x\right) = \sum_{l=0}^{\infty}\sum_{k=2}^{\infty}\frac{[m]_{q_1}^{k-2}x^k}{[k-2]_{q_1}!}\frac{[n]_{q_2}^{l}y^l}{[l]_{q_2}!}q_1^{k(k-1)}q_2^{l(l-1)} \times E_{q_1}\left(-[m]_{q_1}q_1^k x\right)E_{q_2}\left(-[n]_{q_2}q_2^l y\right)\frac{1}{q_1^{2k-2}} + \sum_{l=0}^{\infty}\sum_{k=1}^{\infty}\frac{[m]_{q_1}^{k-1}x^k}{[k-1]_{q_1}!}\frac{[n]_{q_2}^{l}y^l}{[l]_{q_2}!}q_1^{k(k-1)}q_2^{l(l-1)} \times E_{q_1}\left(-[m]_{q_1}q_1^k x\right)E_{q_2}\left(-[n]_{q_2}q_2^l y\right)\frac{1}{[m]_{q_1}q_1^{k-1}} + \frac{2q_1}{[2]_{q_1}[m]_{q_1}}x + \frac{q_1^2}{[3]_{q_1}[m]_{q_1}^2}. \tag{28}$$

Replacing $k$ by $k+2$ and $k$ by $k+1$ in the abovementioned, we obtain

$$S_{m,n}^{q_1,q_2}\left(t^2;x\right) = x^2\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{[m]_{q_1}^{k}x^k}{[k]_{q_1}!}\frac{[n]_{q_2}^{l}y^l}{[l]_{q_2}!}q_1^{(k+2)(k+1)}q_2^{l(l-1)} \times E_{q_1}\left(-[m]_{q_1}q_1^{k+2} x\right)E_{q_2}\left(-[n]_{q_2}q_2^l y\right)\frac{1}{q_1^{2k+2}} + x\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{[m]_{q_1}^{k}x^k}{[k]_{q_1}!}\frac{[n]_{q_2}^{l}y^l}{[l]_{q_2}!}q_1^{k^2}q_2^{l(l-1)} \times E_{q_1}\left(-[m]_{q_1}q_1^{k+1} x\right)E_{q_2}\left(-[n]_{q_2}q_2^l y\right)\frac{1}{[m]_{q_1}} + \frac{2q_1}{[2]_{q_1}[m]_{q_1}}x + \frac{q_1^2}{[3]_{q_1}[m]_{q_1}^2}. \tag{29}$$

Equation (20) implies

$$S_{m,n}^{q_1,q_2}\left(t^2;x\right) = x^2 + \frac{1}{[m]_{q_1}}\left(1 + \frac{2q_1}{[2]_{q_1}}\right)x + \frac{q_1^2}{[3]_{q_1}[m]_{q_1}^2}. \tag{30}$$

Similarly, we write that

$$S_{m,n}^{q_1,q_2}\left(s^2;y\right) = y^2 + \frac{1}{[n]_{q_2}}\left(1 + \frac{2q_2}{[2]_{q_2}}\right)y + \frac{q_2^2}{[3]_{q_2}[n]_{q_2}^2}. \tag{31}$$

So, the proof is completed. □

Similarly, given by the proof of Lemma 1, we calculate $S_{m,n}^{q_1,q_2}(t^3;x)$ and $S_{m,n}^{q_1,q_2}(t^4;x)$, shortly. Since

$$\int_{[l]_{q_2}/q_2^{k-1}[n]_{q_2}}^{[l+1]_{q_2}/q_2^{k-1}[n]_{q_2}}\int_{[k]_{q_1}/q_1^{k-1}[m]_{q_1}}^{[k+1]_{q_1}/q_1^{k-1}[m]_{q_1}} t^3 d_{q_1}^R t d_{q_2}^R s = \frac{q_1q_2}{[m]_{q_1}[n]_{q_2}}\left(\frac{[k]_{q_1}^3}{q_1^{3k-3}[m]_{q_1}^3} + \frac{3[k]_{q_1}^2}{q_1^{2k-2}[m]_{q_1}^2}\frac{q_1}{[2]_{q_1}[m]_{q_1}} + \frac{3[k]_{q_1}}{q_1^{k-1}[m]_{q_1}}\frac{q_1^2}{[3]_{q_1}[m]_{q_1}^2} + \frac{q_1^3}{[4]_{q_1}[m]_{q_1}^3}\right), \tag{32}$$

we write

$$S_{m,n}^{q_1,q_2}\left(t^3;x\right) = \sum_{l=0}^{\infty}\sum_{k=1}^{\infty}\frac{[m]_{q_1}^{k-1}x^k}{[k-1]_{q_1}!}\frac{[n]_{q_2}^{l}y^l}{[l]_{q_2}!}q_1^{k(k-1)}q_2^{l(l-1)} \times E_{q_1}\left(-[m]_{q_1}q_1^k x\right)E_{q_2}\left(-[n]_{q_2}q_2^l y\right) \times \frac{[k-1]_{q_1}^2 + 2q^{k-1}[k-1] + q^{2k-2}}{q_1^{3k-3}[m]_{q_1}^2} + \frac{3q_1}{[2]_{q_1}[m]_{q_1}}\left(x^2 + \frac{1}{[m]_{q_1}}x\right) + \frac{3q_1^2}{[3]_{q_1}[m]_{q_1}^2}x + \frac{q_1^3}{[4]_{q_1}[m]_{q_1}^3}. \tag{33}$$

