Polyhedral hyperbolic metrics on surfaces

François Fillastre

Abstract Let S be a topologically finite surface, and g be a hyperbolic metric on S with a finite number of conical singularities of positive singular curvature, cusps and complete ends of infinite area. We prove that there exists a convex polyhedral surface P in hyperbolic space $\mathbb{H}^3$ and a group G of isometries of $\mathbb{H}^3$ such that the induced metric on the quotient P/G is isometric to g. Moreover, the pair (P, G) is unique among a particular class of convex polyhedra.

Keywords Hyperbolic generalized polyhedra · Equivariant polyhedral realization · Complete hyperbolic metrics · Alexandrov Theorem · Hyperbolic–de Sitter space

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1 Introduction

1.1 Statements

In all the text, $\overline{S}$ is a compact oriented surface of genus g, and the surface S is obtained from the surface $\overline{S}$ by removing $(n+p)$ points and m closed discs. The surface S is said to be of type $(g, n+p, m)$, and we require that S can be endowed with a hyperbolic metric, that is:

$$2g - 2 + n + p + m > 0.$$

We consider on S hyperbolic metrics with n conical singularities of positive curvature, p cusps and m complete hyperbolic ends of infinite area.

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A polyhedron of the hyperbolic space $\mathbb{H}^3$ is *generalized* if some of its vertices lie "outside" $\mathbb{H}^3$ (see Sect. 2 for precise definitions). An *invariant polyhedron* of $\mathbb{H}^3$ is a pair (P, G) where P is a polyhedron and G a discrete group of isometries of $\mathbb{H}^3$ such that $G(P) = P$. Let g be a metric on S. If there exists an invariant polyhedron (P, G) such that the induced metric on $\partial P/G$ is isometric to (S, g), we say that (P, G) *realizes* the metric g. In this paper we prove:

Theorem A *Each hyperbolic metric on S with conical singularities of positive singular curvature, cusps and complete ends of infinite area can be realized by a unique convex generalized hyperbolic polyhedron invariant under the action of a group of isometries acting freely cocompactly on a totally umbilical surface.*

Theorem A can be reformulated as the following three results.

Theorem 1.1 *Suppose S has genus* 0. *Then each hyperbolic metric on S with conical singularities of positive singular curvature, cusps and complete ends of infinite area can be realized by a unique convex generalized hyperbolic polyhedron* (*with a finite number of vertices*).

A *parabolic group* is a discrete group of isometries of $\mathbb{H}^3$ acting freely cocompactly on a horosphere. A *parabolic polyhedron* is an invariant polyhedron (P, G) where G is a parabolic group.

Theorem B *Suppose S has genus* 1. *Then each hyperbolic metric on S with conical singularities of positive singular curvature, cusps and complete ends of infinite area can be realized by a unique convex generalized hyperbolic parabolic polyhedron.*

A *Fuchsian group* is a discrete group of isometries of $\mathbb{H}^3$ acting freely cocompactly on a totally geodesic plane. A *Fuchsian polyhedron* is an invariant polyhedron (P, G) where G is a Fuchsian group.

Theorem B′ *Suppose S has genus* > 1. *Then each complete hyperbolic metric on S with conical singularities of positive singular curvature, cusps and complete ends of infinite area can be realized by a unique convex generalized hyperbolic Fuchsian polyhedron.*

Theorem 1.1 is already known (references are given below). It follows that in this paper we will prove Theorem B and Theorem B′.

Remark In the statements above, uniqueness must be understood as the uniqueness among the class of convex polyhedra described in the statements. Otherwise the statements are false as it is easy to construct other examples of invariant (convex) polyhedra realizing hyperbolic metrics on S. For example one can consider polyhedra invariant under the action of a group of loxodromic isometries acting cocompactly on a surface at constant distance from a geodesic for genus 1, or polyhedra invariant under the action of a quasi-Fuchsian group giving a convex cocompact metric for genus >1. Other examples are provided by groups acting non-cocompactly on the hyperbolic plane. Uniqueness is of course also meant up to congruences.

1.2 Plan of the paper and sketch of the proof

In the remainder of this section, we will give references about particular cases of these statements which are already known, and conclude on some related problems.

To prove Theorems B and B′ we will use the so-called Alexandrov method, or deformation method, or continuity method. It is an adaptation of the method used by Alexandrov in the proof of its famous theorem about the induced metric on the boundary of convex Euclidean polytopes [2]. The general idea is to endow with a suitable topology both the space of polyhedra and the space of metrics, and to use topological arguments to prove that the map given by the induced metric on the polyhedra is a homeomorphism. Actually the topological result lying behind the proof is the Domain Invariance Theorem, see [3].

In Sect. 2 we introduce the "hyperbolic-de Sitter space" in order to describe convex parabolic and Fuchsian generalized polyhedra, that will lead to a parameterization of the spaces of polyhedra. In Sect. 3 we prove a result about infinitesimal rigidity of the polyhedra. It will correspond to the local injectivity of the map "induced metric", that will be introduced in Sect. 5. In this section we also need to prove the properness of this map. In Sect. 4 we parameterize the spaces of metrics with the help of the Teichmüller space. Finally in Sect. 6 we collect all the results above to get first the proofs of Theorems B and B′ and then the proof of Theorem A.

Remark Until now, polyhedral realization statements were usually proved using the Alexandrov method, which relies on a local injectivity statement (sometimes given by a global injectivity statement). There exists a recent method to prove polyhedral realization theorems, called variational method. We refer to [7,13,14,19] for more details. This method does not require a local injectivity statement, and furthermore this one is obtained as a corollary of the proof. In [13], together with Ivan Izmestiev we proved the particular case of Theorem B considering only conical singularities. We used the variational method and then got a local injectivity result for this case. The main idea in the present paper is to note that the local injectivity result needed to prove Theorem B can be obtained in a simple way as a consequence of the one of [13] (Sect. 3).

1.3 Known cases and related results

In the case of genus 0, if the metric has only conical singularities of positive curvature, Theorem 1.1 is the hyperbolic version of the famous Alexandrov Theorem cited above. The case with only cusps was proved in [26]—this reference also contains the uniqueness part of Theorem 1.1. The proof of Theorem 1.1 is contained in [33]. Actually the results in this reference are much more general, see below. For genus > 1, the case with only cusps is done in [36] and the case with cusps and ends of infinite area is done in [35]. The case with only conical singularities of positive curvature is the subject of [12]. These three results are proved using the Alexandrov method, but the way to prove the local injectivity lies on volume of simplices and the Schäfli formula in the two firsts and on the so-called infinitesimal Pogorelov map in the other. The present paper provides another proof of these results. Note that the statement of Theorem B′ contains the case of hyperbolic (smooth) metrics on compact surfaces. In this case the Fuchsian polyhedron (P, G) must be seen as degenerated: P is the totally geodesic plane fixed by G. The theorem then just says that any compact hyperbolic surface has the hyperbolic plane as universal cover. We don't prove this result again here, so we will always assume that $n + m + p > 0$. Concerning the torus, I only know the case with conical singularities done in [13].

Hyperideal convex polyhedra with finite number of vertices (that is with all vertices lying "outside" $\mathbb{H}^3$) were studied in [6], in order to characterize them by their dihedral angles. This kind of characterization is studied for Fuchsian hyperideal convex polyhedra in [35] and [30]. Partial results on uniqueness were found in [6,25,27]. Such problems are in a certain sense "dual" to the results proved here, and strongly related to the Andreev Theorem [4,5,16,29,39]. We refer to these references for more details.

