Convolution of convex valuations

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Abstract We show that the natural "convolution" on the space of smooth, even, translation-invariant convex valuations on a euclidean space V, obtained by intertwining the product and the duality transform of S. Alesker J. Differential Geom. 63: 63–95, 2003; Geom.Funct. Anal. 14:1–26, 2004 may be expressed in terms of Minkowski sum. Furthermore the resulting product extends naturally to odd valuations as well. Based on this technical result we give an application to integral geometry, generalizing Hadwiger's additive kinematic formula for $SO(V)$ Convex Geometry, North Holland, 1993 to general compact groups $G \subset O(V)$ acting transitively on the sphere: it turns out that these formulas are in a natural sense dual to the usual (intersection) kinematic formulas.

1 Introduction

1.1 General introduction

In the series of papers [1–4], S. Alesker has introduced an array of fundamental operations on convex valuations, illuminating and extending many classical constructions.

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In rough terms, the space $\mathrm{Val}(V)$ of continuous translation-invariant valuations on a finite-dimensional euclidean space V turns out to be (among other features) a commutative graded algebra with linear involution $\mathbb{D}$ (cf. Sect. 1.2 below for precise definitions). Thinking of $\mathbb{D}$ as analogous to the Fourier transform, Alesker has asked for a geometric characterization of the resulting "convolution" obtained by intertwining the product with $\mathbb{D}$. Our main technical result (Theorem 1.2 below) is that this convolution is simply an artifact of the Minkowski sum operation.

Our main application (Theorem 1.7) may be described as follows. Alesker has shown that if $G \subset O(V)$ acts transitively on the sphere of V then the dimension of the subspace $\mathrm{Val}^G \subset \mathrm{Val}(V)$ of G-invariant valuations is finite. It follows easily that if $\phi_1, \ldots, \phi_N$ is a basis then given any $\psi \in \mathrm{Val}^G$ there are constants $c_{ij} = c_{ij}^\psi$ such that

$$\int_{\overline{G}} \psi(A \cap \bar{g}B)\, d\bar{g} = \sum_{i,j=1}^{N} c_{ij}\, \phi_i(A)\, \phi_j(B) \tag{1}$$

for any compact convex bodies $A, B \subset V$ (here $\overline{G} = G \ltimes V$ is the group generated by G and the translation group of V; the classical case $G = O(V)$ or $SO(V)$ was given originally by Blaschke).

By the same token there is also another, less well known, array of kinematic formulas for Minkowski addition instead of intersection. Hadwiger worked these out for $G = O(V)$ or $SO(V)$ (cf. [17]). Theorem 1.7 states that for general G the addition and intersection formulas are simply transforms of one another under the involution $\mathbb{D}$, provided all elements of Val^G are even in the sense of Sect. 1.2 below.

The relations established in this paper also furnish the means for a fuller understanding of the integral geometry of the unitary group, originally studied in [2] and [13], including a completely explicit form of the principal kinematic formula for this group. This will be the subject of another paper.

1.2 Definitions and background

1.2.1 Valuations

Throughout the paper, V will denote a finite-dimensional euclidean vector space. By $\mathcal{K}(V)$ we denote the space of nonempty compact convex sets in V, endowed with the Hausdorff metric. The dense subspace $\mathcal{K}^{\mathrm{sm}}(V)$ consists of compact convex sets with smooth and strictly convex boundary.

A (convex) valuation on V is a complex-valued map ϕ on $\mathcal{K}(V)$ such that

$$\phi(K \cup L) + \phi(K \cap L) = \phi(K) + \phi(L)$$

whenever $K, L, K \cup L \in \mathcal{K}(V)$. The valuation ϕ is called translation invariant if $\phi(K + x) = \phi(K)$ for all $x \in V, K \in \mathcal{K}(V)$. We will denote the space of continuous translation invariant valuations by $\mathrm{Val}(V)$ and use the abbreviated term *valuation* to refer to them. However we note that Alesker [5–8, 10] has introduced a much broader notion of valuation that makes sense even on smooth manifolds. The restricted class of valuations considered here is in some sense an infinitesimal version of this broader notion.