Using $[k]_q = [k-1]_q + q^{k-1}$, we have

$$\sum_{l=0}^{\infty}\sum_{k=1}^{\infty}\frac{[m]_{q_1}^{k-1}x^k}{[k-1]_{q_1}!}\frac{[n]_{q_2}^{l}y^l}{[l]_{q_2}!}q_1^{k(k-1)}q_2^{l(l-1)} \times E_{q_1}\left(-[m]_{q_1}q_1^k x\right)E_{q_2}\left(-[n]_{q_2}q_2^l y\right)$$

$$\times \frac{[k-1]_{q_1}^2 + 2q^{k-1}[k-1] + q^{2k-2}}{q_1^{3k-3}[m]_{q_1}^2}$$

$$= \sum_{l=0}^{\infty}\sum_{k=3}^{\infty} \frac{[m]_{q_1}^{k-3} x^k}{[k-3]_{q_1}!} \frac{[n]_{q_2}^l y^l}{[l]_{q_2}!} q_1^{k(k-1)} q_2^{l(l-1)} \times E_{q_1}\left(-[m]_{q_1} q_1^k x\right) E_{q_2}\left(-[n]_{q_2} q_2^l y\right) \frac{1}{q_1^{3k-3}}$$
$$+ \sum_{l=0}^{\infty}\sum_{k=2}^{\infty} \frac{[m]_{q_1}^{k-2} x^k}{[k-2]_{q_1}!} \frac{[n]_{q_2}^l y^l}{[l]_{q_2}!} q_1^{k(k-1)} q_2^{l(l-1)} \times E_{q_1}\left(-[m]_{q_1} q_1^k x\right) E_{q_2}\left(-[n]_{q_2} q_2^l y\right) \frac{q_1^{k-2}}{q_1^{3k-3}[m]_{q_1}}$$
$$+ \sum_{l=0}^{\infty}\sum_{k=2}^{\infty} \frac{[m]_{q_1}^{k-2} x^k}{[k-2]_{q_1}!} \frac{[n]_{q_2}^l y^l}{[l]_{q_2}!} q_1^{k(k-1)} q_2^{l(l-1)} \times E_{q_1}\left(-[m]_{q_1} q_1^k x\right) E_{q_2}\left(-[n]_{q_2} q_2^l y\right) \frac{2q^{k-1}}{q_1^{3k-3}[m]_{q_1}}$$
$$+ \sum_{l=0}^{\infty}\sum_{k=1}^{\infty} \frac{[m]_{q_1}^{k-1} x^k}{[k-1]_{q_1}!} \frac{[n]_{q_2}^l y^l}{[l]_{q_2}!} q_1^{k(k-1)} q_2^{l(l-1)} \times E_{q_1}\left(-[m]_{q_1} q_1^k x\right) E_{q_2}\left(-[n]_{q_2} q_2^l y\right) \frac{q^{2k-2}}{q_1^{3k-3}[m]_{q_1}^2}. \tag{34}$$

Then, we rewrite

$$= x^3 \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \frac{[m]_{q_1}^{k} x^k}{[k]_{q_1}!} \frac{[n]_{q_2}^l y^l}{[l]_{q_2}!} q_1^{(k+3)(k+2)} q_2^{l(l-1)} \times E_{q_1}\left(-[m]_{q_1} q_1^{k+3} x\right) E_{q_2}\left(-[n]_{q_2} q_2^l y\right) \frac{1}{q_1^{3k+6}}$$
$$+ x^2 \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \frac{[m]_{q_1}^{k} x^k}{[k]_{q_1}!} \frac{[n]_{q_2}^l y^l}{[l]_{q_2}!} q_1^{(k+2)(k+1)} q_2^{l(l-1)} \times E_{q_1}\left(-[m]_{q_1} q_1^{k+2} x\right) E_{q_2}\left(-[n]_{q_2} q_2^l y\right) \frac{q_1^k}{q_1^{3k+3}[m]_{q_1}}$$
$$+ x^2 \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \frac{[m]_{q_1}^{k} x^k}{[k]_{q_1}!} \frac{[n]_{q_2}^l y^l}{[l]_{q_2}!} q_1^{(k+2)(k+1)} q_2^{l(l-1)} \times E_{q_1}\left(-[m]_{q_1} q_1^{k+2} x\right) E_{q_2}\left(-[n]_{q_2} q_2^l y\right) \frac{2q^{k+1}}{q_1^{3k+3}[m]_{q_1}}$$
$$+ x \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \frac{[m]_{q_1}^{k} x^k}{[k]_{q_1}!} \frac{[n]_{q_2}^l y^l}{[l]_{q_2}!} q_1^{(k+1)(k)} q_2^{l(l-1)} \times E_{q_1}\left(-[m]_{q_1} q_1^{k+1} x\right) E_{q_2}\left(-[n]_{q_2} q_2^l y\right) \frac{q^{2k}}{q_1^{3k}[m]_{q_1}^2}. \tag{35}$$