1.4 Some open questions

The study of convex polyhedra in hyperbolic space is related to the study of hyperbolic 3-manifolds with convex boundary. Particular case of Theorem 1.1 when the metrics have only cone singularities (i.e. Alexandrov Theorem) is a part of the following question:

Question 1 *Let M be a compact connected* 3*-manifold with boundary, and let M admit a complete hyperbolic convex cocompact metric. Can each hyperbolic cone metric on ∂M with singularities of positive curvature be uniquely extended to a hyperbolic metric on M with convex polyhedral boundary?*

A similar question can be asked for metrics with "ideal" or "hyperideal" boundary. We refer to [35,36] for precise definitions and statements. Theorem B′ provides an example of a weaker statement for all these configurations in the case of "Fuchsian manifolds". Theorem B can be seen as the most simple extension of those questions to non-compact manifolds. The analogous of Question 1 in the case of manifolds with smooth strictly convex boundary was done in [37].

Another way to extend Theorem A would be to study analogous polyhedra in Lorentzian space-forms. The closer to this paper would be to study them in the "hyperbolic-de Sitter space" (see Sect. 2 for a definition). Our proof of the local injectivity (Sect. 3) remains true in this wider case. Then it would remain to parameterize spaces of polyhedra and spaces of metrics, that is a bit more delicate than in our hyperbolic case, as the induced metric on such polyhedra can be Riemannian, Lorentzian or degenerated on different faces and edges. For closed polyhedral surfaces with a finite number of vertices, this is done in [33,34]. Closed polyhedral surfaces with a finite number of vertices in Minkowski space are studied in [34]. It seems that it does not exist yet similar results in the anti-de Sitter space. Space-like convex Fuchsian polyhedra in Minkowski and anti-de Sitter spaces are studied in [11,36,38]. It is possible that there exists convex Fuchsian polyhedra in these spaces for which the induced metric is not everywhere space-like (for example it may contain light-like edges). Convex parabolic polyhedra can be defined in the anti-de Sitter space, but they cannot be space-like.

2 Spaces of polyhedra

2.1 HS-polyhedra

We denote by $\mathbb{R}^4_1$ the Minkowski space of dimension 4, that is the space $\mathbb{R}^4$ endowed with the bilinear form $\langle \cdot, \cdot \rangle_1$ represented by:

$$J := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};$$

it is a flat complete Lorentzian manifold. The hyperbolic space can be seen as the upper-branch of the unitary two-branched hyperboloid:

$$\mathbb{H}^3 = \{x \in \mathbb{R}^4_1 | \, \|x\|^2_1 = -1, \; x_4 > 0\};$$

and the de Sitter space is the unitary one-branched hyperboloid:

$$\mathrm{dS}^3 = \{x \in \mathbb{R}^4_1 | \, \|x\|^2_1 = 1\};$$

both endowed with the induced metric. De Sitter space is a complete simply connected Lorentzian manifold of constant curvature 1 diffeomorphic to $\mathbb{S}^2 \times \mathbb{R}$. We refer to [23] for more details about Lorentzian geometry. The geodesics of the hyperboloids are given by their intersection with the vector planes of $\mathbb{R}^4_1$.

Let us project homeomorphically the hyperboloids of $\mathbb{R}^4_1$ along lines onto the Euclidean unit sphere $\mathbb{S}^3$. We denote by $\mathbb{H}_+$ the image of upper-part of the two-branched hyperboloid (the usual hyperbolic space), and by $\mathbb{H}_-$ the image of the other branch of the hyperboloid. The spheres S_+ and S_- in $\mathbb{S}^3$ delimiting respectively $\mathbb{H}_+$ and $\mathbb{H}_-$ are the images of the light-cone under the projection (S_+ corresponds to the usual boundary at infinity of the hyperbolic space). The image under the projection of the de Sitter space is exactly $\mathbb{S}^3$ less the closures of $\mathbb{H}_+$ and $\mathbb{H}_-$ for the topology of the sphere.

We call *hyperbolic-de Sitter space*, and denote by $\widetilde{\mathrm{HS}}^3$, the sphere $\mathbb{S}^3$ less the spheres S_+ and S_- endowed with the hyperbolic and de Sitter distances induced by the projection described above. Actually it is possible to define this space as a "metric" space, i.e. to define a "distance" between a point in de Sitter space and a point in hyperbolic space, but we do not need it. See [33,34] for more details. The spheres S_+ and S_- are the two components of the *boundary at infinity* $\partial_\infty \widetilde{\mathrm{HS}}^3$. The intersection of a surface with the boundary at infinity is called the *boundary at infinity* of the surface. In this model, the geodesics correspond to the great circles, and for the de Sitter geodesics, the like-type of a geodesic depends if it intersects or not $\mathbb{H}_+$ (or $\mathbb{H}_-$, that is the same): it is space-like if it does not intersect $\mathbb{H}_+$, it is time-like if it intersects $\mathbb{H}_+$ and light-like if it is tangent to S_+ (or S_-, that is the same).

We denote by $\widetilde{\mathrm{HS}}^3_+$ the upper half part of $\widetilde{\mathrm{HS}}^3$ (its intersection with $\{x_4 > 0\}$ in Minkowski space). There exists a more usual model of $\widetilde{\mathrm{HS}}^3_+$, called the *Klein projective model*, and given by the projection $x \mapsto x/x_4$ in $\mathbb{R}^4_1$ of $\mathbb{H}^3$ and dS^3_+ (the half upper part of dS^3) onto the hyperplane $\{x_4 = 1\}$, which is identified with the Euclidean space $\mathbb{R}^3$. This is equivalent to project $\widetilde{\mathrm{HS}}^3_+$ onto $\{x_4 = 1\}$.

Isometries of both hyperbolic and de Sitter spaces are restriction to the hyperboloids of the linear isometries of $\mathbb{R}^4_1$, which form the Lorentz group $\mathcal{L}$. The Lorentz group is the group of isometries of $\widetilde{\mathrm{HS}}^3$. Note that the antipodal map is an isometry of $\widetilde{\mathrm{HS}}^3$. We will consider two kinds of such isometries:

- *null-rotations*, whose restriction to hyperbolic space correspond to parabolic isometries. Each of them (pointwise) fixes a unique light-like vector of $\mathbb{R}^4_1$.
- *boosts*, whose restriction to hyperbolic space correspond to hyperbolic isometries. Each of them leaves invariant two light-like vectors as well as the time-like plane containing them.

We also consider the associated invariant surfaces:

- *horospheres* are the connected surfaces of $\widetilde{\mathrm{HS}}^3$ leaved invariant by all the null-rotations fixing a same light-like vector ℓ.
- *caps* are the connected surfaces of $\widetilde{\mathrm{HS}}^3$ leaved invariant by all the boosts fixing a same time-like hyperplane $P_{\mathbb{H}^2}$.

As well as the groups acting on them:

- a *parabolic group* is a discrete group of isometries of $\widetilde{\mathrm{HS}}^3$ acting freely cocompactly on a horosphere. It contains only null-rotations.
- a *Fuchsian group* is a discrete group of isometries of $\widetilde{\mathrm{HS}}^3$ acting freely cocompactly on a cap. It contains only boosts.

Moreover we include in the definitions of horospheres and caps the parts of the boundary at infinity of $\widetilde{\mathrm{HS}}^3$ leaved invariant by the corresponding isometries.

Definition 2.1

- A *convex polyhedron* in a space-form M is an intersection of half-spaces of M. The number of half-spaces may be infinite, but the intersection is asked to be locally finite: each face must be a polygon with a finite number of vertices, and the number of edges at each vertex must be finite.
- A *convex HS-polyhedron* is a subset of $\widetilde{\mathrm{HS}}^3 \cup \partial_\infty \widetilde{\mathrm{HS}}^3$ that corresponds to a convex polyhedron of $\mathbb{S}^3$.
- A *convex parabolic HS-polyhedron* is a pair (P,G) where P is a convex *HS*-polyhedron, G is a parabolic group and $G(P)=P$.
- A *convex Fuchsian HS-polyhedron* is a pair (P,G) where P is a convex *HS*-polyhedron, G is a Fuchsian group and $G(P)=P$.
- A *convex generalized hyperbolic polyhedron* P is the intersection of $\mathbb{H}^3$ with a convex *HS*-polyhedron such that all the edges of P meet $\mathbb{H}^3$.

