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Research Article

Three-Point Boundary Value Problems of Nonlinear Second-Order q-Difference Equations Involving Different Numbers of q

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We study a new class of three-point boundary value problems of nonlinear second-order q-difference equations. Our problems contain different numbers of q in derivatives and integrals. By using a variety of fixed point theorems (such as Banach's contraction principle, Boyd and Wong fixed point theorem for nonlinear contractions, Krasnoselskii's fixed point theorem, and Leray-Schauder nonlinear alternative) and Leray-Schauder degree theory, some new existence and uniqueness results are obtained. Illustrative examples are also presented.

1. Introduction

The q-difference calculus or quantum calculus is an old subject that was initially developed by Jackson [1], Carmichael [2], Mason [3], and Adams [4], in the first quarter of 20th century, has been developed over the years, for instance, see [5–14] and the references therein. In fact, q-calculus has a rich history, and the details of its basic notions, results, and methods can be found in the text [15]. In recent years, the topic has attracted the attention of several researchers, and a variety of new results can be found in the papers [16–28] and the references cited therein.

In [24], Ahmad et al. studied a boundary value problem of nonlinear q-difference equations with nonlocal boundary conditions given by

$$\begin{gathered} D_q^2 x(t) = f(t, x(t)), \quad t \in I_q^1, \\ \alpha_1 x(0) - \beta_1 D_q x(0) = \gamma_1 x(\eta_1), \\ \alpha_2 x(1) + \beta_2 D_q x(1) = \gamma_2 x(\eta_2), \end{gathered} \tag{1}$$

where $f \in C(I_q^1 \times \mathbb{R}, \mathbb{R})$, $I_q^1 = \{q^n : n \in \mathbb{N}\} \cup \{0, 1\}$, and $q \in (0,1)$ is a fixed constant. The existence of solutions for problem (1) is shown by means of a variety of fixed point theorems such as Banach's contraction principle, Krasnoselskii's fixed point theorem, and Leray-Schauder nonlinear alternative.

Yu and Wang [28] considered a boundary value problem with the nonlinear second-order q-difference equation,

$$\begin{gathered} D_q^2 u(t) + f\left(t, u(t), D_q u(t)\right) = 0, \quad t \in I_q^1, \\ D_q u(0) = 0, \qquad D_q u(1) = \alpha u(1), \end{gathered} \tag{2}$$

where $f \in C(I_q^1 \times \mathbb{R}^2, \mathbb{R})$ and $\alpha \neq 0$ is a fixed number. Existence and uniqueness of the solutions are obtained by means of Banach's contraction principle, Leray-Schauder nonlinear alternative, and Leray-Schauder continuation theorem.

Pongarm et al. [29] considered sequential derivative of nonlinear q-difference equation with three-point boundary conditions,

$$\begin{gathered} D_q\left(D_p + \lambda\right) u(t) = f(t, u(t)), \quad t \in I_q^T = [0, T] \cap I_q^1, \\ u(0) = 0, \qquad u(T) = \alpha \int_0^\eta u(s)\, d_r s, \end{gathered} \tag{3}$$

where $0 < p, q, r < 1$, $f \in C(I_q^T \times \mathbb{R}, \mathbb{R})$, $0 < \eta < T$, and λ, β are given constants. Existence results are proved based on Banach's contraction mapping principle, Krasnoselskii's fixed point theorem, and Leray-Schauder degree theory.

We note that in the above-mentioned papers [24, 28] the q-numbers in the equation and the boundary conditions are the same. As far as we know the paper by Pongarm et al. [29] is the first paper which has different values of the q-numbers in q-derivative and q-integral.

In this paper, we discuss the existence of solutions for the following nonlinear q-difference equation with three-point integral boundary condition

$$\begin{gathered} D_q^2 x(t) = f(t, x(t)), \quad t \in I_q^T, \\ \alpha x(\eta) + \beta D_r x(\eta) = 0, \qquad \int_0^T x(s)\, d_p s = 0, \\ 0 < \eta < T, \end{gathered} \tag{4}$$

where $f \in C(I_q^T \times \mathbb{R}, \mathbb{R})$, $I_q^T = I_q^1 \cap [0, T]$, $I_q^1 = \{q^n : n \in \mathbb{N}\} \cup \{0, 1\}$, $q \in (0, 1)$ is a fixed constant, and $\eta \in I_q^T \setminus \{0, T\} := (0, T)_q$. Also, $0 < p, q, r < 1$, and α, β are given constants such that $\beta \neq \alpha((T/(1+p)) - \eta)$.

It is noteworthy that, in the above problem (4), we have three different values of the q-numbers, in q-derivatives and the q-integral. Moreover, we emphasize the fact that, instead the value $x(0)$ is usually used in the literature, we use the values of the function and its derivative in an intermediate point $\eta \in (0, T)$.

The aim of this paper is to prove some existence and uniqueness results for the boundary value problem (4). Our results are based on Banach's contraction mapping principle, nonlinear contraction, Krasnoselskii's fixed point theorem, Leray-Schauder nonlinear alternative, and Leray-Schauder degree theory.

The rest of the paper is organized as follows. In Section 2, we provide some basic definitions, preliminaries facts, and a lemma, which are used later. The main results are given in Section 3. In the end, Section 4, some results illustrating the results established in this paper are also presented.

2. Preliminaries

Let us recall some basic concepts of q-calculus [15, 18].

Definition 1. For $0 < q < 1$, one defines the q-derivative of a real valued function f as

$$\begin{gathered} D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \quad t \in I_q^1 \setminus \{0\}, \\ D_q f(0) = \lim_{t \to 0} D_q f(t). \end{gathered} \tag{5}$$

The higher-order q-derivatives are given by

$$D_q^0 f(t) = f(t), \qquad D_q^n f(t) = D_q D_q^{n-1} f(t), \quad n \in \mathbb{N}. \tag{6}$$

For $x \geq 0$ one sets $J_x = \{xq^n : n \in \mathbb{N} \cup \{0\}\} \cup \{0\}$ and define, the definite q-integral of a function $f : J_x \to \mathbb{R}$ by

$$I_q f(x) = \int_0^x f(s)\, d_q s = \sum_{n=0}^{\infty} x(1-q) q^n f(xq^n) \tag{7}$$

provided that the series converges.