Finally, we have

$$S_{m,n}^{q_1,q_2}\left(t^3; x\right) = x^3 + \frac{1}{[m]_{q_1}}\left(\frac{1}{q_1} + 2\right)x^2 + \frac{1}{[m]_{q_1}^2}x + \frac{3q_1}{[2]_{q_1}[m]_{q_1}}\left(x^2 + \frac{1}{[m]_{q_1}}x\right) + \frac{3q_1^2}{[3]_{q_1}[m]_{q_1}^2}x + \frac{q_1^3}{[4]_{q_1}[m]_{q_1}^3}. \tag{36}$$

Since

$$\int_{[l]_{q_2}/q_2^{k-1}[n]_{q_2}}^{[l+1]_{q_2}/q_2^{k-1}[n]_{q_2}} \int_{[k]_{q_1}/q_1^{k-1}[m]_{q_1}}^{[k+1]_{q_1}/q_1^{k-1}[m]_{q_1}} t^4 d_{q_1}^R t d_{q_2}^R s = \frac{q_1 q_2}{[m]_{q_1}[n]_{q_2}}\left(\frac{[k]_{q_1}^4}{q_1^{4k-4}[m]_{q_1}^4} + \frac{4[k]_{q_1}^3}{q_1^{3k-3}[m]_{q_1}^3}\frac{q_1}{[2]_{q_1}[m]_{q_1}} + \frac{6[k]_{q_1}^2}{q_1^{2k-2}[m]_{q_1}^2}\frac{q_1^2}{[3]_{q_1}[m]_{q_1}^2} + \frac{4[k]_{q_1}}{q_1^{k-1}[m]_{q_1}}\frac{q_1^3}{[4]_{q_1}[m]_{q_1}^3} + \frac{q_1^4}{[5]_{q_1}[m]_{q_1}^4}\right), \tag{37}$$

we obtain

$$S_{m,n}^{q_1,q_2}\left(t^4; x\right) = x^4 + \frac{1}{[m]_{q_1}}\left(\frac{1}{q_1^2} + \frac{2}{q_1} + 3\right)x^3 + \frac{1}{[m]_{q_1}^2}\left(\frac{1}{q_1^2} + \frac{3}{q_1} + 3\right)x^2 + \frac{1}{[m]_{q_1}^3}x + \frac{4q_1}{[2]_{q_1}[m]_{q_1}}\left(x^3 + \frac{1}{[m]_{q_1}}\left(\frac{1}{q_1} + 2\right)x^2 + \frac{1}{[m]_{q_1}^2}x\right) + \frac{6q_1^2}{[3]_{q_1}[m]_{q_1}^2}\left(x^2 + \frac{1}{[m]_{q_1}}x\right) + \frac{4q_1^3}{[4]_{q_1}[m]_{q_1}^3}x + \frac{q_1^4}{[5]_{q_1}[m]_{q_1}^4}. \tag{38}$$

## 3. Approximation Properties in Polynomial Weighted Spaces

For bivariate operators, the space is considered as follows:

$$C_{p,q}\left(\mathbb{R}_+^2\right) = \left\{ f \in C\left(\mathbb{R}_+^2\right) : \omega_{p,q} f \text{ is uniformly continuous and bounded on } \mathbb{R}_+^2 = [0,\infty) \times [0,\infty) \right\} \tag{39}$$

associated with the weighted function $\omega_{p,q}(x,y) = \omega_p(x)\omega_q(y)$, $(p,q) \in \mathbb{N}_0 \times \mathbb{N}_0$. The weight $\omega_p$ is defined as $\omega_0(t) = 1$, $\omega_p(x) = (1+x^p)^{-1}$. The norm of this space is denoted by $\|\cdot\|_{p,q}$ and is defined by

$$\|f\|_{p,q} = \sup_{(x,y)\in\mathbb{R}_+^2} \omega_{p,q}(x,y)\,|f(x,y)|. \tag{40}$$

Now, we give some useful results given by Agratini [15].

For each $z \in \mathbb{R}_+$, define the function $\varphi_z$ by $\varphi_z^r(t) = (t-z)^r$, $t \in \mathbb{R}_+$, $r \in \mathbb{N}$. For the one-dimensional operator $L_s$, $s \in \mathbb{N}$, and for each $r \in \mathbb{N}$, a polynomial $\Gamma_r$ exists such that

$$s^{r/2}\left(L_s\varphi_z^r\right)(t) \le \Gamma_r(t), \quad \deg(\Gamma_r) \le r. \tag{41}$$

**Theorem 2** (see [15]). *Consider $(p,q) \in \mathbb{N}_0 \times \mathbb{N}_0$. For any $(m,n) \in \mathbb{N}\times\mathbb{N}$, given by*

$$\left(L_{m,n}f\right)(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{m,i}(x)\, b_{n,j}(y)\, f\left(x_{m,i}, y_{n,j}\right), \quad (x,y)\in\mathbb{R}_+^2 \tag{42}$$

*the operator $L_{m,n}$ verifies*

$$\left\|L_{m,n}\left(\frac{1}{\omega_{p,q}};\cdot\right)\right\|_{p,q} \le c(p,q),$$
$$\left\|L_{m,n}(f;\cdot)\right\|_{p,q} \le c(p,q)\,\|f\|_{p,q}, \quad f \in C_{p,q}\left(\mathbb{R}_+^2\right). \tag{43}$$