The definitions of convex parabolic generalized hyperbolic polyhedron and of convex Fuchsian generalized hyperbolic polyhedron are then obvious. A convex generalized hyperbolic polyhedron has three kinds of vertices: *finite vertices* which are in $\mathbb{H}^3$, *ideal vertices* (or sometimes *infinite*) which are on $\partial_\infty \mathbb{H}^3$ and *hyperideal vertices* which are outside $\mathbb{H}^3$. The hyperideal case contains the ideal case, and a vertex which is hyperideal but not ideal is called *strictly hyperideal*. We will speak about *convex umbilical HS-polyhedron* when we speak in the same time about parabolic and Fuchsian polyhedra and about *umbilical group* when we speak in the same time about parabolic and Fuchsian groups.

We want to prove that, up to a global isometry, the convex umbilical HS-polyhedra can be bijectively projected onto the Klein projective model. It means that there exists a global isometry (of the Lorentz group) which sends them into $\widetilde{\mathrm{HS}}^3_+$. It will allow us to parameterize the polyhedra with the help of the Euclidean coordinates of their vertices. As the polyhedra we consider are convex sets in $\mathbb{S}^3$, they are contained in a half-space, but we must check that the hyperplane delimiting this half-space is space-like. This property is clear for closed convex generalized hyperbolic polyhedra with a finite number of vertices, it is why such polyhedra are usually defined directly in the Klein projective model. But it is easy to construct closed convex HS-polyhedron with a finite number of vertices which cannot be bijectively projected into the Klein projective model. They are studied in [34].

2.2 Parabolic polyhedra

Let G be a parabolic group. A horosphere H leaved invariant by G is given by the intersection (supposed non-empty) in the Minkowski space of a unitary hyperboloid with an affine light-like hyperplane of $\mathbb{R}^4_1$. If G fixes the light-like vector ℓ, the affine light-like hyperplane is parallel to the light-like hyperplane L containing ℓ (actually L is the orthogonal $\ell^\perp$ of ℓ for the bilinear form of the Minkowski space). The vector ℓ gives in $\mathbb{S}^3$ a point ℓ_+ on S_+ and a point ℓ_- on S_- which are antipodal. The boundary at infinity of a horosphere is either ℓ_+ either ℓ_-, depending if the hyperplane giving H lies above or below L. The boundary at infinity of H is called the *center* of H. It follows that in the de Sitter space there exists two families of antipodal horospheres constructed from a light-like vector ℓ: the ones centered at ℓ_+ and the ones centered at ℓ_-, see Fig. 1. Remember that we also consider $S_+\setminus\{\ell_+\}$ and $S_-\setminus\{\ell_-\}$ as horospheres leaved invariant by G. The following lemma is straightforward as G has no fixed points in $\widetilde{\mathrm{HS}}^3$.

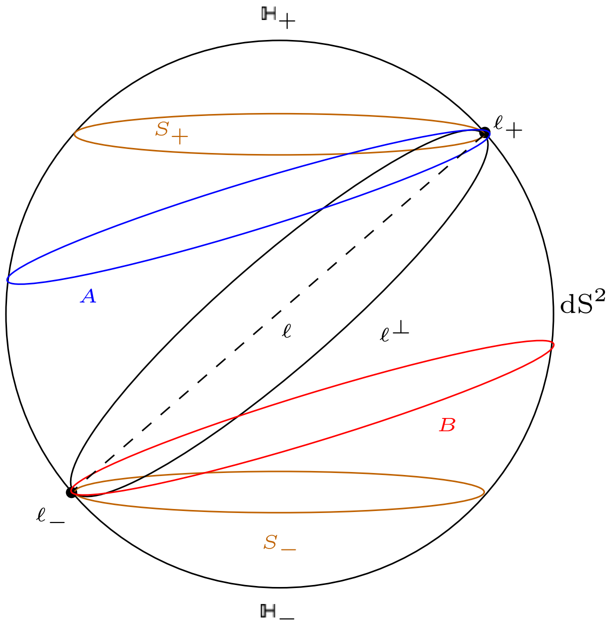

Fig. 1 In the spherical projective model, A is a horosphere of center ℓ_+ and B is a horosphere of center ℓ_- (drawn with one dimension less than the text)

Lemma 2.2 *Let G be a parabolic group which fixes ℓ and let x be a point of $\mathbb{S}^3$. Then the accumulating set of Gx in $\mathbb{S}^3$ is either ℓ_+ either ℓ_- if $x \notin \ell^\perp$. If $x \in \ell^\perp$, the accumulating set is constituted by both ℓ_+ and ℓ_-.*

We denote by $\widetilde{\mathrm{HS}}^3_\ell$ the intersection of the de Sitter space with the (closed) half-space delimited by $\ell^\perp$. The half-space is chosen such that it contains the hyperbolic space $\mathbb{H}_+$.

Lemma 2.3 *Up to a global isometry a convex parabolic HS-polyhedron (P, G) is contained in $\widetilde{\mathrm{HS}}^3_\ell$.*

Proof The polyhedron (P, G) is the convex hull of the union of finitely many orbits of the group G. If the polyhedron is constituted with the orbit of one single point, this one belongs either to a horosphere centered at ℓ_+ either to a horosphere centered at ℓ_- either to $\ell^\perp$, and we are done. We can consider that the polyhedron has at least as vertices the orbit of two points x and y and we suppose that they are living on horospheres in different sides of $\ell^\perp$. As P is convex, there exists a totally geodesic plane M of $\mathbb{S}^3$ such that P is entirely contained in one side of M. As ℓ_+ and ℓ_- are antipodal, they belong to different sides of M (M cannot be $\ell^\perp$ because x and y live in different sides of $\ell^\perp$). But from Lemma 2.2 there exists points in the orbits of x and y as near as ℓ_+ and ℓ_- as we want for the topology of $\mathbb{S}^3$. This contradicts the convexity of P. □

Lemma 2.4 *Up to a global isometry a convex parabolic generalized hyperbolic polyhedron (P, G) can be bijectively projected onto the Klein projective model. Its image is a convex Euclidean polyhedron with vertices lying on ellipsoids of center $(1 - r^2, 0, 0)$ and of radii (r^2, r, r), where r is a positive real number. The vertices accumulate on the point of tangency of the ellipsoids with the unit sphere.*

Proof From Lemma 2.3, up to a global isometry (P, G) is entirely contained in $\widetilde{\mathrm{HS}}^3_\ell$. As (P, G) is now required to have all its edges meeting the hyperbolic space, then it cannot have

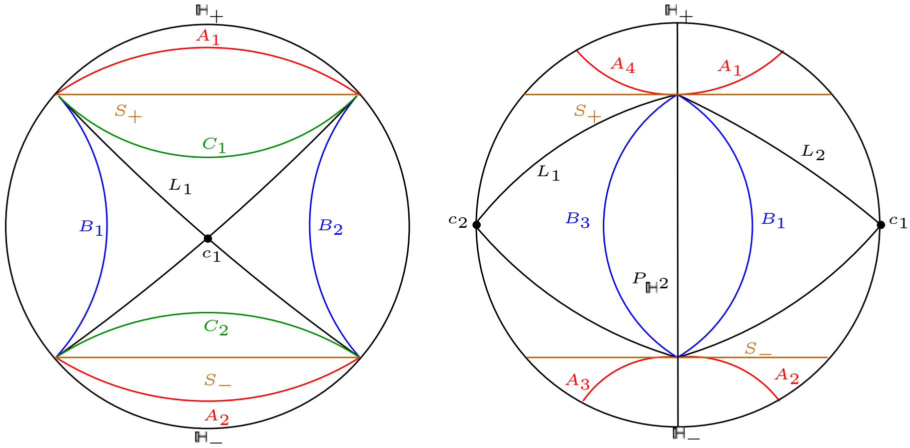

Fig. 2 In the spherical projective model, A_i are (parts of) hyperbolic caps, B_i are time-like caps, C_i are space-like caps and L_i are light-like caps

any vertex on $\ell^\perp$, as in this case the edge between the vertex and ℓ_+ must be an edge of P. But this edge would be light-like and then P cannot be a generalized hyperbolic polyhedron.