For $a, b \in J_x$, one sets

$$\begin{aligned} \int_a^b f(s)\, d_q s &= I_q f(b) - I_q f(a) \\ &= (1-q) \sum_{n=0}^{\infty} q^n [b f(bq^n) - a f(aq^n)]. \end{aligned} \tag{8}$$

Note that for $a, b \in J_x$, one has $a = xq^{n_1}$, $b = xq^{n_2}$ for some $n_1, n_2 \in \mathbb{N}$; thus, the definite integral $\int_a^b f(s) d_q s$ is just a finite sum, so no question about convergence is raised.

One notes that

$$D_q I_q f(x) = f(x), \tag{9}$$

while if f is continuous at $x = 0$, then

$$I_q D_q f(x) = f(x) - f(0). \tag{10}$$

In q-calculus, the product rule and integration by parts formula are

$$\begin{aligned} D_q(gh)(t) &= (D_q g(t)) h(t) + g(qt) D_q h(t), \\ \int_0^x f(t) D_q g(t)\, d_q t &= [f(t) g(t)]_0^x - \int_0^x D_q f(t) g(qt)\, d_q t. \end{aligned} \tag{11}$$

Further, reversing the order of integration is given by

$$\int_0^t \int_0^s f(r)\, d_q r\, d_q s = \int_0^t \int_{qr}^t f(r)\, d_q s\, d_q r. \tag{12}$$

In the limit $q \to 1$, the above results correspond to their counterparts in standard calculus.

Lemma 2. *Let $0 < p, q, r < 1$ and $\eta \in (0, T)_q$. Then, for any $y \in C(I_q^T, \mathbb{R})$, the boundary value problem,*

$$D_q^2 x(t) = y(t), \quad t \in I_q^T, \tag{13}$$

$$\alpha x(\eta) + \beta D_r x(\eta) = 0, \qquad \int_0^T x(s)\, d_p s = 0, \tag{14}$$

is equivalent to the integral equation

$$\begin{aligned} x(t) = & \int_0^t (t-qs)\, y(s)\, d_q s - \frac{(1+p)\,t - T}{\Omega} \\ & \times \left[\alpha \int_0^{\eta} (\eta - qs)\, y(s)\, d_q s + \beta \int_0^{r\eta} y(s)\, d_q s \right. \\ & \left. + \frac{\beta}{(1-r)\,\eta} \int_{r\eta}^{\eta} (\eta - qs)\, y(s)\, d_q s \right] \\ & + \frac{1+p}{T\Omega} (\alpha (t-\eta) - \beta) \int_0^T \int_0^s (s-qv)\, y(v)\, d_q v d_p s, \end{aligned} \tag{15}$$

where

$$\Omega = (\alpha\eta + \beta)(1+p) - \alpha T \neq 0. \tag{16}$$

Proof. Taking double q-integral for (13), we have

$$x(t) = \int_0^t \int_0^s y(v)\, d_q v\, d_q s + c_1 t + c_2. \tag{17}$$

By changing the order of q-integration, we have

$$\begin{aligned} x(t) &= \int_0^t \int_{qv}^t y(v)\, d_q s\, d_q v + c_1 t + c_2 \\ &= \int_0^t (t-qs)\, y(s)\, d_q s + c_1 t + c_2. \end{aligned} \tag{18}$$

In particular, for $t = \eta$, we get

$$x(\eta) = \int_0^{\eta} (\eta - qs)\, y(s)\, d_q s + \eta c_1 + c_2. \tag{19}$$

Taking r-derivative for (18), for $t \neq 0$, we obtain

$$\begin{aligned} & D_r x(t) \\ & = D_r \left[\left[\int_0^t (t-qs)\, y(s)\, d_q s + c_1 t + c_2 \right] \right] \\ & = \frac{1}{(1-r)\,t} \left[\left[\int_0^t (t-qs)\, y(s)\, d_q s - \int_0^{rt} (rt - qs)\, y(s)\, d_q s \right] + c_1 \right. \\ & = \int_0^{rt} y(s)\, d_q s + \int_{rt}^t \frac{t-qs}{(1-r)\,t} y(s)\, d_q s + c_1. \end{aligned} \tag{20}$$

For $t = 0$, we have

$$\begin{aligned} D_r x(0) &= \lim_{t \to 0} D_r x(0) \\ &= \lim_{t \to 0} \frac{t(1-q)}{1-r} \sum_{n=0}^{\infty} q^n \left(1 - q^{n+1}\right) \\ & \quad \times \left[h(tq^n) - r^2 h(rtq^n) \right] + c_1 \\ &= c_1. \end{aligned} \tag{21}$$

Therefore,

$$D_r x(\eta) = \int_0^{r\eta} y(s)\, d_q s + \int_{r\eta}^{\eta} \frac{\eta - qs}{(1-r)\,\eta} y(s)\, d_q s + c_1. \tag{22}$$

Now, using the first condition of (14) with (19), (22), we have

$$\begin{aligned} (\alpha\eta + \beta)\, c_1 + \alpha c_2 = & -\alpha \int_0^{\eta} (\eta - qs)\, y(s)\, d_q s \\ & - \beta \int_0^{r\eta} y(s)\, d_q s \\ & - \frac{\beta}{(1-r)\,\eta} \int_{r\eta}^{\eta} (\eta - qs)\, y(s)\, d_q s. \end{aligned} \tag{23}$$

Taking the p-integral for (18) from 0 to t, we obtain

$$\int_0^t x(s)\, d_p s = \int_0^t \int_0^s (s-qv)\, y(v)\, d_q v\, d_p s + \frac{t^2}{1+p} c_1 + t c_2. \tag{24}$$

Substituting $t = T$ in (24) and using the second condition of (14), we get

$$\frac{T^2}{1+p} c_1 + T c_2 = - \int_0^T \int_0^s (s-qv)\, y(v)\, d_q v\, d_p s. \tag{25}$$