**Theorem 3** (see [15]). *Let $(p,q) \in \mathbb{N}_0 \times \mathbb{N}_0$. For any $(m,n) \in \mathbb{N}\times\mathbb{N}$, the operator $L_{m,n}$ given by (42) satisfies*

$$\begin{aligned}\omega_{p,q}(x,y)\left|\left(L_{m,n}f\right)(x,y) - f(x,y)\right| \\ \le c(p,q)\,\omega_f\left(\frac{\phi(x)}{\sqrt{m}}, \frac{\phi(y)}{\sqrt{n}}\right),\end{aligned} \tag{44}$$

*$(x,y)\in\mathbb{R}_+^2$, where $\phi$ is given by*

$$\phi(t) = \sqrt{\Gamma_2(t)} + \sqrt{\Gamma_2(t) + \sum_{k=0}^{p}\binom{p}{k}\Gamma_{k+2}(t)}, \tag{45}$$

*with the polynomials $\Gamma_\nu(t)$, $\nu = \overline{2, p+2}$, being indicated at (41), and $c(p,q)$ is a suitable constant.*

**Theorem 4** (see [15]). *Let $(p,q) \in \mathbb{N}_0 \times \mathbb{N}_0$. Let the operator $L_{m,n}$, $(m,n)\in\mathbb{N}\times\mathbb{N}$, be defined by (42). For any $(x,y)\in\mathbb{R}_+^2$ the pointwise convergence takes place*

$$\lim_{m,n\to\infty}\left(L_{m,n}f\right)(x,y) = f(x,y), \quad f \in C_{p,q}\left(\mathbb{R}_+^2\right). \tag{46}$$

*If $K_1$, $K_2$ are compact intervals included in $\mathbb{R}_+$, then (46) holds uniformly on the domain $K_1 \times K_2$.*

In the latter paper, we use the weight function $\rho(x,y) = (1+x^2+y^2)^{-1}$ instead of $\omega_{p,q}(x,y)$ and instead of the space $C_{p,q}(\mathbb{R}_+^2)$, the space $C_2(\mathbb{R}_+^2)$ associated with the weighted function $\rho(x,y)$ is used. We denote the norm of this space by $\|\cdot\|_2$.

**Lemma 5.** *The operator $S_{m,n}^{q_1,q_2}$, $(m,n)\in\mathbb{N}\times\mathbb{N}$, $q_1,q_2\in(0,1)$, given by (10) verifies*

$$\left\|S_{m,n}^{q_1,q_2}\left(\frac{1}{\rho(t,s)};\cdot\right)\right\|_2 \le \frac{10}{3}, \tag{47}$$

$$\left\|S_{m,n}^{q_1,q_2}(f;\cdot)\right\|_2 \le \frac{10}{3}\|f\|_2, \quad f\in C_2\left(\mathbb{R}_+^2\right). \tag{48}$$

*Proof.* Since $q_1,q_2\in(0,1)$ and by Lemma 1, we have

$$\begin{aligned}\rho(x,y)\,S_{m,n}^{q_1,q_2}\left(\frac{1}{\rho(t,s)};\cdot\right) \\ \le \rho(x,y)\left(1+x^2+y^2+\frac{7}{3}(x+y)\right).\end{aligned} \tag{49}$$

So, inequality (47) is proved. Since the operator $S_{m,n}^{q_1,q_2}$ is linear and positive and by using (47)

$$\begin{aligned}\rho(x,y)\,S_{m,n}^{q_1,q_2}(f;\cdot) &\le \rho(x,y)\,S_{m,n}^{q_1,q_2}\left(\rho(t,s)|f|\frac{1}{\rho(t,s)};\cdot\right) \\ &\le \frac{10}{3}\|f\|_2,\end{aligned} \tag{50}$$

we obtain inequality (48). □

The Steklov function associated with $f\in C(\mathbb{R}_+^2)$ is given as follows:

$$f_{h,\delta}(x,y) = \frac{1}{h\delta}\int_0^h du\int_0^\delta f(x+u,y+v)\,dv, \quad (x,y)\in\mathbb{R}_+^2, \tag{51}$$

where $h,\delta>0$.

The modulus of smoothness function associated with any function $f\in C_2(\mathbb{R}_+^2)$ is given by

$$\omega_f(h,\delta) = \sup_{\substack{0\le u\le h\\0\le v\le\delta}}\left\|f(x+u,y+v)-f(x,y)\right\|_2, \quad (x,y)\in\mathbb{R}_+^2. \tag{52}$$

One can see that

$$\begin{aligned}\|f-f_{h,\delta}\|_2 &= \sup_{(x,y)\in\mathbb{R}_+^2}\rho(x,y)\left|\frac{1}{h\delta}\int_0^h du\int_0^\delta f(x+u,y+v) - f(x,y)\,dv\right| \\ &\le \sup_{(x,y)\in\mathbb{R}_+^2}\rho(x,y)\sup_{\substack{0\le u\le h\\0\le v\le\delta}} f(x+u,y+v)-f(x,y) \\ &= \sup_{\substack{0\le u\le h\\0\le v\le\delta}}\left\|f(x+u,y+v)-f(x,y)\right\|_2 \\ &= \omega_f(h,\delta).\end{aligned} \tag{53}$$