If P is not entirely contained in the interior of $\widetilde{\mathrm{HS}}^3_+$, consider a vertex x of P belonging to $\mathrm{dS}^3 \backslash \mathrm{dS}^3_+$ and taken among the most far vertices from the equator of $\mathbb{S}^3$. Consider a boost B along the line passing through ℓ_+ and x, such that ℓ_+ is the attractive point, and such that B sends x to a point in the interior of $\widetilde{\mathrm{HS}}^3_+$. Such a B exists as x lies in the interior of $\widetilde{\mathrm{HS}}^3_\ell$. It is clear that the isometry B sends P to a convex parabolic generalized hyperbolic polyhedron contained in the interior of $\widetilde{\mathrm{HS}}^3_+$ which can be bijectively projected in the Klein projective model. All the vertices of its image are lying on the images of horospheres. A direct computation shows that these images have the announced shape. □

The following lemma is then obvious:

Lemma 2.5 *Let (P, G) be a convex parabolic generalized hyperbolic polyhedron of center ℓ_+. The orthogonal projection of ∂P onto any horosphere H of center ℓ_+ along the lines starting from ℓ_+ is a homeomorphism.*

2.3 Fuchsian polyhedra

Let G be a Fuchsian group leaving invariant a totally geodesic surface $P_{\mathbb{H}^2}$ of $\mathbb{H}^3$. Up to global isometries, we will always consider that $P_{\mathbb{H}^2}$ is given in the Minkowski space of dimension 4 by the intersection of the hyperbolic space with the hyperplane $\{x_1 = 0\}$. In the Klein projective model $P_{\mathbb{H}^2}$ is sent to the horizontal plane, and its boundary at infinity is the horizontal circle on the sphere. We will also denote by $P_{\mathbb{H}^2}$ the hyperplane defining the surface in $\mathbb{H}^3$ as well as the intersection of the hyperplane with $\widetilde{\mathrm{HS}}^3$. The group G also fixes the vector ${}^t(1,0,0,0) \in \mathbb{R}^4_1$ and then the corresponding point c_1 of dS^3 (as well as its antipodal c_2), and it also fixes all the time-like affine hyperplanes parallel to $P_{\mathbb{H}^2}$ in $\mathbb{R}^4_1$.

In $\mathbb{H}_+$ and $\mathbb{H}_-$ the caps fixed by G, called *hyperbolic caps*, are the totally umbilical surfaces at constant distance from $P_{\mathbb{H}^2}$ and their induced metric has constant negative sectional curvature. In the Klein projective model they correspond to the part of ellipsoids of radii $(1, 1, r)$, $r < 1$, contained in one side of $P_{\mathbb{H}^2}$. Caps of dS^3 are of three kinds (see Fig. 2):

- *light-like caps:* they are the intersections between dS^3 and the hyperplanes parallel to $P_{\mathbb{H}^2}$ passing through the points c_1 and c_2. They give the light-cone of c_1 and the one of c_2. Their boundary at infinity is the one of $P_{\mathbb{H}^2}$. In the Klein projective model, c_1 is sent to infinity and a component of its light-cone is sent to the upper-part of the vertical cylinder tangent to the unit sphere;
- *space-like caps:* they are the intersection between dS^3 and the hyperplanes parallel to $P_{\mathbb{H}^2}$ passing through the points ${}^t(x, 0, 0, 0)$, $x > 1$. For each x it gives two congruent space-like totally umbilical surfaces at constant distance from c_1, contained inside the light-cone of c_1. Their induced metric has negative sectional curvature. Their boundary at infinity is one component of the one of $P_{\mathbb{H}^2}$. In the Klein projective model the one contained in dS^3_+ is sent to the upper half-part of an ellipsoid of radii $(1, 1, r)$, $r > 1$. Two others families are given by considering the planes passing through the points ${}^t(x, 0, 0, 0)$, $x < -1$;
- *time-light caps:* they are the intersection between dS^3 and the hyperplanes parallel to $P_{\mathbb{H}^2}$ passing through the points ${}^t(x, 0, 0, 0)$, $0 < x < 1$. For each x it gives a time-like totally umbilical surface at constant distance from c_1, lying outside the light-cone of c_1. The induced metric has positive sectional curvature. The boundary at infinity is the one of $P_{\mathbb{H}^2}$. In the Klein projective model, the upper half-part of such surface is sent to the upper half-part of a one-sheeted hyperboloid of radii $(1, 1, r)$, $r > 0$. One other family is given by considering the planes passing through the points ${}^t(x, 0, 0, 0)$, $-1 < x < 0$.

Remember that we also consider parts of S_+ and S_- contained in one side of $P_{\mathbb{H}^2}$ as caps.

Lemma 2.6 *Let G be a Fuchsian group which fixes c_1 and let x be a point of $\mathbb{S}^3$ which is not c_1 or its antipodal c_2. Then the accumulating set of Gx in $\mathbb{S}^3$ is the boundary at infinity of the cap containing x. If x belongs to a light-like cap, the accumulating set depends on the choice of x as it can also contain c_1 or c_2.*

Proof By definition all the elements of Gx lie on the same cap. It follows that, if x is not on a light-like cap, Gx accumulates on a part of the boundary at infinity of the cap has G has no fixed point on the cap. If x lies on a light-like cap, a sequence of Gx can also converges to the point fixed by G. It occurs if x lies on a time-like geodesic plane invariant under the action of an element of G. It is not always the case as G is discrete.

If x is not c_1 or c_2, the accumulating set of Gx can be seen as the closure of the set of the points on S_+ and S_- fixed by the elements of G. Up to antipodals, this set does not depend on the choice of the point x. If x belongs to $\mathbb{H}_+$ or $\mathbb{H}_-$, such set is known as the limit set of G, and this one is the entire boundary at infinity of $P_{\mathbb{H}^2}$ as G is cocompact, see e.g. [20]. □

Lemma 2.7 *Up to a global isometry a convex Fuchsian HS-polyhedron (P, G) is entirely contained in the convex hull in $\mathbb{S}^3$ of the future cone of c_1.*

Proof We first prove that (P, G) is entirely contained in one side of $P_{\mathbb{H}^2}$. Actually the proof is the same as in the parabolic case. If the polyhedron is constituted with the orbit of one single point which belongs to a cap we are done as each cap is entirely contained in one side of $P_{\mathbb{H}^2}$. It the point belongs to the boundary at infinity it is easy to see that it also remains in one side of $P_{\mathbb{H}^2}$. We can suppose that the polyhedron has at least as vertices the orbit of two points x and y, living on caps in different sides of $P_{\mathbb{H}^2}$. As P is convex, there exists a totally geodesic plane M of $\mathbb{S}^3$ such that P is entirely contained in one side of M. The plane M cannot be $P_{\mathbb{H}^2}$ because x and y live in different sides of $P_{\mathbb{H}^2}$. But for the topology of $\mathbb{S}^3$, there exists points in the orbits of x and y as near as $P_{\mathbb{H}^2}$ as we want, because the orbits accumulate on the intersection of $P_{\mathbb{H}^2}$ with one of the boundaries at infinity. This contradicts the convexity of P.

Now we can use another projective model for the hyperbolic-de Sitter space: it is the projection of the part of $\widetilde{\mathrm{HS}}^3$ delimited by $P_{\mathbb{H}^2}$ and containing c_1 onto the hyperplane parallel to $P_{\mathbb{H}^2}$ and passing through c_1. The target space is naturally isometric to $\mathbb{R}^3_1$. The point c_1 is sent to the origin, its light-cone to the light-cone of $\mathbb{R}^3_1$. A half-part of $\mathbb{H}_+$ (resp. $\mathbb{H}_-$) is sent onto the interior of the upper-branch (resp. lower-branch) of the unitary two-sheeted hyperboloid. The de Sitter space is sent outside these hyperboloids. We now know that, up to a global isometry, (P, G) can be bijectively sent onto this model. The map from one model to the other sends convex sets to convex sets. This model is easily seen from Fig. 2.