Solving the system of linear equations (23) and (25) for the unknown constants c_1 and c_2, we have

$$\begin{aligned} c_1 = & -\frac{1+p}{\Omega} \left[\alpha \int_0^{\eta} (\eta - qs)\, y(s)\, d_q s \right. \\ & + \beta \int_0^{r\eta} y(s)\, d_q s + \frac{\beta}{(1-r)\,\eta} \\ & \left. \times \int_{r\eta}^{\eta} (\eta - qs)\, y(s)\, d_q s \right] \\ & + \frac{\alpha(1+p)}{T\Omega} \int_0^T \int_0^s (s-qv)\, y(v)\, d_q v\, d_p s, \\ c_2 = & -\frac{(\alpha\eta + \beta)(1+p)}{T\Omega} \int_0^T \int_0^s (s-qv)\, y(v)\, d_q v\, d_p s \\ & + \frac{T}{\Omega} \left[\alpha \int_0^{\eta} (\eta - qs)\, y(s)\, d_q s + \beta \int_0^{r\eta} y(s)\, d_q s \right. \\ & \left. + \frac{\beta}{(1-r)\,\eta} \int_{r\eta}^{\eta} (\eta - qs)\, y(s)\, d_q s \right], \end{aligned} \tag{26}$$

where Ω is defined by (16). Substituting the values of c_1 and c_2 in (18), we obtain (15). This completes the proof. □

Let $\mathscr{C} = C(I_q^T, \mathbb{R})$ denotes the Banach space of all the continuous functions from I_q^T to $\mathbb{R}$ endowed with the norm defined by $\|x\| = \sup\{|x(t)|, t \in I_q^T\}$. Define an operator $A : \mathscr{C} \to \mathscr{C}$ by

$$\begin{aligned}(Ax)(t) = {} & \int_0^t (t-qs) f(s,x(s))\, d_q s - \frac{(1+p)t - T}{\Omega} \\ & \times \left[\alpha \int_0^\eta (\eta - qs) f(s,x(s))\, d_q s \right. \\ & \quad + \beta \int_0^{r\eta} f(s,x(s))\, d_q s \\ & \quad \left. + \frac{\beta}{(1-r)\eta} \int_{r\eta}^\eta (\eta - qs) f(s,x(s))\, d_q s \right] \\ & + \frac{(1+p)(\alpha(t-\eta) - \beta)}{T\Omega} \\ & \times \int_0^T \int_0^s (s - qv) f(v, x(v))\, d_q v\, d_p s. \end{aligned} \tag{27}$$

Observe that the problem (4) has solutions if and only if the operator A has fixed points.

For the sake of convenience, we set a constant Λ as

$$\begin{aligned}\Lambda = {} & \frac{T^2}{1+q} + \frac{pT}{|\Omega|} \left[\frac{|\alpha| \eta^2}{1+q} + |\beta| r\eta \right. \\ & \left. + \frac{|\beta| \eta (1 + q(2+r))}{1+q} \right] \\ & + \frac{(1+p)(|\alpha|(T-\eta) + |\beta|) T^2}{(1+q)(1+p+p^2)|\Omega|}. \end{aligned} \tag{28}$$

3. Main Results

Now, we are in the position to establish the main results. Our first result is based on Banach's fixed point theorem.

Theorem 3. *Assume that $f : I_q^T \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the conditions*

(H_1) $|f(t,x) - f(t,y)| \le L|x-y|$, *for all $t \in I_q^T$ and $x, y \in \mathbb{R}$,*

(H_2) $L\Lambda < 1$,

where L is a Lipschitz constant, and Λ is defined by (28).

Then, the boundary value problem (4) *has a unique solution.*

Proof. We transform the boundary value problem (4) into a fixed point problem $x = Ax$, where $A : \mathscr{C} \to \mathscr{C}$ is defined by (27). Assume that $\sup_{t \in I_q^T} |f(t,0)| = M$, and choose a constant R satisfying

$$R \ge \frac{M\Lambda}{1 - L\Lambda}. \tag{29}$$

Now, we will show that $AB_R \subset B_R$, where $B_R = \{x \in \mathscr{C} : \|x\| \le R\}$. For any $x \in B_R$, we have

$$\begin{aligned}\|Ax\| = {} & \sup_{t \in I_q^T} \left| \int_0^t (t-qs) f(s,x(s))\, d_q s - \frac{(1+p)t - T}{\Omega} \right. \\ & \times \left[\alpha \int_0^\eta (\eta - qs) f(s,x(s))\, d_q s \right. \\ & \quad + \beta \int_0^{r\eta} f(s,x(s))\, d_q s + \frac{\beta}{(1-r)\eta} \\ & \quad \left. \times \int_{r\eta}^\eta (\eta - qs) f(s,x(s))\, d_q s \right] \\ & + \frac{(1+p)(\alpha(t-\eta) - \beta)}{T\Omega} \\ & \left. \times \int_0^T \int_0^s (s-qv) f(v,x(v))\, d_q v\, d_p s \right| \\ \le {} & \sup_{t \in I_q^T} \left\{ \int_0^t (t - qs) \right. \\ & \times (|f(s,x(s)) - f(s,0)| + |f(s,0)|)\, d_q s \\ & + \frac{|(1+p)t - T|}{|\Omega|} \\ & \times \left[|\alpha| \int_0^\eta (\eta - qs) (|f(s,x(s)) - f(s,0)| \right. \\ & \qquad + |f(s,0)|)\, d_q s \\ & \quad + |\beta| \int_0^{r\eta} (|f(s,x(s)) - f(s,0)| \\ & \qquad + |f(s,0)|)\, d_q s \\ & \quad + \frac{|\beta|}{(1-r)\eta} \\ & \quad \times \int_{r\eta}^\eta (\eta - qs)(|f(s,x(s)) - f(s,0)| \\ & \qquad \left. + |f(s,0)|)\, d_q s \right] \\ & + \frac{(1+p)|\alpha(t-\eta) - \beta|}{T|\Omega|} \\ & \times \int_0^T \int_0^s (s - qv) \\ & \qquad \times (|f(v,x(s)) - f(v,0)| \\ & \qquad \left. + |f(v,0)|)\, d_q v\, d_p s \right\} \\ \le {} & \sup_{t \in I_q^T} \left\{ \int_0^t (t-qs)(LR + M)\, d_q s + \frac{|(1+p)t - T|}{|\Omega|} \right. \end{aligned}$$