A valuation φ is said to have *parity* $\epsilon = \pm$ if $\varphi(-K) = \epsilon\varphi(K)$ for all $K \in \mathcal{K}(V)$ (more colloquially we will call such valuations *even* and *odd*, respectively); it is said to have degree k if $\varphi(tK) = t^k\varphi(K)$ for all $K \in \mathcal{K}(V)$ and all $t \geq 0$. Put $\mathrm{Val}_k^\epsilon(V) \subset \mathrm{Val}(V)$

for the subspace of valuations of degree k and parity ϵ. A theorem of McMullen [16] states that

$$\text{Val}(V) = \bigoplus_{0 \leq k \leq n, \, \epsilon = \pm} \text{Val}_k^\epsilon(V), \qquad (2)$$

where $\dim V = n$. It follows that $\text{Val}(V)$ carries a natural Banach space structure, with norm

$$\| \varphi \| := \sup_{K \subset B} |\varphi(K)|, \qquad (3)$$

where $B \subset V$ is a fixed bounded set with nonempty interior. Furthermore Val_0 and Val_n are both one-dimensional, spanned by the Euler characteristic χ and the volume vol, respectively.

Note that $GL(V)$ acts continuously on $\text{Val}(V)$ and on each $\text{Val}_k^\epsilon(V)$ by $g \cdot \varphi(K) := \varphi(g^{-1}K), g \in GL(V), \varphi \in \text{Val}(V), K \in \mathcal{K}(V)$. If the map $g \mapsto g\phi$ is a smooth map $GL(V) \to \text{Val}(V)$, then the valuation ϕ is said to be $GL(V)$-smooth. The subspace of $GL(V)$-smooth valuations, endowed with the C^∞ topology obtained by identifying ϕ with the map $g \mapsto g\phi$, is denoted by $\text{Val}^{\text{sm}}(V)$.

Given $A \in \mathcal{K}$ we define $\mu_A \in \text{Val}$ by

$$\mu_A(K) := \text{vol}(A + K), \qquad (4)$$

where $A + K := \{a + k : a \in A, k \in K\}$ is the Minkowski sum. If $A \in \mathcal{K}^{\text{sm}}$ then $\mu_A \in \text{Val}^{\text{sm}}$, and if A is antipodally symmetric then $\mu_A \in \text{Val}^+$. McMullen conjectured, and Alesker [1] proved, that

Theorem 1.1 *The linear span $\langle \mu_A : A \in \mathcal{K} \rangle$ is dense in* Val. *The linear span $\langle \mu_A : A \in \mathcal{K}^{\text{sm}} \rangle$ is dense in* Val^{sm}.

1.2.2 Alesker product and Crofton measures

The Alesker product is defined by putting

$$\mu_A \cdot \mu_B(K) = \text{vol}(\Delta(K) + A \times B), \quad K \in \mathcal{K}, \qquad (5)$$

where $\Delta \colon V \to V \times V$ denotes the diagonal embedding, then extending to all pairs $\phi, \psi \in \text{Val}^{\text{sm}}$ by continuity [4]. Using Fubini's theorem this may also be expressed

$$\mu_A \cdot \varphi(K) := \int_V \varphi(K \cap (x - A)) \, dx, \qquad (6)$$

where $x - A := \{x - a : a \in A\}$. This endows the space $\text{Val}^{\text{sm}}(V)$ of smooth valuations with the structure of a commutative graded algebra. The Euler characteristic χ is the unit element for this product. If $W \subset V$ is a subspace then the restriction map

$$r_W \colon \text{Val}^{\text{sm}}(V) \to \text{Val}^{\text{sm}}(W) \qquad (7)$$

is a homomorphism of algebras.