The following inequalities verify

$$\left\|\frac{\partial}{\partial x} f_{h,\delta}\right\|_2 \le \frac{2}{h}\omega_f(h,\delta), \qquad \left\|\frac{\partial}{\partial y} f_{h,\delta}\right\|_2 \le \frac{2}{\delta}\omega_f(h,\delta), \tag{54}$$

where $h, \delta > 0$. In order to justify these inequalities, one can see that

$$\begin{aligned}
&\left\|\frac{\partial}{\partial x} f_{h,\delta}\right\|_2 \\
&= \sup_{(x,y)\in\mathbb{R}_+^2} \rho(x,y) \frac{1}{h\delta} \left| \int_0^\delta f(x+h, y+v) - f(x, y+v)\, dv \right| \\
&= \sup_{(x,y)\in\mathbb{R}_+^2} \rho(x,y) \frac{1}{h\delta} \left| \int_0^\delta f(x+h, y+v) - f(x,y) + f(x,y) - f(x, y+v)\, dv \right| \\
&\le \frac{1}{h\delta} \sup_{(x,y)\in\mathbb{R}_+^2} \rho(x,y) \int_0^\delta \left| f(x+h, y+v) - f(x,y) \right| dv \\
&\quad + \frac{1}{h\delta} \sup_{(x,y)\in\mathbb{R}_+^2} \rho(x,y) \int_0^\delta \left| f(x, y+v) - f(x,y) \right| dv \\
&\le \frac{2}{h} \sup_{(x,y)\in\mathbb{R}_+^2} \rho(x,y) \sup_{\substack{0\le u\le h \\ 0\le v\le\delta}} f(x+u, y+v) - f(x,y) \\
&= \frac{2}{h}\omega_f(h,\delta).
\end{aligned} \tag{55}$$

**Theorem 6.** *For any $(m,n) \in \mathbb{N}\times\mathbb{N}$ and $f \in C_2(\mathbb{R}_+^2)$, the operator $S_{m,n}^{q_1,q_2}$ $q_1, q_2 \in (0,1)$ given by* (10) *satisfies*

$$\begin{aligned}
&\rho(x,y) \left| \left(S_{m,n}^{q_1,q_2} f\right)(x,y) - f(x,y) \right| \\
&\quad \le \frac{10}{3}\omega_f\left(\frac{\phi(x,q_1)}{[m]_{q_1}}, \frac{\phi(y,q_2)}{[n]_{q_2}}\right),
\end{aligned} \tag{56}$$

*where $(x,y) \in \mathbb{R}_+^2$, $\phi(x,q) = q/[2]_q + \Gamma(x,q)$.*

*Proof.* For any $(m,n) \in \mathbb{N}\times\mathbb{N}$, we can write

$$\begin{aligned}
&\rho(x,y) \left| \left(S_{m,n}^{q_1,q_2} f\right)(x,y) - f(x,y) \right| \\
&\quad \le \rho(x,y) \left| S_{m,n}^{q_1,q_2}\left(f - f_{h,\delta}; x, y\right) \right| \\
&\qquad + \rho(x,y) \left| \left(S_{m,n}^{q_1,q_2} f_{h,\delta}\right)(x,y) - f_{h,\delta}(x,y) \right| \\
&\qquad + \rho(x,y) \left| f_{h,\delta}(x,y) - f(x,y) \right|.
\end{aligned} \tag{57}$$

Inequalities (48) and (53) imply that

$$\begin{aligned}
&\rho(x,y) \left| S_{m,n}^{q_1,q_2}\left(f - f_{h,\delta}; x, y\right) \right| \\
&\quad \le \left\| S_{m,n}^{q_1,q_2}\left(f - f_{h,\delta}; x, y\right) \right\|_2 \\
&\quad \le \frac{10}{3} \left\| f - f_{h,\delta} \right\|_2 \le \frac{10}{3}\omega_f(h,\delta).
\end{aligned} \tag{58}$$

Let $C_2^1(\mathbb{R}_+^2)$ be the class of all functions in which partial derivatives belong to $C_2(\mathbb{R}_+^2)$. Since $f_{h,\delta} \in C_2^1\mathbb{R}_+^2$ and $S_{m,n}^{q_1,q_2}$ given by (10) is linear and monotone, we get

$$\begin{aligned}
&\rho(x,y) \left| \left(S_{m,n}^{q_1,q_2} f_{h,\delta}\right)(x,y) - f_{h,\delta}(x,y) \right| \\
&\quad = \rho(x,y) \left| S_{m,n}^{q_1,q_2}\left( \int_x^t \frac{\partial}{\partial u} f_{h,\delta}(u,s)\, du; x, y \right) \right| \\
&\qquad + \rho(x,y) \left| S_{m,n}^{q_1,q_2}\left( \int_y^s \frac{\partial}{\partial v} f_{h,\delta}(x,s)\, dv; x, y \right) \right| \\
&\quad \le \rho(x,y) S_{m,n}^{q_1,q_2}\left( \left| \int_x^t \frac{\partial}{\partial u} f_{h,\delta}(u,s)\, du \right|; x, y \right) \\
&\qquad + \rho(x,y) S_{m,n}^{q_1,q_2}\left( \left| \int_y^s \frac{\partial}{\partial v} f_{h,\delta}(x,s)\, dv \right|; x, y \right).
\end{aligned} \tag{59}$$