$$
\begin{aligned}
&\times\left[|\alpha|\int_0^{\eta}(\eta-qs)(LR+M)\,d_qs+|\beta|\right.\\
&\qquad\times\int_0^{r\eta}(LR+M)\,d_qs\\
&\qquad\left.+\frac{|\beta|}{(1-r)\eta}\int_{r\eta}^{\eta}(\eta-qs)(LR+M)\,d_qs\right]\\
&+\frac{(1+p)|\alpha(t-\eta)-\beta|}{T|\Omega|}\\
&\left.\times\int_0^T\int_0^s(s-qv)(LR+M)\,d_qv\,d_ps\right\}\\
\le{}&(LR+M)\left\{\frac{T^2}{1+q}+\frac{pT}{|\Omega|}\right.\\
&\qquad\times\left[\frac{|\alpha|\eta^2}{1+q}+|\beta|r\eta\right.\\
&\qquad\left.+\frac{|\beta|\eta(1+q(2+r))}{1+q}\right]\\
&\qquad\left.+\frac{(1+p)(|\alpha|(T-\eta)+|\beta|)T^2}{(1+q)(1+p+p^2)|\Omega|}\right\}\\
={}&(LR+M)\Lambda\le R.
\end{aligned}
\tag{30}
$$

Therefore, $AB_R\subset B_R$.

Next, we will show that A is a contraction. For any $x,y\in\mathscr{C}$ and for each $t\in I_q^T$, we have

$$
\begin{aligned}
&\|Ax-Ay\|\\
&=\sup_{t\in I_q^T}|(Ax)(t)-(Ay)(t)|\\
&\le\sup_{t\in I_q^T}\left|\int_0^t(t-qs)|f(s,x(s))-f(s,y(s))|\,d_qs\right.\\
&\qquad-\frac{(1+p)t-T}{\Omega}\\
&\qquad\times\left[\alpha\int_0^{\eta}(\eta-qs)\,|f(s,x(s))\right.\\
&\qquad\qquad-f(s,y(s))|\,d_qs\\
&\qquad+\beta\int_0^{r\eta}|f(s,x(s))\\
&\qquad\qquad-f(s,x_2(s))|\,d_qs\\
&\qquad+\frac{\beta}{(1-r)\eta}\\
&\qquad\times\int_{r\eta}^{\eta}(\eta-qs)\,|f(s,x(s))\\
&\qquad\qquad\left.-f(s,y(s))|\,d_qs\right]\\
&\qquad+\frac{(1+p)(\alpha(t-\eta)-\beta)}{T\Omega}\\
&\qquad\times\int_0^T\int_0^s(s-qv)\\
&\qquad\qquad\left.\times|f(v,x(v))-f(v,y(v))|\,d_qv\,d_ps\right|\\
&\le\sup_{t\in I_q^T}\left\{L\|x-y\|\int_0^t(t-qs)\,d_qs+L\|x-y\|\right.\\
&\qquad\times\frac{|(1+p)t-T|}{|\Omega|}\left[|\alpha|\int_0^{\eta}(\eta-qs)\,d_qs\right.\\
&\qquad\qquad+|\beta|\int_0^{r\eta}d_qs+\frac{|\beta|}{(1-r)\eta}\\
&\qquad\qquad\left.\times\int_{r\eta}^{\eta}(\eta-qs)\,d_qs\right]\\
&\qquad+\frac{(1+p)(|\alpha|(t-\eta)+|\beta|)}{T|\Omega|}L\|x-y\|\\
&\qquad\left.\times\int_0^T\int_0^s(s-qv)\,d_qv\,d_ps\right\}\\
&\le L\|x-y\|\left\{\frac{T^2}{1+q}+\frac{pT}{|\Omega|}\right.\\
&\qquad\times\left[\frac{|\alpha|\eta^2}{1+q}+|\beta|r\eta\right.\\
&\qquad\left.+\frac{|\beta|\eta(1+q(2+r))}{1+q}\right]\\
&\qquad\left.+\frac{(1+p)(|\alpha|(T-\eta)+|\beta|)T^2}{(1+q)(1+p+p^2)|\Omega|}\right\}\\
&=L\Lambda\|x-y\|.
\end{aligned}
\tag{31}
$$

Since $L\Lambda<1$, A is a contraction. Thus, the conclusion of the theorem follows by Banach's contraction mapping principle. This completes the proof. □

Next, we can still deduce the existence and uniqueness of a solution to the boundary value problem (4). We will use nonlinear contraction to accomplish this.

Definition 4. Let E be a Banach space and let $F:E\to E$ be a mapping. F is said to be a nonlinear contraction if there exists a continuous nondecreasing function $\Psi:R^+\to R^+$

such that $\Psi(0) = 0$ and $\Psi(\rho) < \rho$ for all $\rho > 0$ with the following property:

$$\|Fx - Fy\| \leq \Psi(\|x - y\|), \quad \forall x, y \in E. \tag{32}$$

Lemma 5 (Boyd and Wong [30]). *Let E be a Banach space and let $F : E \to E$ be a nonlinear contraction. Then, F has a unique fixed point in E.*

Theorem 6. *Suppose that*

(H_3) *there exists a continuous function $h : I_q^T \to \mathbb{R}^+$ such that*

$$|f(t,x) - f(t,y)| \leq h(t) \frac{|x-y|}{G + |x-y|} \tag{33}$$

for all $t \in I_q^T$ and $x, y \geq 0$, where

$$\begin{aligned} G = {} & \int_0^T (T - qs) h(s) d_q s + \frac{pT}{|\Omega|} \\ & \times \left[|\alpha| \int_0^\eta (\eta - qs) h(s) d_q s + |\beta| \int_0^{r\eta} h(s) d_q s \right. \\ & \left. + \frac{|\beta|}{(1-r)\eta} \int_{r\eta}^\eta (\eta - qs) h(s) d_q s \right] \\ & + \frac{(1+p)(|\alpha|(T-\eta) + |\beta|)}{T|\Omega|} \\ & \times \int_0^T \int_0^s (s - qv) h(v) d_q v d_p s \end{aligned} \tag{34}$$

and Ω is defined in (16).

Then, the boundary value problem (4) *has a unique solution.*

Proof. Let the operator $A : \mathscr{C} \to \mathscr{C}$ be defined as (27). We define a continuous nondecreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\Psi(\rho) = \frac{G\rho}{G + \rho}, \quad \forall \rho \geq 0, \tag{35}$$

such that $\Psi(0) = 0$ and $\Psi(\rho) < \rho$, for all $\rho > 0$.