The condition to be convex can be rephrased as: the convex hull of the orbit of a vertex cannot contain any other vertex. Suppose P has a vertex on a time-like cap of dS^3. In the model described above such a cap is represented as a one-branched hyperboloid. As the orbit of the vertex go to infinity ($P_{\mathbb{H}^2}$ is sent to infinity in this model), the convex hull of the orbit of the vertex is the entire space. It follows that P cannot have a vertex on a time-like cap. Then—up to an isometry—the vertices are inside or on the light-cone of c_1. For the same argument than above they must be all inside or on the same component of the light-cone. □

Lemma 2.8 *Up to a global isometry a convex Fuchsian generalized hyperbolic polyhedron* (P, G) *can be bijectively projected into the Klein projective model. Its image is a convex Euclidean polyhedron with vertices lying on the intersection of the ellipsoids of center* 0 *and radii* $(1, 1, r)$*, where r is a positive real number, with the open upper half-space.*

Proof From Lemma 2.7 we just need to check that P cannot have any vertex on a light-like cap. Suppose that there exists such a vertex x. As in the Klein projective model the accumulating set of Gx is the horizontal circle, P must contain the convex hull of this circle together with x. In particular it contains the piece of line along which x is projected onto the horizontal circle. This line belongs to the light-cone of c_1 and P is contained inside this light-cone: the line is an edge of P, but it is light-like, then it cannot meet the hyperbolic space, that contradicts the fact that P is a generalized hyperbolic polyhedron. A direct computation shows that the images of hyperbolic and space-like caps have the announced shape. □

The following is then obvious.

Lemma 2.9 *Let* (P, G) *be a convex Fuchsian generalized hyperbolic polyhedron. The orthogonal projection of* ∂P *onto* $P_{\mathbb{H}^2}$ *along the lines orthogonal to* $P_{\mathbb{H}^2}$ *is a homeomorphism.*

2.4 Polyhedral embedding

An *equivariant polyhedral embedding* of $\overline{S}$ in $\widetilde{\mathrm{HS}}^3$ is a pair $(\overline{\phi}, \rho)$ where

- $\overline{\phi}$ is a polyhedral embedding of the universal cover $\widetilde{\overline{S}}$ of $\overline{S}$ into $\widetilde{\mathrm{HS}}^3$
- ρ is a representation of the fundamental group Γ of $\overline{S}$ into $\mathcal{L}$

such that $\overline{\phi}$ is *equivariant* under the action of Γ:

$$\forall \gamma \in \Gamma, \forall x \in \widetilde{\overline{S}}, \overline{\phi}(\gamma x) = \rho(\gamma)\overline{\phi}(x).$$

An equivariant polyhedral embedding of S in $\mathbb{H}^3$ is the restriction to the hyperbolic space of an equivariant polyhedral embedding of $\overline{S}$ in $\widetilde{\mathrm{HS}}^3$, which is such that all the edges of the image of $\overline{S}$ meet the hyperbolic space. The equivariant polyhedral embedding is *convex* if its image is a convex set. It is *parabolic* if $\rho(\Gamma)$ is parabolic and *Fuchsian* if $\rho(\Gamma)$ is Fuchsian. It is *umbilical* if it is parabolic or Fuchsian—this is determined by the genus of S. It is clear that the image of a convex parabolic (resp. Fuchsian) polyhedral embedding bounds a convex parabolic (resp. Fuchsian) generalized hyperbolic polyhedron (P, G). Conversely, due to Lemmas 2.5 and 2.9, the canonical embedding of ∂P in $\mathbb{H}^3$ together with the action of G gives a convex umbilical polyhedral embedding of S.

We denote by $\mathcal{P}(n, m, p)$ the set of convex umbilical polyhedral embeddings of S in the hyperbolic space constituted with the orbits of n finite vertices, p ideal vertices and m strictly hyperideal vertices, modulo isotopies of S and isometries of $\mathbb{H}^3$. More precisely, the equivalence relation is the following: let (ϕ_1, ρ_1) and (ϕ_2, ρ_2) be two elements of $\mathcal{P}(n, m, p)$. We say that (ϕ_1, ρ_1) and (ϕ_2, ρ_2) are equivalent if there exists

- a homeomorphism h of S isotopic to the identity, such that if h_t is the isotopy (i.e. $t \in [0, 1]$, $h_0 = h$ and $h_1 = id$), then h_t fixes pointwise the ideal boundary of S for all t,
- a hyperbolic isometry I,

such that, for a lift $\widetilde{h}$ of h to $\widetilde{S}$ we have

$$\phi_2 \circ \widetilde{h} = I \circ \phi_1.$$

As two lifts of h only differ by conjugation by elements of Γ, using the equivariance property of the embedding, it is easy to check that the definition of the equivalence relation does not depend on the choice of the lift.

As Lemmas 2.4 and 2.8 say that the image of a convex umbilical polyhedral embedding of S can be drawn in the Euclidean space we have:

Lemma 2.10 *Endowed with the topology given by the Euclidean coordinates in the Klein projective model of the vertices in a fundamental domain, the space* $\mathcal{P}(n, m, p)$ *is a non-empty open subset of the manifold* $\mathbb{R}^{6g-6+3(n+m)} \times (\mathbb{S}^2)^p$.

Proof It is easy to construct an element of $\mathcal{P}$. One could start with the convex hull of the orbit of points on the unit sphere in the Klein projective model, and slightly push some points as well as their iterates outside or inside the ball, in such a manner that a point and its iterates all belong to the same cap or horosphere. Then we take the convex hull of all the points obtained in this way. If the move is sufficiently small, all the points are extremal points for the convex hull, and the resulting polyhedron is invariant by construction.

Let (P, G) be a convex umbilical polyhedron. It is determined by the coordinates in Euclidean space of vertices of P contained in a fundamental domain for the action of G and the data of this fundamental domain. The positions of the vertices give parameters living in $\mathbb{R}^{n+m} \times (\mathbb{S}^2)^p$. The fundamental domain corresponds to an element of the Teichmüller space of $\overline{S}$, that gives $(6g - 6)$ parameters (in the case of the torus, an element of the Teichmüller space is determined by the position of one vertex). It is clear that for any little change of the parameters we stay in $\mathcal{P}$:

- finite and strictly hyperideal vertices belong to open sets of $\mathbb{R}^3$, ideal vertices belong to open sets of $\mathbb{S}^2$ (as described in Lemmas 2.4 and 2.8);
- convexity is an open condition;
- Teichmüller space is an open set (one can consider for example the topology given by the fundamental domains. If S has genus > 1 these ones can be described using the so-called “canonical polygons”, see [8,40] or [12]);
- the condition that the edges meet the hyperbolic space is an open condition. □

3 Infinitesimal rigidity

The results of this section will be used to prove Lemma 5.1.

A *Killing field* of a (Riemannian or Lorentzian) space-form M is a vector field of M such that the elements of its local 1-parameter group are isometries. An *infinitesimal isometric deformation* of a polyhedral surface in a space-form of dimension 3 is the data of

- a triangulation of the polyhedral surface given by a triangulation of each face, such that no new vertex arises,
- a Killing field on each face of the triangulation such that two Killing fields on two adjacent triangles are equal on the common edge.

An infinitesimal isometric deformation is called *trivial* if it is the restriction to the polyhedral surface of a global Killing field. Let $(\phi, \rho) \in \mathcal{P}(n, m, p)$ and (ϕ_t, ρ_t) be a path in $\mathcal{P}(n, m, p)$ with $(\phi, \rho) = (\phi_0, \rho_0)$ such that the induced metric is preserved at the first order at $t = 0$. Up to global isometries we consider that the representations always fix the same objects.