Then, by definition of norm $\|\cdot\|_2$ and the first mean value theorem for integration, we have

$$\begin{aligned}
\left| \int_x^t \frac{\partial}{\partial u} f_{h,\delta}(u,s)\, du \right| &\le \left| \int_x^t \rho(u,s) \left| \frac{\partial}{\partial u} f_{h,\delta}(u,s) \right| \frac{du}{\rho(u,s)} \right| \\
&\le \left\| \frac{\partial}{\partial x} f_{h,\delta} \right\|_2 \left| \int_x^t \frac{du}{\rho(u,s)} \right| \\
&= \left\| \frac{\partial}{\partial x} f_{h,\delta} \right\|_2 |t-x| \frac{1}{\rho(\xi,s)}, \quad \xi \in (x,t) \\
&\le \left\| \frac{\partial}{\partial x} f_{h,\delta} \right\|_2 |t-x| \left( \frac{1}{\rho(x,s)} + \frac{1}{\rho(t,s)} \right).
\end{aligned} \tag{60}$$

Following the same way, one finds

$$\begin{aligned}
&\left| \int_y^s \frac{\partial}{\partial v} f_{h,\delta}(x,s)\, dv \right| \\
&\quad \le \left\| \frac{\partial}{\partial y} f_{h,\delta} \right\|_2 |s-y| \left( \frac{1}{\rho(x,y)} + \frac{1}{\rho(x,s)} \right).
\end{aligned} \tag{61}$$

Using inequalities (59), (61), and (60), since $S_{m,n}^{q_1,q_2}$ is linear and monotone, we get

$$\begin{aligned}\rho(x,y)\left|\left(S_{m,n}^{q_1,q_2}f_{h,\delta}\right)(x,y)-f_{h,\delta}(x,y)\right| &\leq \rho(x,y)\left\|\frac{\partial}{\partial x}f_{h,\delta}\right\|_2 S_{m,n}^{q_1,q_2}\left(\frac{|t-x|}{\rho(x,s)};x,y\right)\\&+\rho(x,y)\left\|\frac{\partial}{\partial x}f_{h,\delta}\right\|_2 S_{m,n}^{q_1,q_2}\left(\frac{|t-x|}{\rho(t,s)};x,y\right)\\&+\rho(x,y)\left\|\frac{\partial}{\partial y}f_{h,\delta}\right\|_2 S_{m,n}^{q_1,q_2}\left(\frac{|s-y|}{\rho(x,y)};x,y\right)\\&+\rho(x,y)\left\|\frac{\partial}{\partial y}f_{h,\delta}\right\|_2 S_{m,n}^{q_1,q_2}\left(\frac{|s-y|}{\rho(x,s)};x,y\right).\end{aligned}\qquad(62)$$

From Lemmas 1 and 5 and (36), we can write the following:

$$\begin{aligned}\rho(x,y)\left|\left(S_{m,n}^{q_1,q_2}f_{h,\delta}\right)(x,y)-f_{h,\delta}(x,y)\right| &\leq \left\|\frac{\partial}{\partial x}f_{h,\delta}\right\|_2\frac{10}{3}\frac{q_1}{[2]_{q_1}[m]_{q_1}}\\&+\left\|\frac{\partial}{\partial x}f_{h,\delta}\right\|_2\frac{10}{3}\frac{\Gamma(x,q_1)}{[m]_{q_1}}\\&+\left\|\frac{\partial}{\partial y}f_{h,\delta}\right\|_2\frac{q_2}{[2]_{q_2}[n]_{q_2}}\\&+\left\|\frac{\partial}{\partial y}f_{h,\delta}\right\|_2\frac{10}{3}\frac{\Gamma(y,q_2)}{[n]_{q_2}},\end{aligned}\qquad(63)$$

where

$$\Gamma(x,q)=\left(\frac{1}{q}+1+\frac{q}{[2]_q}\right)x^2+\left(1+\frac{3q}{[2]_q}+\frac{2q^2}{[3]_q}\right)x+\frac{q^3}{[4]_q}\qquad(64)$$

is a polynomial of degree 2. Then, by inequalities (54), we have

$$\begin{aligned}\rho(x,y)\left|\left(S_{m,n}^{q_1,q_2}f_{h,\delta}\right)(x,y)-f_{h,\delta}(x,y)\right| &\leq\left[\frac{2}{h}\frac{10}{3}\left(\frac{q_1}{[2]_{q_1}[m]_{q_1}}+\frac{\Gamma(x,q_1)}{[m]_{q_1}}\right)\right.\\&\left.+\frac{2}{\delta}\left(\frac{q_2}{[2]_{q_2}[n]_{q_2}}+\frac{10}{3}\frac{\Gamma(y,q_2)}{[n]_{q_2}}\right)\right]\omega_f(h,\delta).\end{aligned}\qquad(65)$$

Finally, we write from (53)

$$\rho(x,y)\left|f_{h,\delta}(x,y)-f(x,y)\right|\leq\left\|f_{h,\delta}-f\right\|_2\leq\omega_f(h,\delta).\qquad(66)$$

If we go back to (57) and take $h=\phi(x,q_1)/[m]_{q_1}$, $\delta=\phi(y,q_2)/[n]_{q_2}$, then the proof is completed. □

We replace $q_1$ and $q_2$ in (10) by sequences $(q_{1,m})$, $(q_{2,n})$ so that

$$\lim_{m\to\infty}q_{1,m}=1,\qquad \lim_{n\to\infty}q_{2,n}=1,$$
$$\lim_{m\to\infty}\frac{1}{[n]_{q_{1,m}}}=0,\qquad \lim_{n\to\infty}\frac{1}{[n]_{q_{2,n}}}=0.\qquad(67)$$

So, $(h,\delta)\to(0^+,0^+)$ as $n,m\to\infty$. Knowing that modulus of smoothness function $\omega_f$ satisfies the property $\lim_{(h,\delta)\to(0^+,0^+)}\omega_f(h,\delta)=0$, from Theorem 6, we deduce the following result.