Let $x, y \in \mathscr{C}$. Then, we get

$$|f(s, x(s)) - f(s, y(s))| \leq \frac{h(s)}{G} \Psi(\|x - y\|). \tag{36}$$

Thus,

$$\begin{aligned} & |Ax(t) - Ay(t)| \\ & \leq \int_0^t (t - qs) h(s) \frac{|x(s) - y(s)|}{G + |x(s) - y(s)|} d_q s \\ & \quad + \frac{|(1+p)t - T|}{|\Omega|} \left[|\alpha| \int_0^\eta (\eta - qs) \right. \\ & \qquad \times h(s) \frac{|x(s) - y(s)|}{G + |x(s) - y(s)|} d_q s \\ & \qquad + |\beta| \int_0^{r\eta} h(s) \frac{|x(s) - y(s)|}{G + |x(s) - y(s)|} d_q s \\ & \qquad + \frac{|\beta|}{(1-r)\eta} \int_{r\eta}^\eta (\eta - qs) \\ & \qquad \left. \times h(s) \frac{|x(s) - y(s)|}{G + |x(s) - y(s)|} d_q s \right] \\ & \quad + \frac{(1+p)|\alpha(t - \eta) - \beta|}{T|\Omega|} \\ & \quad \times \int_0^T \int_0^s (s - qv) \\ & \qquad \times h(v) \frac{|x(v) - y(v)|}{G^* + |x(v) - y(v)|} d_q v d_p s \\ & \leq \left\{ \int_0^T (T - qs) h(s) d_q s + \frac{pT}{|\Omega|} \right. \\ & \quad \times \left[|\alpha| \int_0^\eta (\eta - qs) h(s) d_q s + |\beta| \int_0^{r\eta} h(s) d_q s \right. \\ & \qquad \left. + \frac{|\beta|}{(1-r)\eta} \int_{r\eta}^\eta (\eta - qs) h(s) d_q s \right] \\ & \quad + \frac{(1+p)(|\alpha|(T-\eta) + |\beta|)}{T|\Omega|} \\ & \quad \left. \times \int_0^T \int_0^s (s - qv) h(v) d_q v d_p s \right\} \\ & \qquad \times \frac{\|x - y\|}{G + \|x - y\|} \\ & = \frac{G\|x - y\|}{G + \|x - y\|}, \quad \forall t \in I_q^T. \end{aligned} \tag{37}$$

This implies that $\|Ax - Ay\| \leq \Psi(\|x - y\|)$. Hence, A is a nonlinear contraction. Therefore, by Lemma 5, the operator A has a unique fixed point in $\mathscr{C}$, which is a unique solution of problem (4). $\square$

The third result is based on the following Krasnoselskii fixed point theorem [31].

Theorem 7. *Let K be a bounded closed convex and nonempty subset of a Banach space X. Let A, B be operators such that:*

(i) *$Ax + By \in K$ whenever $x, y \in K$,*

(ii) *A is compact and continuous,*

(iii) *B is a contraction mapping.*

Then, there exists $z \in K$ such that $z = Az + Bz$.

Theorem 8. *Assume that (H_1) and (H_2) hold. In addition one supposes that:*

(H_4) *$|f(t,x)| \le \mu(t)$, for all $(t,x) \in I_q^T \times \mathbb{R}$, with $\mu \in L^1(I_q^T, \mathbb{R}^+)$.*

If

$$\Lambda < 1, \tag{38}$$

where Λ is given by (28), then the boundary value problem (4) has at least one solution on I_q^T.

Proof. Setting $\max_{t\in I_q^T}|\mu(t)| = \|\mu\|$ and choosing a constant

$$R \ge \|\mu\| \Lambda, \tag{39}$$

we consider $B_R = \{x \in \mathscr{C} : \|x\| \le R\}$.

In view of Lemma 2, we define the operators $\mathscr{F}_1$ and $\mathscr{F}_2$ on the ball B_R as

$$\begin{aligned} (\mathscr{F}_1 x)(t) &= \int_0^t (t-qs) f(s,x(s))\, d_q s, \\ (\mathscr{F}_2 x)(t) &= -\frac{(1+p)t - T}{\Omega} \left[\alpha \int_0^\eta (\eta - qs) f(s,x(s))\, d_q s + \beta \int_0^{r\eta} f(s,x(s))\, d_q s + \frac{\beta}{(1-r)\eta} \times \int_{r\eta}^{\eta} (\eta - qs) f(s,x(s))\, d_q s \right] \\ &\quad + \frac{(1+p)(\alpha(t-\eta) - \beta)}{T\Omega} \times \int_0^T \int_0^s (s - qv) f(v,x(v))\, d_q v\, d_p s. \end{aligned} \tag{40}$$

For $x, y \in B_R$, by computing directly, we have

$$\begin{aligned} \|\mathscr{F}_1 x + \mathscr{F}_2 y\| &\le \|\mu\| \int_0^t (t-qs)\, d_q s + \|\mu\| \frac{|(1+p)t - T|}{|\Omega|} \times \left[|\alpha| \int_0^\eta (\eta - qs)\, d_q s + |\beta| \int_0^{r\eta} d_q s + \frac{|\beta|}{(1-r)\eta} \int_{r\eta}^{\eta} (\eta - qs)\, d_q s \right] \\ &\quad + \|\mu\| \frac{(1+p)|\alpha(t-\eta) - \beta|}{T|\Omega|} \times \int_0^T \int_0^s (s-qv)\, d_q v\, d_p s \\ &\le \|\mu\| \Lambda \le R. \end{aligned} \tag{41}$$

Therefore, $\mathscr{F}_1 x + \mathscr{F}_2 y \in B_R$. Condition (38) implies that $\mathscr{F}_2$ is a contraction mapping. Next, we will show that $\mathscr{F}_1$ is compact and continuous. Continuity of f coupled with the assumption (H_3) implies that the operator $\mathscr{F}_1$ is continuous and uniformly bounded on B_R. We define $\sup_{(t,x)\in I_q^T \times B_R} |f(t,x)| = f_{\max} < \infty$. For $t_1, t_2 \in I_q^T$ with $t_1 \le t_2$ and $x \in B_R$, we have

$$\begin{aligned} |\mathscr{F}_1 x(t_2) - \mathscr{F}_1 x(t_1)| &= \left| \int_0^{t_2} (t_2 - qs) f(s,x(s))\, d_q s - \int_0^{t_1} (t_1 - qs) f(s,x(s))\, d_q s \right| \\ &= \left| \int_0^{t_1} (t_2 - t_1) f(s,x(s))\, d_q s + \int_{t_1}^{t_2} (t_2 - qs) f(s,x(s))\, d_q s \right| \\ &\le \left| t_2^2 - t_1^2 \right| \left(\frac{1+2q}{1+q} \right) f_{\max}. \end{aligned} \tag{42}$$