We denote

$$Z(\phi(x)) := \frac{d}{dt}\phi_t(x)_{|t=0} \in T_{\phi(x)}\mathbb{H}^3$$

and

$$\dot{\rho}(\gamma)(\phi(x)) = \frac{d}{dt}\rho_t(\gamma)(\phi(x))_{|t=0} \in T_{\rho(\gamma)\phi(x)}\mathbb{H}^3.$$

The vector field Z has a property of equivariance under $\rho(\Gamma)$:

$$Z(\rho(\gamma)\phi(x)) = \dot{\rho}(\gamma)(\phi(x)) + d\rho(\gamma) \cdot Z(\phi(x)).$$

This can be written

$$Z(\rho(\gamma)\phi(x)) = d\rho(\gamma) \cdot (d\rho(\gamma)^{-1}\dot{\rho}(\gamma)(\phi(x)) + Z(\phi(x))) \tag{1}$$

and $d\rho(\gamma)^{-1}\dot{\rho}(\gamma)$ is a Killing field of $\widetilde{\mathrm{HS}}^3$, because it is the derivative of a path in SO(2, 1) (we must multiply by $d\rho(\gamma)^{-1}$, because $\dot{\rho}(\gamma)$ is not a vector field). We denote this Killing field by $\vec{\rho}(\gamma)$. Equation 1 can be written, if $y = \phi(x)$,

$$Z(\rho(\gamma)y) = d\rho(\gamma) \cdot (\vec{\rho}(\gamma) + Z)(y). \tag{2}$$

A *parabolic deformation* is an infinitesimal isometric deformation Z on a parabolic polyhedron which verifies Eq. 2, where $\vec{\rho}(\gamma)$ is a *parabolic Killing field*, that is a Killing field of $\widetilde{\mathrm{HS}}^3$ which restriction to each horosphere fixed by $\rho(\Gamma)$ gives a Killing field of $\mathbb{R}^2$.

A *Fuchsian deformation* is an infinitesimal isometric deformation Z on a Fuchsian polyhedron which verifies Eq. 2, where $\vec{\rho}(\gamma)$ is a *Fuchsian Killing field*, that is a Killing field of $\widetilde{\mathrm{HS}}^3$ which restriction to each space-like and hyperbolic caps fixed by $\rho(\Gamma)$ gives a Killing field of $\mathbb{H}^2$.

A parabolic polyhedron is *parabolic infinitesimally rigid* if all its parabolic deformations are trivial and a Fuchsian polyhedron is *Fuchsian infinitesimally rigid* if all its Fuchsian deformations are trivial.

We want to prove.

Theorem C *Convex parabolic generalized hyperbolic polyhedra are parabolic infinitesimally rigid.*

Theorem C′ *Convex Fuchsian generalized hyperbolic polyhedra are Fuchsian infinitesimally rigid.*

Actually Theorem C′ is already known, because it is directly deduced from other known cases (convex Fuchsian polyhedra in Minkowski space [38], convex Fuchsian polyhedra with finite vertices in hyperbolic space [12] or in de Sitter space [11]) using the so-called "infinitesimal Pogorelov maps". We refer to [11] for a complete discussion about Fuchsian infinitesimal rigidity.

We need to prove Theorem C. It will be deduced from

Theorem 3.1 [13] *Convex parabolic polyhedra with finite vertices in $\mathbb{H}^3$ are parabolic infinitesimally rigid.*

Proof of Theorem C We consider horospheres with center the light-like vector of $\mathbb{R}^4_1$

$$\ell := \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

We denote by H_0 the light-like hyperplane containing ℓ. It is the hyperplane tangent to the light-cone along the vector ℓ. We denote by H_t the affine light-like hyperplane parallel to H_0 and passing though the point $(0, 0, 0, t)$, $t > 0$. We denote by H_t^h the horosphere of $\mathbb{H}^3$ obtained as the intersection of $\mathbb{H}^3$ and H_t, and by H_t^s the horosphere of dS^3 obtained as the intersection of dS^3 and H_t.

As the totally geodesic subspaces of both $\mathbb{H}^3$ and dS^3 are defined by the intersections of the spaces with hyperplanes of $\mathbb{R}^4_1$, a convex generalized hyperbolic polyhedron is uniquely defined by a convex (polyhedral) cone in $\mathbb{R}^4_1$. Moreover if the polyhedron is invariant under isometries the Lorentz group, the corresponding cone is also invariant under the action of the extension of these isometries to the Minkowski space. It follows that we can speak about *parabolic convex (polyhedral) cones* of the Minkowski space of dimension 4. A cone is called *hyperbolic* if it lies entirely inside the future cone of the origin of $\mathbb{R}^4_1$ (i.e. the intersection of the cone with $\mathbb{H}^3$ is a convex hyperbolic polyhedron with finite vertices). Each horosphere H_t^h (resp. H_t^s) gives a convex (smooth) cone which is (a half-part of) the set of zeros of the quadratic form q_t^h (resp. q_t^s), where:

$$q_t^h := \begin{pmatrix} t^2 & 0 & 0 & 0 \\ 0 & t^2 & 0 & 0 \\ 0 & 0 & t^2+1 & -1 \\ 0 & 0 & -1 & 1-t^2 \end{pmatrix}; \; q_t^s := \begin{pmatrix} t^2 & 0 & 0 & 0 \\ 0 & t^2 & 0 & 0 \\ 0 & 0 & t^2-1 & 1 \\ 0 & 0 & 1 & -t^2-1 \end{pmatrix}.$$

We will denote in the same way the quadratic forms q_t^s and q_t^h and the cones given by the set of their zeros. As a convex parabolic HS-polyhedron (P, G) is constituted as the union of finitely many orbits, there exists a cone q_l^s, $0 < l < 1$, such that the cone of P lies in the interior of q_l^s. We introduce the following linear transformation of $\mathbb{R}^4$:

$$A := \begin{pmatrix} l & 0 & 0 & 0 \\ 0 & l & 0 & 0 \\ 0 & 0 & \dfrac{l^2}{\sqrt{l^2+1}} & 0 \\ 0 & 0 & -\dfrac{1}{\sqrt{l^2+1}} & \sqrt{l^2+1} \end{pmatrix}$$

which sends the cone q_l^s to the light-cone of $\mathbb{R}^4_1$. Note that A preserves the direction of ℓ as well as H_0. Hence A sends horospheres of center ℓ to horospheres of center ℓ, and it sends obviously convex cones to convex cones. The following properties are directly checked by matrix multiplications:

- A sends a cone q_t^s, $t > l$, to a cone q_r^h;
- A sends a cone q_t^h to a cone q_r^h;
- A sends the light-cone of $\mathbb{R}^4_1$ to a cone q_r^h.
- let B be a null-rotation fixing ℓ. Recall that it has the form, with x and y real numbers (see e.g. [21]):

$$B := \begin{pmatrix} 1 & 0 & -x & x \\ 0 & 1 & -y & y \\ x & y & 1-\dfrac{x^2+y^2}{2} & \dfrac{x^2+y^2}{2} \\ x & y & -\dfrac{x^2+y^2}{2} & 1+\dfrac{x^2+y^2}{2} \end{pmatrix}.$$

  Then

$$C := ABA^{-1}$$

  is a null-rotation fixing ℓ.

It follows that A sends (P, G) to a convex parabolic hyperbolic cone.