**Theorem 7.** *Let $(m,n)\in\mathbb{N}\times\mathbb{N}$ and let $(q_{1,m})$, $(q_{2,n})$ be sequences in the interval $(0,1)$ satisfying (67). Let the operator $S_{m,n}^{q_1,q_2}$ given by (10) and $f\in C_2(\mathbb{R}_+^2)$ $q_R$-integrable function. For any $(x,y)\in\mathbb{R}_+^2$ the pointwise convergence takes place*

$$\lim_{m,n\to\infty}\left(S_{m,n}^{q_{1,m},q_{2,n}}f\right)(x,y)=f(x,y).\qquad(68)$$

*If $K_1$, $K_2$ are compact intervals included in $\mathbb{R}_+$, then (68) holds uniformly on the domain $K_1\times K_2$.*

## 4. Voronovskaya-Type Theorem

We will prove the Voronovskaya-type theorem.

**Theorem 8.** *Let $(q_{1,n})$, $(q_{2,n})$ be sequences in the interval $(0,1)$ satisfying (67). Suppose that $f\in C_2^2(\mathbb{R}_+^2)$ is the class of all functions in which the second partial derivatives belong to $C_2(\mathbb{R}_+^2)$ and $q_R$-integrable function. Then, for every $(x,y)\in\mathbb{R}_+^2$, one has*

$$\begin{aligned}\lim_{n\to\infty}n\left\{S_{n,n}^{q_{1,n},q_{2,n}}(f;x,y)-f(x,y)\right\}&=\frac{x}{2}f_{xx}(x,y)+\frac{y}{2}f_{yy}(x,y)\\&+\frac{1}{2}f_x(x,y)+\frac{1}{2}f_y(x,y).\end{aligned}\qquad(69)$$

*Proof.* Let $f\in C_2^2(\mathbb{R}_+^2)$ and $q_R$-integrable function and let $(x_0,y_0)\in\mathbb{R}_+^2$ be fixed point. Then, by the Taylor formula, we can write

$$\begin{aligned}f(t,s)&=f(x_0,y_0)+f_x(x_0,y_0)(t-x_0)+f_y(x_0,y_0)(s-y_0)\\&+\frac{1}{2}\left\{f_{xx}(x_0,y_0)(t-x_0)^2+2f_{xy}(x_0,y_0)\right.\\&\left.\times(t-x_0)(s-y_0)+f_{yy}(x_0,y_0)(s-y_0)^2\right\}\\&+\varphi(t,s;x_0,y_0)\sqrt{(t-x_0)^4+(s-y_0)^4},\end{aligned}\qquad(70)$$

where $(t, s) \in \mathbb{R}_+^2$, $\varphi(t, s) = \varphi(t, s; x_0, y_0)$ belongs to $C_2(\mathbb{R}_+^2)$, and $\lim_{(t,s)\to(x_0,y_0)} \varphi(t, s) = 0$ for $n \in \mathbb{N}$. From the linearity of $S_{n,n}^{q_1,q_2}$, we have

$$\begin{aligned} S_{n,n}^{q_1,q_2}(f(t,s); x_0, y_0) &= f(x_0, y_0) + f_x(x_0, y_0) S_{n,n}^{q_1,q_2}(t - x_0; x_0) \\ &\quad + f_y(x_0, y_0) S_{n,n}^{q_1,q_2}(s - y_0; y_0) \\ &\quad + \frac{1}{2}\{f_{xx}(x_0, y_0) S_{n,n}^{q_1,q_2}((t - x_0)^2; x_0) \\ &\qquad + 2 f_{xy}(x_0, y_0) S_{n,n}^{q_1,q_2}(t - x_0; x_0) S_{n,n}^{q_1,q_2}(s - y_0; y_0) \\ &\qquad + f_{yy}(x_0, y_0) S_{n,n}^{q_1,q_2}((s - y_0)^2; y_0)\} \\ &\quad + S_{n,n}^{q_1,q_2}\left(\varphi(t,s)\sqrt{(t - x_0)^4 + (s - y_0)^4}; x_0, y_0\right). \end{aligned} \tag{71}$$

By Lemma 1 and since the sequences $q_{1,n}$, $q_{2,n}$ satisfy (67), we obtain

$$\begin{aligned} \lim_{n\to\infty} n S_{n,n}^{q_{1,n},q_{2,n}}(t - x_0; x_0) &= \frac{1}{2} = \lim_{n\to\infty} n S_{n,n}^{q_{1,n},q_{2,n}}(s - y_0; y_0), \\ \lim_{n\to\infty} n S_{n,n}^{q_{1,n},q_{2,n}}((t - x_0)^2; x_0) &= x_0, \\ \lim_{n\to\infty} n S_{n,n}^{q_{1,n},q_{2,n}}((s - y_0)^2; y_0) &= y_0. \end{aligned} \tag{72}$$