Actually, as $t_2 - t_1 \to 0$, the right-hand side of the above inequality tends to be zero. So, $\mathscr{F}_1$ is relatively compact on B_R. Hence, by the Arzelá-Ascoli Theorem, $\mathscr{F}_1$ is compact on B_R. Therefore, all the assumptions of Theorem 7 are satisfied, and the conclusion of Theorem 7 implies that the boundary value problem (4) has at least one solution on I_q^T. This completes the proof. □

As the fourth result, we prove the existence of solutions of (4) by using Leray-Schauder nonlinear alternative.

Theorem 9 (Nonlinear Alternative for Single Valued Maps [32]). *Let E be a Banach space, C a closed convex subset of E, U an open subset of C, and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then, either*

(i) F *has a fixed point in* $\overline{U}$ *or*

(ii) *there is a* $u \in \partial U$ *(the boundary of* U *in* C*) and* $\lambda \in (0,1)$ *with* $u = \lambda F(u)$.

Theorem 10. *Assume that:*

(H_5) *there exists a continuous nondecreasing function* $\psi : [0,\infty) \to (0,\infty)$ *and a function* $z \in L^1(I_q^T, \mathbb{R}^+)$ *such that*

$$\left|f(t,u)\right| \le z(t)\,\psi(\|u\|), \quad \text{for each } (t,u) \in I_q^T \times \mathbb{R}; \tag{43}$$

(H_6) *there exists a constant* $M > 0$ *such that*

$$\frac{M}{\psi(M)\,\|z\|_{L^1}\Lambda} > 1. \tag{44}$$

Then, the boundary value problem (4) *has at least one solution on* I_q^T.

Proof. We will show that *A maps bounded sets (balls) into bounded sets in* $\mathcal{C}$. For a positive number ρ, let $B_\rho = \{x \in C(I_q^T,\mathbb{R}) : \|x\| \le \rho\}$ be a bounded ball in $C(I_q^T,\mathbb{R})$. Then, for $t \in I_q^T$, we have

$$\begin{aligned}
&|(Ax)(t)| \\
&\le \int_0^t (t-qs)\left|f(s,x(s))\right| d_qs + \frac{\left|(1+p)t-T\right|}{|\Omega|} \\
&\quad\times\left[|\alpha|\int_0^\eta (\eta-qs)\left|f(s,x(s))\right| d_qs \right.\\
&\qquad + |\beta|\int_0^{r\eta}\left|f(s,x(s))\right| d_qs + \frac{|\beta|}{(1-r)\eta} \\
&\qquad \left.\times\int_{r\eta}^\eta (\eta-qs)\left|f(s,x(s))\right| d_qs\right] \\
&\quad + \frac{(1+p)\left|\alpha(t-\eta)-\beta\right|}{T|\Omega|} \\
&\quad\times\int_0^T\int_0^s (s-qv)\left|f(v,x(v))\right| d_qv\, d_ps \\
&\le \psi(\|x\|)\int_0^t (t-qs)\,z(s)\, d_qs + \frac{\psi(\|x\|)\left|(1+p)t-T\right|}{|\Omega|} \\
&\quad\times\left[|\alpha|\int_0^\eta (\eta-qs)\,z(s)\, d_qs + |\beta|\int_0^{r\eta} z(s)\, d_qs\right.\\
&\qquad\left. + \frac{|\beta|}{(1-r)\eta}\int_{r\eta}^\eta (\eta-qs)\,z(s)\, d_qs\right] \\
&\quad + \frac{\psi(\|x\|)(1+p)\left|\alpha(t-\eta)-\beta\right|}{T|\Omega|} \\
&\quad\times\int_0^T\int_0^s (s-qv)\,z(s)\, d_qv\, d_ps \\
&\le \psi(\|x\|)\|z\|_{L^1}\int_0^t (t-qs)\, d_qs \\
&\quad + \frac{\psi(\|x\|)\|z\|_{L^1}\left|(1+p)t-T\right|}{|\Omega|} \\
&\quad\times\left[|\alpha|\int_0^\eta (\eta-qs)\, d_qs + |\beta|\int_0^{r\eta} d_qs\right.\\
&\qquad\left. + \frac{|\beta|}{(1-r)\eta}\int_{r\eta}^\eta (\eta-qs)\, d_qs\right] \\
&\quad + \frac{\psi(\|x\|)\|z\|_{L^1}(1+p)\left|\alpha(t-\eta)-\beta\right|}{T|\Omega|} \\
&\quad\times\int_0^T\int_0^s (s-qv)\, d_qv d_ps \\
&\le \frac{\psi(\|x\|)\|z\|_{L^1}T^2}{1+q} \\
&\quad + \frac{\psi(\|x\|)\|z\|_{L^1}pT}{|\Omega|}\left[\frac{|\alpha|\eta^2}{1+q} + |\beta|r\eta\right.\\
&\qquad\left. + \frac{|\beta|\eta(1+q(2+r))}{1+q}\right] \\
&\quad + \frac{\psi(\|x\|)\|z\|_{L^1}(1+p)(|\alpha|(T-\eta)+|\beta|)T^2}{(1+q)(1+p+p^2)|\Omega|} \\
&= \psi(\|x\|)\|z\|_{L^1}\Lambda.
\end{aligned} \tag{45}$$

