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Research Article

Solvability of a Class of Generalized Neumann Boundary Value Problems for Second-Order Nonlinear Difference Equations

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This paper is motivated by Rachnkovab and Tisdell (2006) and Anderson et al. (2007). New sufficient conditions for the existence of at least one solution of the generalized Neumann boundary value problems for second order nonlinear difference equations $\nabla\Delta x(k) = f(k, x(k), x(k+1))$, $k \in [1, n-1]$, $x(0) = ax(1)$, $x(n) = bx(n-1)$, are established.

1. Introduction

Recently, there have been many papers discussed the solvability of two-point or multipoint boundary value problems for second-order or higher-order difference equations, we refer the readers to the text books [1, 2] and papers [3–8] and the references therein.

In a recent paper [3], Anderson et al. studied the following problem:

$$\begin{gathered}\nabla\Delta y(k) = f\big(k, y(k), \Delta y(k)\big), \quad k = 1, \ldots, n-1, \\ \Delta y(0) = \Delta y(n) = 0,\end{gathered} \tag{1.1}$$

where

$$\Delta y(k) = \begin{cases} y(k+1) - y(k), & \text{for } k = 0, \ldots, n-1, \\ 0, & \text{for } k = n, \end{cases}$$
$$\nabla \Delta y(k) = \begin{cases} y(k+1) - 2y(k) + y(k-1), & \text{for } k = 1, \ldots, n-1, \\ 0, & \text{for } k = 0 \text{ or } n. \end{cases} \tag{1.2}$$

The following result was proved.

Theorem ART

Suppose that f is continuous and there exist constants $\alpha \le 0$, $K \ge 0$ such that

$$\left| f(t,p,q) - p \right| \le \alpha \left[2pf(t,p,q) + q^2 \right] + K, \quad (t,p,q) \in \{1, \ldots, n-1\} \times R^2. \tag{$*$}$$

Then BVP(1.1) has at least one solution.

The methods in [3] involved new inequalities on the right-hand side of the difference equation and Schaefer's Theorem in the finite-dimensional space setting.

In [7], the following discrete boundary value problem (BVP) involving second order difference equations and two-point boundary conditions

$$\frac{\nabla \Delta y_k}{h^2} = f\left(t_k, y_k, \frac{\Delta y_k}{h}\right), \quad k = 1, \ldots, n-1,$$
$$y_0 = 0, \qquad y_n = 0, \tag{1.3}$$

was studied, where $n \ge 2$ an integer, f is continuous, scalar-valued function, the step size is $h = N/n$ with N a positive constant, the grid points are $t_k = kh$ for $k = 0, \ldots, n$. The differences are given by

$$\Delta y_k = \begin{cases} y_{k+1} - y_k, & k = 0, \ldots, n-1, \\ 0, & k = n, \end{cases}$$
$$\nabla \Delta y_k = \begin{cases} y_{k+1} - 2y_k + y_{k-1}, & k = 1, \ldots, n-1, \\ 0, & k = 0 \text{ or } k = n. \end{cases} \tag{1.4}$$

The following two results were proved in [7].

Theorem RT

Let f be continuous on $[0, N] \times R^2$ and $\alpha,\ \beta$, and, K be nonnegative constants. If there exist $c, d \in [0, 1)$ such that

$$\left|f(t,u,v)\right| \leq \alpha|u|^c + \beta|v|^d + K, \quad (t,u,v) \in [0,N] \times R^2, \tag{1.5}$$

then the discrete BVP(1.3) has at least one solution.

Theorem RT

Let f be continuous on $[0, N] \times R^2$ and α, β, and K nonnegative constants. If

$$\left|f(t,u,v)\right| \leq \alpha|u| + \beta|v| + K, \quad (t,u,v) \in [0,N] \times R^2, \tag{1.6}$$

$$\frac{\alpha N^2}{8} + \frac{\beta N}{2} < 1, \tag{1.7}$$

then the discrete BVP(1.3) has at least one solution.

In paper [8], Cabada and Otero-Espinar studied the existence of solutions of a class of nonlinear second-order difference problem with Neumann boundary conditions by using upper and lower solution methods. Assuming the existence of a pair of ordered lower and upper solutions γ and β, they obtained optimal existence results for the case $\gamma \leq \beta$ and even for $\gamma \geq \beta$.