A Killing field of $\mathbb{H}^3$ or dS^3 is the restriction to these spaces of a unique Killing field of $\mathbb{R}^4_1$. Let Z be a vector field of $\mathbb{R}^4_1$. We denote by dZ the differential of Z at the point x. The vector field Z is a Killing field if and only if, for all vector X based at x:

$$\langle dZ(X), X\rangle_1 = 0.$$

We define the vector field $\tilde{Z}$ as being at the point $\tilde{x} := Ax$ the vector $NZ(x)$, where

$$N := J\,{}^tA^{-1}J.$$

If $\tilde{X} = AX$ is a vector based at $\tilde{x}$, we have:

$$\begin{aligned} \langle d\tilde{Z}(\tilde{X}), \tilde{X}\rangle_1 &= {}^t d\tilde{Z}(\tilde{X})J\tilde{X} = {}^t(NdZ(X))JAX \\ &= {}^t dZ(X)JA^{-1}JJAX = \langle dZ(X), X\rangle_1. \end{aligned} \tag{3}$$

It follows that the map $Z \mapsto \tilde{Z}$ sends an infinitesimal isometric deformation of a convex polyhedral cone to an infinitesimal isometric deformation of a convex polyhedral hyperbolic cone (its image by A), and one is trivial when the other is. Hence to prove Theorem C it remains to check that $Z \mapsto \tilde{Z}$ sends parabolic deformations to parabolic deformations, as we know by Theorem 3.1 that parabolic deformations of convex parabolic hyperbolic cones are trivial. Let Z be a parabolic deformation of a convex parabolic cone. It verifies, for some null-rotation B:

$$Z(Bx) = B(Z(x) + K(x))$$

where $K(x)$ is a parabolic Killing field, and we want to prove that there exists a parabolic Killing field $\tilde{K}$ such that:

$$\tilde{Z}(C\tilde{x}) = C(\tilde{Z}(\tilde{x}) + \tilde{K}(\tilde{x})).$$

A direct computation shows that

$$CN = NB$$

and then we get, with $x = A^{-1}\tilde{x}$:

$$\tilde{Z}(C\tilde{x}) = NZ(Bx) = NB(Z(x) + K(x))$$
$$= CN(Z(x) + K(x)) = C(\tilde{Z}(\tilde{x}) + NK(x)).$$

We define the vector field $\tilde{K}$ at the point $\tilde{x}$ as $NK(x)$. By (3) we know that $\tilde{K}$ is a Killing field as K is. It remains to check that, for all $\tilde{x}$, $\tilde{K}(\tilde{x})$ is tangent to the horosphere of center ℓ passing through $\tilde{x}$. Horospheres have the property that all geodesics starting from their center intersect them orthogonally. It follows that it suffices to check that $\tilde{K}(\tilde{x})$ is orthogonal to the plane spanned by $\tilde{x}$ and ℓ. The vector $K(x)$ satisfies this property, then it is orthogonal to both x and ℓ. A computation analogous to Eq. 3 shows that $\tilde{K}$ is also orthogonal to both $\tilde{x}$ and ℓ (ℓ is an eigenvector of A). □

Remark A similar proof might work for spherical and Fuchsian polyhedra.

The property of the map $Z \mapsto \tilde{Z}$ to send Killing field on Killing field is just a particular expression of the Darboux–Sauer Theorem, which says that "infinitesimal rigidity is a projective property" [10,31,32]. See also e.g. [9].

In this proof we never used the condition that the edges of the polyhedral surface intersect the hyperbolic space. Actually we proved the parabolic infinitesimal rigidity for a convex parabolic HS-polyhedra. In particular we proved the parabolic infinitesimal rigidity for convex parabolic polyhedra in de Sitter space.

4 Spaces of metrics

We denote by $\widetilde{\mathcal{M}}(n, p, m)$ the space of hyperbolic metrics on S with n conical singularities with positive singular curvature, p cusps and m complete hyperbolic ends of infinite area. Cusps and conical points are marked in the following sense: if, for a metric of $\widetilde{\mathcal{M}}(n, p, m)$, the neighborhood of $x \in \overline{S}$ is isometric to the neighborhood of the apex of a convex cone then any hyperbolic metric on S for which the neighborhood of x is isometric to a cusp does not belong to $\widetilde{\mathcal{M}}(n, p, m)$, and vice-versa. We define $\mathcal{M}(n, p, m)$ as the quotient of $\widetilde{\mathcal{M}}(n, p, m)$ by the isotopies of S which fix pointwise the ideal boundary of S. We want to prove:

Lemma 4.1 *The space $\mathcal{M}(n, p, m)$ is a connected and simply connected manifold of dimension $6g - 6 + 3(n + m) + 2p$.*

Note that $\mathcal{M}(0, p, m)$ is the Teichmüller space $T_g(p, m)$ of a surface of finite topological type (g, p, m). In this case the lemma above is well-known (see e.g. [1,22]). If the metric has conical singularities, we can use the following theorem, which is a particular simple case of the results of [17,18]:

Theorem 4.2 *Hyperbolic metrics on a topologically finite surface with a finite number of conical singularities, cusps and complete ends of infinite area are uniquely determined by the conformal structure of the surface and the values of the cone-angles.*

It follows that $\mathcal{M}(n, p, m)$ is in bijection with the product of $T_g(n + p, m)$ and $]0, 2\pi[^n$ (the values of the cone-angles—there is no Gauss–Bonnet condition on them as we are restricted to positive singular curvature). We endow $\mathcal{M}(n, p, m)$ with the topology induced by the bijection, what obviously gives Lemma 4.1.

There exists another way to prove Lemma 4.1, which is used in [33] for the case of the sphere (without cusps), but the arguments does not depend on the genus. See also [26] for a related construction in the case of the sphere with cusps. Analogous arguments in a close context where used for example in [24,28,36]. The idea lies on the fact that $\mathcal{M}(n, p, m)$ is locally parameterized by the edge lengths of triangulations of the metrics. Then it is not hard to continuously "smooth" the cone-angles, and the conclusion follows from the connectedness and simply connectedness of $\mathcal{M}(0, p, m)$.

5 The map "induced metric"

5.1 Local injectivity

Let $(\phi, \rho) \in \mathcal{P}(n, m, p)$. The induced metric on $\phi(S)$ is isometric to a hyperbolic metric smooth everywhere except at the vertices, which provide cone angles of positive curvature (two faces sharing an edge can be unfolded in the plane and then the induced metric is not singular at the edges). By Lemmas 2.5 and 2.9 the induced metric on $\phi(S)/\rho(\Gamma)$ belongs to $\mathcal{M}(n, p, m)$. We denote by $\mathcal{I}$ the map from $\mathcal{P}$ to $\mathcal{M}$ obtained in this way. The determining fact, which uses the results of Sect. 3, is:

Lemma 5.1 *The map $\mathcal{I}$ is a local homeomorphism.*

Proof The map $\mathcal{I}$ is obviously continuous. Moreover Theorems C and C′ gives the local injectivity of $\mathcal{I}$. This last fact is very classical, see e.g. [15]. It is used for example in [12]. □

5.2 Properness

We will prove that $\mathcal{I}$ is proper in the following way: if $(\phi_k, \rho_k)_k$ is a sequence in $\mathcal{P}(n, m, p)$ such that $(\mathcal{I}(\phi_k, \rho_k))_k$ converges in $\mathcal{M}(n, m, p)$, then there exists a subsequence of $(\phi_k, \rho_k)_k$ converging in $\mathcal{P}(n, m, p)$. We must prove the convergence of the sequence of representations and the convergence of the sequence of coordinates of the vertices in $\mathbb{R}^3$. In all the proof below, we always assume that convergence is up to the extraction of a subsequence and we denote $\phi_k(S)$ by P_k.

5.2.1 Fuchsian case

We begin with the Fuchsian case as it is the most familiar. The properness is proved in [12] if the metric has only conical singularities, in [36] if it has only ideal vertices, and in [35] if it has only strictly hyperideal vertices. Actually the arguments we need here are all contained in these references. For convenience we repeat them. The proof can be decomposed in three steps.