Consequently,

$$\|Ax\| \le \psi(\|x\|)\|z\|_{L^1}\Lambda. \tag{46}$$

Next, we will show that *A maps bounded sets into equicontinuous sets of* $C(I_q^T,\mathbb{R})$. Let $t_1, t_2 \in I_q^T$ with $t_1 \le t_2$ and $x \in B_\rho$. Then, we have

$$\begin{aligned}
&\left|(Ax)(t_2)-(Ax)(t_1)\right| \\
&\le \left|\int_0^{t_2} (t_2-qs)\left|f(s,x(s))\right| d_qs\right.\\
&\qquad\left. - \int_0^{t_1} (t_1-qs)\left|f(s,x(s))\right| d_qs\right| \\
&\quad + \frac{(1+p)\left|t_2-t_1\right|}{|\Omega|}\left[|\alpha|\int_0^\eta (\eta-qs)\left|f(s,x(s))\right| d_qs\right.
\end{aligned}$$

$$
\begin{aligned}
&\quad + |\beta| \int_0^{r\eta} |f(s, x(s))| \, d_q s \\
&\quad + \frac{|\beta|}{(1-r)\eta} \int_{r\eta}^{\eta} (\eta - qs) \\
&\quad \times |f(s, x(s))| \, d_q s \Big] \\
&+ \frac{(1+p)|\alpha| |t_2 - t_1|}{T|\Omega|} \\
&\quad \times \int_0^T \int_0^s (s - qv) |f(v, x(v))| \, d_q v \, d_p s \\
&\le \int_0^{t_1} |t_2 - t_1| z(s) \psi(\rho) \, d_q s \\
&\quad + \int_{t_1}^{t_2} (t_2 - qs) z(s) \psi(\rho) \, d_q s \\
&\quad + \frac{(1+p)|t_2 - t_1|}{|\Omega|} \Big[|\alpha| \int_0^{\eta} (\eta - qs) z(s) \psi(\rho) \, d_q s \\
&\quad + |\beta| \int_0^{r\eta} z(s) \psi(\rho) \, d_q s + \frac{|\beta|}{(1-r)\eta} \\
&\quad \times \int_{r\eta}^{\eta} (\eta - qs) z(s) \psi(\rho) \, d_q s \Big] \\
&+ \frac{(1+p)|\alpha| |t_2 - t_1|}{T|\Omega|} \\
&\quad \times \int_0^T \int_0^s (s - qv) z(s) \psi(\rho) \, d_q v \, d_p s.
\end{aligned} \tag{47}
$$

As $t_2 - t_1 \to 0$, the right-hand side of the above inequality tends to zero independently of $x \in B_\rho$. As A satisfies the above assumptions; therefore, it follows by the Arzelá-Ascoli theorem that $A : C(I_q^T, \mathbb{R}) \to C(I_q^T, \mathbb{R})$ is completely continuous.

Let x be a solution. Then, for $t \in I_q^T$ and following the similar computations as in the first step, we have

$$|x(t)| \le \psi(\|x\|) \|z\|_{L^1} \Lambda. \tag{48}$$

Consequently, we have

$$\frac{\|x\|}{\psi(\|x\|) \|z\|_{L^1} \Lambda} \le 1. \tag{49}$$

In view of (H_5), there exists M such that $\|x\| \ne M$. Let us set

$$U = \left\{ x \in C\left(I_q^T, \mathbb{R}\right) : \|x\| < M \right\}. \tag{50}$$

Note that the operator $A : \overline{U} \to C(I_q^T, \mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \lambda A x$ for some $\lambda \in (0, 1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Theorem 9), we deduce that A has a fixed point $x \in \overline{U}$ which is a solution of the problem (4). This completes the proof. □

Finally, we prove that problem (4) has at least one solution on I_q^T by using Leray-Schauder degree theory.

Theorem 11. *Let $f : I_q^T \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that:*

(H_7) *there exist constants $0 \le \kappa < \Lambda^{-1}$, where Λ is given by* (28) *and $N > 0$ such that $|f(t, x)| \le \kappa |x| + N$ for all $t \in I_q^T$, $x \in \mathscr{C}$.*

Then, the boundary value problem (4) *has at least one solution.*

Proof. Let us define an operator $A : \mathscr{C} \to \mathscr{C}$ as (27). We wish to prove that there exists at least one solution $x \in \mathscr{C}$ of the fixed point equation

$$x = Ax. \tag{51}$$

We define a ball $B_R \subset \mathscr{C}$, with a constant radius $R > 0$ given by

$$B_R = \left\{ x \in \mathscr{C} : \max_{t \in I_q^T} |x(t)| < R \right\}. \tag{52}$$

Then, it is sufficient to show that $A : \overline{B}_R \to \mathscr{C}(I_q^T)$ satisfies

$$x \ne \lambda A x, \quad \forall x \in \partial B_R, \quad \forall \lambda \in [0, 1]. \tag{53}$$

Now, we set

$$H(\lambda, x) = \lambda A x, \quad x \in \mathscr{C}, \ \lambda \in [0, 1]. \tag{54}$$

Then, by the Arzelá-Ascoli theorem, we conclude that a continuous map h_λ defined by $h_\lambda(x) = x - H(\lambda, x) = x - \lambda A x$ is completely continuous. If (53) holds, then the following Leray-Schauder degrees are well defined. From the homotopy invariance of topological degree, it follows that

$$
\begin{aligned}
\deg(h_\lambda, B_R, 0) &= \deg(I - \lambda A, B_R, 0) = \deg(h_1, B_R, 0) \\
&= \deg(h_0, B_R, 0) = \deg(I, B_R, 0) = 1 \ne 0, \\
&\qquad 0 \in B_R,
\end{aligned} \tag{55}
$$

where I denotes the unit operator. By the nonzero property of Leray-Schauder degree, $h_1(x) = x - Ax = 0$ for at least one $x \in B_R$. Let us assume that $x = \lambda A x$ for some $\lambda \in [0, 1]$. Then, for all $t \in I_q^T$, we obtain

$$
\begin{aligned}
&|x(t)| \\
&\quad = |\lambda (Ax)(t)| \\
&\quad \le \int_0^t (t - qs) |f(s, x(s))| \, d_q s + \frac{|(1+p)t - T|}{|\Omega|} \\
&\qquad \times \Big[|\alpha| \int_0^{\eta} (\eta - qs) |f(s, x(s))| \, d_q s
\end{aligned}
$$