In this paper, we study the following boundary value problem for second-order nonlinear difference equation

$$\begin{gathered} \nabla\Delta x(k) = f(k, x(k), x(k+1)), \quad k \in [1, n-1], \\ x(0) = ax(1), \qquad x(n) = bx(n-1), \end{gathered} \tag{1.8}$$

where $a, b \in R$, $n \geq 2$ is an integer, and f is continuous, scalar-valued function. We note that when $a = b = 1$, BVP(1.8) becomes the following BVP:

$$\begin{gathered} \nabla\Delta x(k) = f(k, x(k), x(k+1)), \quad t \in [1, T-1], \\ \Delta x(0) = 0 = \Delta x(n-1), \end{gathered} \tag{1.9}$$

which is called Neumann boundary value problem of difference equation and is a special case of BVP(1.1). When $a = b = 0$, BVP(1.8) is changed to

$$\begin{gathered} \nabla\Delta x(k) = f(k, x(k), x(k+1)), \quad t \in [1, T-1], \\ x(0) = 0 = x(n), \end{gathered} \tag{1.10}$$

which is the so-called Dirichlet problem for discrete difference equations and is a special case of BVP(1.3).

The purpose of this paper is to improve the assumptions $(*)$, (1.5), and (1.6) in the results in paper [3, 5, 7–9], by using Mawhin's continuation theorem of coincidence degree, to establish sufficient conditions for the existence of at least one solution of BVP(1.8). It is interesting that we allow f to be sublinear, at most linear or superlinear.

This paper is organized as follows. In Section 2, we make the main results, and in Section 3, we give some examples, which cannot be solved by theorems in [5, 7, 9], to illustrate the main results presented in Section 3.

2. Main Results

To get the existence results for solutions of BVP(1.8), we need the following fixed point theorems.

Let X and Y be Banach spaces, $L : D(L) \subset X \to Y$ a Fredholm operator of index zero, and $P : X \to X$, $Q : Y \to Y$ projectors such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$, $X = \operatorname{Ker} L \oplus \operatorname{Ker} P$, $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$. It follows that $L|_{D(L)\cap \operatorname{Ker} P} : D(L)\cap \operatorname{Ker} P \to \operatorname{Im} L$ is invertible; we denote the inverse of that map by K_p.

If Ω is an open bounded subset of X, $D(L) \cap \overline{\Omega} \neq \emptyset$, the map $N : X \to Y$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I - Q)N : \overline{\Omega} \to X$ is compact.

Lemma 2.1 (see [9]). *Let L be a Fredholm operator of index zero, and let N be L-compact on Ω. Assume that the following conditions are satisfied:*

(i) *$Lx \neq \lambda Nx$ for every $(x, \lambda) \in [(D(L) \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0, 1)$;*

(ii) *$Nx \notin \operatorname{Im} L$ for every $x \in \operatorname{Ker} L \cap \partial\Omega$;*

(iii) *$\deg(\wedge QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) \neq 0$, where $\wedge : \operatorname{Ker} L \to Y/\operatorname{Im} L$ is the isomorphism.*

Then the equation $Lx = Nx$ has at least one solution in $D(L) \cap \overline{\Omega}$.

Lemma 2.2 (see [9]). *Let X and Y be Banach spaces. Suppose $L : D(L) \subset X \to Y$ is a Fredholm operator of index zero with $\operatorname{Ker} L = \{0\}$, $N : X \to Y$ is L-compact on any open bounded subset of X. If $0 \in \Omega \subset X$ is an open bounded subset and $Lx \neq \lambda Nx$ for all $x \in D(L) \cap \partial\Omega$ and $\lambda \in [0, 1]$, then there is at least one $x \in \Omega$ so that $Lx = Nx$.*

Let $X = R^{n+1}$, $Y = R^{n-1}$ be endowed with the norms

$$\|x\| = \max_{n\in[0,n]} |x(n)|, \qquad \|y\| = \max_{k\in[1,n-1]} |y(k)| \tag{2.1}$$

for $x \in X$ and $y \in Y$, respectively. It is easy to see that X and Y are Banach spaces. Choose $D(L) = \{x \in X : x(0) = ax(1),\ x(n) = bx(n-1)\}$. Let $L : X \to Y$, $Lx(k) = \nabla\Delta x(k)$, $x \in D(L)$, and $N : X \to Y$ by $Nx(k) = f(k, x(k), x(k+1))$.