(i) *The sequence of representations converges.* To the sequence $(\phi_k, \rho_k)_k$ is associated a sequence $(t_k)_k$ in the Teichmüller space T_g of $\overline{S}$, with $(\overline{S}, t_k)$ isometric to $(P_{\mathbb{H}^2}/\rho_k(\Gamma))$. Suppose that the sequence of representations diverges. This implies that the sequence $(t_k)_k$ diverges and it is a well-known fact of Teichmüller theory that in this case there exists a closed geodesic on $\overline{S}$ whose lengths go to infinity for the metrics t_k. But the orthogonal projection of the polyhedra onto $P_{\mathbb{H}^2}$ is contracting, that means that on $P_k/\rho_k(\Gamma)$ the lengths of the same curve on the P_k go to infinity, that is impossible as the sequence of induced metric converges.

(ii) *The distance to $P_{\mathbb{H}^2}$ is uniformly bounded.* As $(\rho_k)_k$ converges the lengths of any closed curve in $\overline{S}$ for the metrics t_k remain bounded from below by a positive constant c. Then the lengths of the same curves on the P_k are bounded from below by c times the inverse of the factor of contraction of the orthogonal projection. Suppose that the polyhedra go far from $P_{\mathbb{H}^2}$. This factor will becomes arbitrary large, and then the lengths of the curves will go to infinity that is impossible. This proves that the distance to $P_{\mathbb{H}^2}$ is uniformly bounded from above. It is also bounded from below as the P_k are convex polyhedral caps above the plane $P_{\mathbb{H}^2}$, and as the values of the cone-angles on the P_k are uniformly bounded.

(iii) *The sequence of the coordinates of the vertices converges.* First we need to "normalize" the sequence of polyhedra in order to avoid trivial divergences of the sequence of polyhedral embeddings (typically we want to avoid one vertex to be sent onto its iterates). It suffices to compose the ϕ_k with hyperbolic isometries such that a point x_k on P_k always stay on the same line orthogonal to $P_{\mathbb{H}^2}$. It follows by (ii) that for a k sufficiently large we can assume that x_k remain fixed for all k, and we now denote this point by x. Moreover all the vertices in a same fundamental domain than x are lying inside a Euclidean cylinder orthogonal to $P_{\mathbb{H}^2}$. Otherwise the projection onto $P_{\mathbb{H}^2}$ of the fundamental domain will give a diverging sequence of representation, that contradicts (i).

Hyperideal vertices do not collapse. Suppose that v_1 and v_2 collapse. Then choose a closed curve γ on the surface going through the point corresponding to x such that v_1 and v_2 belong to different components of the complement of γ. In $\mathbb{H}^3$ γ gives curves γ_k on the polyhedra joining x to one of its iterate. When v_1 goes near v_2 (for the Euclidean topology), the geodesic joining them becomes closer to the ideal boundary, and then the γ_k must approach the boundary at infinity, that obliges their lengths to go to infinity, that's impossible.

The Euclidean coordinates of finite vertices have a converging subsequence as they must be at bounded hyperbolic distance from x. Moreover they cannot collapse. Otherwise suppose that two vertices are arbitrarily close in $\mathbb{R}^3$. As they remain in a compact of $\mathbb{H}^3$, they also must be arbitrarily close in $\mathbb{H}^3$. But that is impossible because their distances on the polyhedra are uniformly bounded and because the polyhedra are convex.

The last thing to prove is that the Euclidean distance between strictly hyperideal vertices v_k and $P_{\mathbb{H}^2}$ are uniformly bounded from above. Otherwise the de Sitter distances between v_k and c_1 go to 0, but this is impossible. To see this we use the model described in the proof of Lemma 2.7, where c_1 corresponds to the origin in the Minkowski space of dimension 3. We see a sequence of (closure of) fundamental domains on P_k for the action of $\rho_k(\Gamma)$ as a sequence $(D_k)_k$ of convex isometric space-like embeddings of the disc, with $(n+m+p)$ singular points. Each D_k must stay out of the light-cone of its vertices, and inside the light-cone of c_1. It follows that if the v_k go to c_1, then the D_k will be in an arbitrarily neighborhood of a light-cone for k sufficiently large. But this is impossible: a light-cone (without its vertex) is a smooth surface, and it cannot be approximate by polyhedral surfaces with a fixed number of vertices.

5.2.2 Parabolic case

The proof of the properness in the parabolic case is very close to the one of the Fuchsian case. Let H be an arbitrarily horosphere of the hyperbolic space, with same center ℓ as the convex parabolic polyhedra (ϕ_k, ρ_k). We normalize $(\phi_k)_k$ as follows. We choose a point s on S and we compose ϕ_k with a parabolic isometry (fixing ℓ) such that the orthogonal projection of $\phi_k(s)$ onto H always give the same point x. With this normalization we avoid some trivial divergences has explained in (iii) above.

(i) *The sequence of representations converges.* Denote by y_k and z_k the orthogonal projections onto H of two iterates of $\phi_k(s)$ under the action of two generators of the fundamental group of the torus. Together with x they give an Euclidean parallelogram Q_k on H. We project those parallelograms onto a horosphere H_k concentric to H which is such that the image of Q_k has area 1. We keep the notations Q_k, x, y_k, z_k for the objects projected onto H_k. If the sequence of representations diverges, we can suppose that the lengths of the Euclidean segments xy_k of H_k go to infinity. As the area is fixed this implies that the lengths of xz_k go to 0. For k sufficiently large, if P_k lies "above" H_k (i.e. some vertices are outside the convex hull of the orbit of x), then as the orthogonal projection onto H_k is contracting, the lengths on the P_k of the curves corresponding to xy_k will go to infinity, that is impossible as the sequence of induced metrics converge. If P_k lies "below" H_k then it is not hard to see that, using the convexity of P_k, the lengths on P_k of the curves corresponding to xz_k are arbitrarily near 0, that is also impossible. It could also occur that the angle between xy_k and xz_k degenerates to a flat angle, that is also forbidden by the convergence of the induced metrics.

(ii) *The distance to H is uniformly bounded.* The argument to prove that the distance is bounded from above is the same as in the Fuchsian case, as it uses only the facts that the sequence of representations converges and that the orthogonal projection is contracting. The argument to prove that the distance in bounded from below is similar: if the P_k go far from H "below" H, then the projection onto H is dilating and as the sequence of representations converges, this will imply that there exists curves on the P_k corresponding to some closed curve on S whose lengths go to 0.

(iii) *The sequence of the coordinates of the vertices converges.* Recall that we look at convex parabolic polyhedra up to hyperbolic isometries fixing ℓ which are such that the polyhedra can be bijectively projected into the Klein projective model. It follows that strictly hyperideals vertices cannot "go to infinity". The other arguments are similar to the ones used in the Fuchsian case.

6 Proofs of theorems

6.1 Proof of Theorems B and B′

We have proved:

✓ $\mathcal{P}$ is a non-empty metric space (Lemma 2.10);
✓ $\mathcal{M}$ is a connected metric space (Lemma 4.1);
✓ $\mathcal{I}$ is a local homeomorphism (Lemma 5.1);
✓ $\mathcal{I}$ is proper (Subsection 5.2).

It follows that $\mathcal{I}$ is a finite-sheeted covering map. But $\mathcal{M}$ is also simply connected (Lemma 4.1). It follows that $\mathcal{I}$ is a homeomorphism between $\mathcal{P}$ and $\mathcal{M}$, that gives Theorems B and B′.

6.2 Proof of Theorem A

There exists only three kinds of totally umbilical surfaces in the hyperbolic space: they are contained in a sphere, a horosphere or a totally geodesic plane. If we consider that a cocompact group acts on them, they must be complete: they are a sphere, a horosphere or a totally geodesic plane. If the totally umbilical surface is the sphere, the only group acting freely on it is the trivial one, and we are in the case of Theorem 1.1. If the surface is a horosphere, the group must be parabolic in the sense we defined it and we are in the case of Theorem B. If the surface is a totally geodesic plane, the group must be Fuchsian in the sense we defined it and we are in the case of Theorem B′.

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