$$
\begin{aligned}
&+|\beta|\int_0^{r\eta}|f(s,x(s))|\,d_qs+\frac{|\beta|}{(1-r)\eta}\\
&\times\int_{r\eta}^{\eta}(\eta-qs)|f(s,x(s))|\,d_qs\Bigg]\\
&+\frac{(1+p)|\alpha(t-\eta)-\beta|}{T|\Omega|}\int_0^T\int_0^s(s-qv)\\
&\times|y(v,x(v))|\,d_qv\,d_ps\\
&\le(\kappa|x|+N)\int_0^t(t-qs)\,d_qs+(\kappa|x|+N)\\
&\times\frac{|(1+p)t-T|}{|\Omega|}\\
&\times\Bigg[|\alpha|\int_0^{\eta}(\eta-qs)\,d_qs+|\beta|\int_0^{r\eta}d_qs\\
&+\frac{|\beta|}{(1-r)\eta}\int_{r\eta}^{\eta}(\eta-qs)\,d_qs\Bigg]\\
&+(\kappa|x|+N)\frac{(1+p)|\alpha(t-\eta)-\beta|}{T|\Omega|}\\
&\times\int_0^T\int_0^s(s-qv)\,d_qv\,d_ps\\
&\le(\kappa|x|+N)\left\{\frac{T^2}{1+q}+\frac{pT}{|\Omega|}\left[\frac{|\alpha|\eta^2}{1+q}+|\beta|r\eta\right.\right.\\
&\left.+\frac{|\beta|\eta(1+q(2+r))}{1+q}\right]\\
&\left.+\frac{(1+p)(|\alpha|(T-\eta)+|\beta|)T^2}{(1+q)(1+p+p^2)|\Omega|}\right\}\\
&=(\kappa|x|+N)\Lambda.
\end{aligned}
\tag{56}
$$

Taking norm $\sup_{t\in I_q^T}|x(t)|=\|x\|$ and solving it for $\|x\|$, this yields

$$\|x\|\le\frac{N\Lambda}{1-\kappa\Lambda}.\tag{57}$$

Let $R=(N\Lambda/(1-\kappa\Lambda))+1$, then (53) holds. This completes the proof ☐

4. Examples

In this section, we illustrate our main results with some examples. Let us consider the following boundary value problem of nonlinear second-order q-difference equations with three-point boundary conditions

$$
\begin{gathered}
D_{1/2}^2x(t)=f(t,x(t)),\qquad t\in I_{1/2}^{1/2}=I_{1/2}^1\cap\left[0,\frac{1}{2}\right],\\
\frac{2}{3}x\left(\frac{1}{8}\right)-\frac{1}{3}D_{3/4}x\left(\frac{1}{8}\right)=0,\qquad\int_0^{1/2}x(s)\,d_{1/4}s=0.
\end{gathered}
\tag{58}
$$

Here, we have $q=1/2$, $p=1/4$, $r=3/4$, $T=1/2$, $\alpha=2/3$, $\beta=-1/3$, and $\eta=1/8$. We find that

$$
\begin{aligned}
\Lambda&=\frac{T^2}{1+q}+\frac{pT}{|\Omega|}\left[\frac{|\alpha|\eta^2}{1+q}+|\beta|r\eta+\frac{|\beta|\eta(1+q(2+r))}{1+q}\right]\\
&+\frac{(1+p)(|\alpha|(T-\eta)+|\beta|)T^2}{(1+q)(1+p+p^2)|\Omega|}\\
&=\frac{1}{6}+\frac{6}{31}\left[\frac{1}{144}+\frac{1}{32}+\frac{19}{288}\right]+\frac{280}{1953}\\
&\approx0.33019713.
\end{aligned}
\tag{59}
$$

(a) Let $f:I_{1/2}^{1/2}\times\mathbb{R}\to\mathbb{R}$ be a continuous function given by

$$f(t,x)=\frac{e^{-\sin^2t}}{1+e^{\cos^2t}}\cdot\frac{|x(t)|}{1+|x(t)|}.\tag{60}$$

Since, $|f(t,x)-f(t,y)|\le(1/2)|x-y|$, then (H_1) is satisfied with $L=1/2$. We can find that

$$L\Lambda\approx0.16509857<1.\tag{61}$$

Hence, by Theorem 3, problem (58) with $f(t,x)$ given by (60) has a unique solution on $I_{1/2}^{1/2}$.

(b) If $f:I_{1/2}^{1/2}\times\mathbb{R}\to\mathbb{R}$ is a continuous function given by

$$f(t,x)=\frac{(t+1)|x|}{1+|x|}.\tag{62}$$

Choosing $h(t)=t+1$, we find that

$$
\begin{aligned}
G&=\int_0^T(T-qs)h(s)\,d_qs+\frac{pT}{|\Omega|}\\
&\times\Bigg[|\alpha|\int_0^{\eta}(\eta-qs)h(s)\,d_qs\\
&+|\beta|\int_0^{r\eta}(s+1)\,d_qs+\frac{|\beta|}{(1-r)\eta}\\
&\times\int_{r\eta}^{\eta}(\eta-qs)(s+1)\,d_qs\Bigg]\\
&+\frac{(1+p)(|\alpha|(T-\eta)+|\beta|)}{T|\Omega|}\\
&\times\int_0^T\int_0^s(s-qv)(v+1)\,d_qv\,d_ps\\
&\approx0.40987235.
\end{aligned}
\tag{63}
$$

Here,

$$\left|f(t,x)-f(t,y)\right| \leq \frac{(1+t)\left|x-y\right|}{0.40987235+\left|x-y\right|}. \qquad (64)$$

Therefore, by Theorem 6, the problem (58) with $f(t,x)$ given by (62) has a unique solution on $I_{1/2}^{1/2}$.

(c) Consider a continuous function $f : I_{1/2}^{1/2} \times \mathbb{R} \to \mathbb{R}$ given by

$$f(t,x) = \sin 2x + \frac{3}{e^{-x^2}+t+2}. \qquad (65)$$

We can show that

$$\left|f(t,x)\right| = \left|\sin 2x + \frac{3}{e^{-x^2}+t+2}\right| \leq 2\left\|x\right\| + \frac{3}{2}, \qquad (66)$$

with

$$\kappa = 2 < \frac{1}{\Lambda} \approx 3.02849389, \qquad (67)$$

and $N = 3/2$. By Theorem 11, the problem (58) with the $f(t,x)$ given by (65) has at least one solution on $I_{1/2}^{1/2}$.

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