Consider the following problem:

$$\nabla\Delta x(k) = 0, \qquad x(0) = ax(1), \qquad x(n) = bx(n-1). \tag{2.2}$$

It is easy to see that problem (2.2) has a unique solution $x(k) = 0$ if and only if

$$(1-a)[(n-1)b-n] \neq a(1-b). \tag{2.3}$$

If (2.3) holds, we call BVP(1.8) at nonresonance case. If $(1-a)[(n-1)b-n] = a(1-b)$, then problem (2.2) has infinite nontrivial solutions. At this case, we call BVP(1.8) at resonance case. In this paper, we establish sufficient conditions for the existence of solutions of BVP(1.8) at nonresonance case, that is, $(1-a)[(n-1)b-n] \neq a(1-b)$, and at resonance case, $a = b = 1$. It is similar to get existence results for the existence of solutions at resonance case when $(1-a)[(n-1)b-n] = a(1-b)$ and $a \neq 1$, $b \neq 1$.

Lemma 2.3. *Suppose $a = b = 1$. Then the following results are valid.*

(i) $\operatorname{Ker} L = \{x = (c, \ldots, c) \in X : c \in R\}$.

(ii) $\operatorname{Im} L = \{y \in Y : \sum_{i=1}^{n-1} y(i) = 0\}$.

(iii) *L is a Fredholm operator of index zero.*

(iv) *There are projectors $P : X \to X$ and $Q : Y \to Y$ such that* $\operatorname{Ker} L = \operatorname{Im} P$, $\operatorname{Ker} Q = \operatorname{Im} L$. *Furthermore, let $\Omega \subset X$ be an open bounded subset with $\overline{\Omega} \cap D(L) \neq \emptyset$; then N is L-compact on $\overline{\Omega}$.*

(v) *$x \in D(L)$ is a solution of $L(x) = N(x)$ which implies that x is a solution of BVP*(1.8).

The projectors $P : X \to X$ and $Q : Y \to Y$, the isomorphism $\wedge : \operatorname{Ker} L \to Y/\operatorname{Im} L$, and the generalized inverse $K_p : \operatorname{Im} L \to D(L) \cap \operatorname{Im} P$ are as follows:

$$\begin{aligned}
Px(n) &= x(1), \\
Qy(n) &= \frac{1}{n-1}\sum_{i=1}^{n-1} y(i), \\
\wedge(c) &= c, \\
K_p y(n) &= \sum_{s=1}^{k}\sum_{i=1}^{s} y(i).
\end{aligned} \tag{2.4}$$

Lemma 2.4. *Suppose $(1-a)[(n-1)b-n] \neq a(1-b)$. Then the following results are valid.*

(i) *$x \in D(L)$ is a solution of $L(x) = N(x)$ which implies that x is a solution of BVP*(1.8).

(ii) $\operatorname{Ker} L = \{0\}$.

(iii) *L is a Fredholm operator of index zero, N is L-compact on each open bounded subset of X.*

Suppose

(A) there exist numbers $\beta > 0$, $\theta \geq 1$, nonnegative sequences $p(n)$, $q(n)$, $r(n)$, functions $g(n, x, y)$, $h(n, x, y)$ such that $f(n, x, y) = g(n, x, y) + h(n, x, y)$ and

$$\begin{aligned}
g(n, x, y)x &\geq \beta |x|^{\theta+1}, \\
|h(n, x, y)| &\leq p(n)|x|^{\theta} + q(n)|y|^{\theta} + r(n),
\end{aligned} \tag{2.5}$$

for all $n \in \{1, \ldots, n-1\}$, $(x, y) \in R^2$;

(B) there exists a constant $M > 0$ so that

$$c\left[\sum_{i=1}^{n-1} f(n,c,c)\right] > 0 \tag{2.6}$$

for all $|c| > M$ or

$$c\left[\sum_{i=1}^{n-1} f(n,c,c)\right] < 0 \tag{2.7}$$

for all $|c| > M$.

Theorem L

Suppose $a^2 \leq 1,\ b^2 \leq 1$, and that (A) and (B) hold. Then BVP(1.8) has at least one solution if

$$\|p\| + \|q\| \max\left\{|b|^{\theta+1}, 1\right\} < \beta. \tag{2.8}$$

Proof. To apply Lemma 2.1, we consider $Lx = \lambda Nx$ for $\lambda \in [0,1]$.