By the Hölder inequality, we have

$$\begin{aligned} &\left| S_{n,n}^{q_1,q_2}\left(\varphi(t,s)\sqrt{(t - x_0)^4 + (s - y_0)^4}; x_0, y_0\right)\right| \\ &\quad \le \{S_{n,n}^{q_1,q_2}(\varphi^2(t,s); x_0, y_0)\}^{1/2} \\ &\qquad \times \{S_{n,n}^{q_1,q_2}((t - x_0)^4; x_0) + S_{n,n}^{q_1,q_2}((s - y_0)^4; y_0)\}^{1/2}. \end{aligned} \tag{73}$$

By properties of $\varphi$ and Theorem 7, we get

$$\lim_{n\to\infty} S_{n,n}^{q_{1,n},q_{2,n}}(\varphi^2(t,s); x_0, y_0) = \varphi^2(x_0, y_0) = 0. \tag{74}$$

From the foregoing facts and using (12) and (38), we obtain

$$\lim_{n\to\infty} n S_{n,n}^{q_{1,n},q_{2,n}}\left(\varphi(t,s)\sqrt{(t - x_0)^4 + (s - y_0)^4}; x_0, y_0\right) = 0. \tag{75}$$

Then, using (72) and (75), we reproduce from (71)

$$\begin{aligned} \lim_{n\to\infty} n\{S_{n,n}^{q_{1,n},q_{2,n}}(f(t,s); x_0, y_0) - f(x_0, y_0)\} &= \frac{x_0}{2} f_{xx}(x_0, y_0) + \frac{y_0}{2} f_{yy}(x_0, y_0) \\ &\quad + \frac{1}{2} f_x(x, y) + \frac{1}{2} f_y(x, y). \end{aligned} \tag{76}$$

Thus, the proof is completed for $f \in C_2^2(\mathbb{R}_+^2)$. □

## References


[1] A. Lupaş, "A $q$-analogue of the Bernstein operator," in *Seminar on Numerical and Statistical Calculus (Cluj-Napoca, 1987)*, vol. 87 of *Preprint*, pp. 85–92, University of Cluj-Napoca, Cluj-Napoca, Romania, 1987.

[2] G. M. Phillips, "Bernstein polynomials based on the $q$-integers," *Annals of Numerical Mathematics*, vol. 4, no. 1–4, pp. 511–518, 1997.

[3] T. Trif, "Meyer-König and Zeller operators based on the $q$-integers," *Revue d'Analyse Numérique et de Théorie de l'Approximation*, vol. 29, no. 2, pp. 221–229, 2000.

[4] O. Doğru and O. Duman, "Statistical approximation of Meyer-König and Zeller operators based on $q$-integers," *Publicationes Mathematicae Debrecen*, vol. 68, no. 1-2, pp. 199–214, 2006.

[5] A. Aral and O. Doğru, "Bleimann, Butzer, and Hahn operators based on the $q$-integers," *Journal of Inequalities and Applications*, vol. 2007, Article ID 79410, 12 pages, 2007.

[6] N. I. Mahmudov and P. Sabancıgil, "$q$-parametric Bleimann Butzer and Hahn operators," *Journal of Inequalities and Applications*, vol. 2008, Article ID 816367, 15 pages, 2008.

[7] A. Aral, "A generalization of Szász-Mirakyan operators based on $q$-integers," *Mathematical and Computer Modelling*, vol. 47, no. 9-10, pp. 1052–1062, 2008.

[8] N. I. Mahmudov, "Approximation by the $q$-Szász-Mirakjan operators," *Abstract and Applied Analysis*, vol. 2012, Article ID 754217, 16 pages, 2012.

[9] N. I. Mahmudov, "On $q$-parametric Szász-Mirakjan operators," *Mediterranean Journal of Mathematics*, vol. 7, no. 3, pp. 297–311, 2010.

[10] O. Agratini and C. Radu, "On $q$-Baskakov-Mastroianni operators," *The Rocky Mountain Journal of Mathematics*, vol. 42, no. 3, pp. 773–790, 2012.

[11] N. I. Mahmudov, "Statistical approximation of Baskakov and Baskakov-Kantorovich operators based on the $q$-integers," *Central European Journal of Mathematics*, vol. 8, no. 4, pp. 816–826, 2010.

[12] D. D. Stancu, "A new class of uniform approximating polynomial operators in two and several variables," in *Proceedings of the Conference on the Constructive Theory of Functions*, pp. 443–455, Akadémiai Kiadó, Budapest, Hungary, 1972.

[13] D. Barbosu, "Some generalized bivariate Bernstein operators," *Mathematical Notes*, vol. 1, no. 1, pp. 3–10, 2000.

[14] O. Doğru and V. Gupta, "Korovkin-type approximation properties of bivariate $q$-Meyer-König and Zeller operators," *Calcolo*, vol. 43, no. 1, pp. 51–63, 2006.

[15] O. Agratini, "Bivariate positive operators in polynomial weighted spaces," *Abstract and Applied Analysis*, vol. 2013, Article ID 850760, 8 pages, 2013.

[16] S. Marinković, P. Rajković, and M. Stanković, "The inequalities for some types of $q$-integrals," *Computers & Mathematics with Applications*, vol. 56, no. 10, pp. 2490–2498, 2008.

[17] G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, vol. 71 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, UK, 1999.

[18] V. Kac and P. Cheung, *Quantum Calculus*, Universitext, Springer, New York, NY, USA, 2002.