Step 1. Let $\Omega_1 = \{x \in X : Lx = \lambda Nx, \lambda \in [0,1]\}$. For $x \in \Omega_1$, we have

$$\begin{gathered}
x(k+1) - 2x(k) + x(k-1) = \lambda f(k, x(k), x(k+1)), \quad k \in [1, n-1], \\
x(0) = ax(1), \\
x(n) = bx(n-1).
\end{gathered} \tag{2.9}$$

So

$$[x(k+1) - 2x(k) + x(k-1)]x(k) = \lambda f(k, x(k), x(k+1))x(k), \quad k \in [1, n-1]. \tag{2.10}$$

It is easy to see that

$$\begin{aligned}
&2\sum_{n=1}^{n-1}[x(k+1) - 2x(k) + x(k-1)]x(k) \\
&\quad = \sum_{n=1}^{n-1}\Big(-[x(k+1)]^2 + 2x(k)x(k+1) - [x(k)]^2 - [x(k-1)]^2 + 2x(k-1)x(k) - [x(k)]^2 \\
&\qquad\qquad + [x(k+1)]^2 - 2[x(k)]^2 + [x(k-1)]^2\Big) \\
&\quad = \sum_{n=1}^{n-1}\Big(-[x(k+1) - x(k)]^2 - [x(k-1) - x(k)]^2 + [x(k+1)]^2 - 2[x(k)]^2 + [x(k-1)]^2\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{n-1}\left(-[x(k+1)-x(k)]^2-[x(k-1)-x(k)]^2\right) \\
&\quad + \left([x(n)]^2-[x(n-1)]^2-[x(1)]^2+[x(0)]^2\right) \\
&= \sum_{n=1}^{n-1}\left(-[x(k+1)-x(k)]^2-[x(k-1)-x(k)]^2\right) \\
&\quad + \left(\left(b^2-1\right)[x(n-1)]^2+\left(a^2-1\right)[x(1)]^2\right).
\end{aligned}
\tag{2.11}
$$

Since $a^2 \leq 1,\ b^2 \leq 1$, we get

$$
\sum_{n=1}^{n-1}[x(k+1)-2x(k)+x(k-1)]x(k) \leq 0. \tag{2.12}
$$

So, we get

$$
\lambda\sum_{n=1}^{n-1} f(k,x(k),x(k+1))x(k) \leq 0. \tag{2.13}
$$

Then

$$
\sum_{n=1}^{n-1}\left[g(k,x(k),x(k+1))+h(k,x(k),x(k+1))\right]x(k) \leq 0. \tag{2.14}
$$

It follows that

$$
\begin{aligned}
\beta\sum_{k=1}^{n-1}|x(k)|^{\theta+1} &\leq -\sum_{n=1}^{n-1}h(k,x(k),x(k+1))x(k) \\
&\leq \sum_{n=1}^{n-1}\left[p(k)|x(k)|^{\theta+1}+q(k)|x(k+1)|^{\theta}|x(k)|+r(k)|x(k)|\right] \\
&\leq \|p\|\sum_{n=1}^{n-1}|x(k)|^{\theta+1}+\|q\|\sum_{k=1}^{n-1}|x(k+1)|^{\theta}|x(k)|+\sum_{k=1}^{n-1}r(k)|x(k)|.
\end{aligned}
\tag{2.15}
$$

For $x_i \geq 0,\ y_i \geq 0$, Holder's inequality implies

$$
\sum_{i=1}^{s}x_iy_i \leq \left(\sum_{i=1}^{s}x_i^p\right)^{1/p}\left(\sum_{i=1}^{s}y_i^q\right)^{1/q}, \quad \frac{1}{p}+\frac{1}{q}=1,\ q>0,\ p>0. \tag{2.16}
$$

It follows that

$$\begin{aligned}
&\beta\sum_{k=1}^{n-1}|x(k)|^{\theta+1}\\
&\quad\leq \|p\|\sum_{n=1}^{n-1}|x(k)|^{\theta+1}+\|q\|\left(\sum_{k=1}^{n-1}|x(k+1)|^{\theta+1}\right)^{\theta/(\theta+1)}\left(\sum_{k=1}^{n-1}|x(k)|^{\theta+1}\right)^{1/(\theta+1)}\\
&\qquad+\|r\|\sum_{k=1}^{n-1}|x(k)|\\
&\quad= \|r\|(n-1)^{\theta/(\theta+1)}\left(\sum_{k=1}^{n-1}|x(k)|^{\theta+1}\right)^{1/(\theta+1)}+\|p\|\sum_{n=1}^{n-1}|x(k)|^{\theta+1}\\
&\qquad+\|q\|\left(|b|^{\theta+1}|x(1)|^{\theta+1}+\sum_{k=1}^{n-2}|x(k+1)|^{\theta+1}\right)^{\theta/(\theta+1)}\left(\sum_{k=1}^{n-1}|x(k)|^{\theta+1}\right)^{1/(\theta+1)}\\
&\quad\leq \|r\|(n-1)^{\theta/(\theta+1)}\left(\sum_{k=1}^{n-1}|x(k)|^{\theta+1}\right)^{1/(\theta+1)}+\|p\|\sum_{n=1}^{n-1}|x(k)|^{\theta+1}\\
&\qquad+\|q\|\max\left\{|b|^{\theta+1},1\right\}\left(\sum_{k=1}^{n-1}|x(k)|^{\theta+1}\right)^{\theta/(\theta+1)}\left(\sum_{k=1}^{n-1}|x(k)|^{\theta+1}\right)^{1/(\theta+1)}\\
&\quad= \|r\|(n-1)^{\theta/(\theta+1)}\left(\sum_{k=1}^{n-1}|x(k)|^{\theta+1}\right)^{1/(\theta+1)}+\|p\|\sum_{n=1}^{n-1}|x(k)|^{\theta+1}\\
&\qquad+\|q\|\max\left\{|b|^{\theta+1},1\right\}\sum_{k=1}^{n-1}|x(k)|^{\theta+1}.
\end{aligned} \tag{2.17}$$

It follows from (2.8) that there exists a constant $M_1 > 0$ such that

$$\sum_{k=1}^{n-1}|x(k)|^{\theta+1}\leq M_1. \tag{2.18}$$

Hence $|x(k)| \leq (M_1/(n-1))^{1/(\theta+1)}$ for all $k \in \{1,\dots,n-1\}$. Hence $\|x\| \leq (M_1/(n-1))^{1/(\theta+1)}$. So Ω_1 is bounded.

Step 2. Prove that the set $\Omega_2 = \{x \in \operatorname{Ker} L : N(x) \in \operatorname{Im} L\}$ is bounded.

For $x \in \operatorname{Ker} L$, we have $x(k) = c$ for $k \in [0,n]$. Thus we have $Nx(k) = f(k,c,c)$. $N(x,y) \in \operatorname{Im} L$ implies that

$$\sum_{k=1}^{n-1} f(n,c,c) = 0. \tag{2.19}$$

It follows from condition (B) that $|c| \leq M$. Thus Ω_2 is bounded.

Step 3. Prove the set $\Omega_3 = \{x \in \operatorname{Ker} L : \pm\lambda \wedge (x) + (1-\lambda)QN(x) = 0,\ \exists\lambda \in [0,1]\}$ is bounded.

If the first inequality of (B) holds, let

$$\Omega_3 = \{x \in \operatorname{Ker} L : \lambda \wedge (x) + (1-\lambda)QN(x) = 0,\ \exists\lambda \in [0,1]\}. \tag{2.20}$$

We will prove that Ω_3 is bounded. For $x(k) = c$ for $k \in [0,n]$ such that $x \in \Omega_3$, and $\lambda \in [0,1]$, we have

$$-(1-\lambda)\sum_{k=1}^{n-1} f(n,c,c) = \lambda c(n-1). \tag{2.21}$$

If $\lambda = 1$, then $c = 0$. If $\lambda \neq 1$, then

$$0 > -(1-\lambda)c\sum_{k=1}^{n-1} f(n,c,c) = \lambda c^2 T \geq 0, \tag{2.22}$$

a contradiction.

If the second inequality of (B) holds, let

$$\Omega_3 = \{x \in \operatorname{Ker} L : -\lambda \wedge (x) + (1-\lambda)QN(x) = 0,\ \exists\lambda \in [0,1]\}. \tag{2.23}$$

Similarly, we can get a contradiction. So Ω_3 is bounded.

Step 4. Obtain open bounded set Ω such (i), (ii), and (iii) of Lemma 2.1.

In the following, we will show that all conditions of Lemma 2.1 are satisfied. Set Ω an open bounded subset of X such that $\Omega \supset \bigcup_{i=1}^{3} \overline{\Omega_i}$. We know that L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$. By the definition of Ω, we have $\Omega \supset \overline{\Omega_1}$ and $\Omega \supset \overline{\Omega_2}$, thus $L(x) \neq \lambda N(x)$ for $x \in (D(L) \setminus \operatorname{Ker} L) \cap \partial\Omega$ and $\lambda \in (0,1)$; $N(x) \notin \operatorname{Im} L$ for $x \in \operatorname{Ker} L \cap \partial\Omega$.

In fact, let $H(x,\lambda) = \pm\lambda \wedge (x) + (1-\lambda)QN(x)$. According the definition of Ω, we know $\Omega \supset \overline{\Omega_3}$, thus $H(x,\lambda) \neq 0$ for $x \in \partial\Omega \cap \operatorname{Ker} L$, thus by homotopy property of degree,

$$\begin{aligned}
\deg(QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) &= \deg(H(\cdot,0), \Omega \cap \operatorname{Ker} L, 0) \\
&= \deg(H(\cdot,1), \Omega \cap \operatorname{Ker} L, 0) \\
&= \deg(\pm\wedge, \Omega \cap \operatorname{Ker} L, 0) \neq 0.
\end{aligned} \tag{2.24}$$

Thus by Lemma 2.1, $L(x) = N(x)$ has at least one solution in $D(L) \cap \overline{\Omega}$, which is a solution of BVP(1.8). The proof is completed. □

Theorem L

Suppose $a^2 \leq 1$, $b^2 \leq 1$, $(1-a)[(n-1)b-n] \neq a(1-b)$, and that (A) holds. Then BVP(1.8) has at least one solution if (2.8) holds.

Proof. To apply Lemma 2.2, we consider $Lx = \lambda Nx$ for $\lambda \in [0,1]$. Let $\Omega_1 = \{x \in X : Lx = \lambda Nx, \lambda \in [0,1]\}$. For $x \in \Omega_1$, we get (2.9) and (2.10). using the methods in the proof of Theorem LX1, we get that Ω_1 is bounded. Let Ω be a nonempty open bounded subset of X such that $\Omega \supset \overline{\Omega_1}$ centered at zero. It is easy to see that L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$. One can see that $Lx \neq \lambda Nx$ for all $x \in D(L) \cap \partial\Omega$ and $\lambda \in [0,1]$. Thus, from Lemma 2.2, $Lx = Nx$ has at least one solution $x \in D(L) \cap \overline{\Omega}$, so x is a solution of BVP(1.8). The proof is complete. □

3. An Example

In this section, we present an example to illustrate the main results in Section 2.

Example 3.1. Consider the following problem:

$$\begin{gathered}
x(k+1) - 2x(k) + x(k-1) = \beta[x(k)]^{2m+1} + p(k)[x(k)]^{2m+1} + q(k)[x(k+1)]^{2m+1} + r(k), \\
k \in [1, n-1], \\
x(0) = ax(1), \\
x(n) = bx(n-1),
\end{gathered} \tag{3.1}$$

where $n \geq 2$, $m \geq 1$ are integers and $\beta > 0$, $p(n)$, $q(n)$, $r(n)$ are sequences. Corresponding to the assumptions of Theorem L1, we set

$$\begin{gathered}
f(k,x,y) = \beta x^{2m+1} + p(k)x^{2m+1} + q(k)y^{2m+1} + r(k), \\
g(k,x,y) = \beta x^{2m+1}, \\
h(k,x,y) = p(k)x^{2m+1} + q(k)y^{2m+1} + r(k),
\end{gathered} \tag{3.2}$$

and $\theta = 2m+1$. It is easy to see that (A) holds, and

$$f(n,c,c) = c^{2m+1}\beta + p(k)c^{2m+1} + q(k)c^{2m+1} + r(k) \tag{3.3}$$

implies that there is $M > 0$ such that $c\sum_{i=1}^{n-1}[c^{2m+1}\beta + p(k)c^{2m+1} + q(k)c^{2m+1} + r(k)] > 0$ for all $n \in [1, n-1]$ and $|c| > M$.

It follows from Theorem L2 that (3.1) has at least one solution if $a^2 \leq 1$, $b^2 \leq 1$, $(1-a)[(n-1)b - n] \neq a(1-b)$ and $\|p\| + \|q\| \max\{|b|^{\theta+1}, 1\} < \beta$. BVP(3.1) has at least one solution if $a = b = 1$ and $\|p\| + \|q\| \max\{|b|^{\theta+1}, 1\} < \beta$.

Remark 3.2. It is easy to see that BVP(3.1) when $a = b = 0$ cannot be solved by using theorems obtained in paper [7]. BVP(3.1) when $a = b = 1$ cannot be solved by the results obtained in paper [3].

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