In the case of even valuations the product admits an alternative expression as follows. Given a linear subspace $P \subset V$, put $\pi_P \colon V \to P$ for the orthogonal projection. If $\phi \in \text{Val}_k^{+,sm}(V)$ then there exists a smooth measure m_ϕ on $\text{Gr}_k(V)$ such that

$$\phi(K) = \int_{\text{Gr}_k(V)} \text{vol}_k(\pi_P(K)) \, dm_\phi(P). \qquad (8)$$

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(If $2 \le k \le n - 2$ then this measure is not unique.) In particular, if K is included in an affine subspace of dimension $< k$ then $\phi(K) = 0$. We will call such a measure a Crofton measure for ϕ. If $\psi \in \text{Val}_l^{+,\text{sm}}(V)$ and $k + l \le n$ then $\phi \cdot \psi \in \text{Val}_{k+l}^{+,\text{sm}}(V)$ is the valuation with Crofton measure

$$m_{\phi \cdot \psi} = \sigma_*(\sin m_\phi \times m_\psi), \tag{9}$$

where $\sigma \colon \text{Gr}_k(V) \times \text{Gr}_l(V) \setminus \Delta \to \text{Gr}_{k+l}(V)$ is the sum map $\sigma(P, Q) := P + Q$, Δ is the null set of pairs of planes (P, Q) that fail to meet transversely, and $\sin \colon \text{Gr}_k \times \text{Gr}_l \to \mathbb{R}_{\ge 0}$ is the function determined by the relation

$$\text{vol}_{k+l}(A + B) = \sin(E, F)\,\text{vol}_k(A)\,\text{vol}_l(B)$$

for convex sets $A \subset E$, $B \subset F$. The valuation that results is independent of the choices of Crofton measures m_ϕ, m_ψ. For more details, in the context of the more general nontranslation-invariant case, the reader is referred to [10].

1.2.3 The Klain function and the $\mathbb{D}$ transform

Let $\phi \in \text{Val}_k^+(V)$. Then the restriction of ϕ to a k-dimensional subspace L is a multiple of the Lebesgue (or k-dimensional Hausdorff) measure on L. Put $\text{Kl}_\phi(L)$ to be the proportionality factor; the resulting continuous function $\text{Kl}_\phi \colon \text{Gr}_k(V) \to \mathbb{R}$ is called the *Klain function* of ϕ. Klain [14] showed that the map $\text{Kl} \colon \text{Val}_k^+(V) \to C(\text{Gr}_k(V))$ is an injection.

The unique valuation $\phi \in \text{Val}_k^+(V)$ whose Klain function is identically 1 is called the kth intrinsic volume and denoted by μ_k. These valuations are $SO(V)$-invariant and, by Hadwiger's Characterization Theorem [15], span the space of all continuous $SO(V)$-invariant valuations.

Denote by $\perp \colon \text{Gr}_k \to \text{Gr}_{n-k}$ the orthogonal complement map. The $\mathbb{D}$ transform of a smooth, even and translation invariant valuation $\phi \in \text{Val}_k^{+,\text{sm}}(V)$ is defined as the unique valuation $\mathbb{D}\phi \in \text{Val}_{n-k}^{+,\text{sm}}(V)$ with Klain function

$$\text{Kl}_{\mathbb{D}\phi} = \text{Kl}_\phi \circ \perp . \tag{10}$$

Equivalently, if m_ϕ is a Crofton measure for ϕ then the pushforward measure

$$m_{\mathbb{D}\phi} := \perp_* m_\phi \tag{11}$$

is a Crofton measure for $\mathbb{D}\phi$.

1.3 Statement of results

1.3.1 The main theorem

Thinking of $\mathbb{D}$ as analogous to the Fourier transform, it is natural to define the *convolution* of $\varphi, \psi \in \text{Val}^{+,\text{sm}}(V)$ by

$$\varphi * \psi := \mathbb{D}(\mathbb{D}\varphi \cdot \mathbb{D}\psi). \tag{12}$$

Alesker has asked for a geometric characterization of the convolution. Our main theorem gives an answer in terms of the action on the valuations $\mu_A, A \in \mathcal{K}(V)$, and shows that it extends in a natural way to all smooth valuations, regardless of parity.

Theorem 1.2 *If A, B are smooth, strictly convex and antipodally symmetric then*

$$\mu_A * \mu_B = \mu_{A+B}. \tag{13}$$

In fact there exists a unique bilinear, continuous product $*$: $\mathrm{Val}^{\mathrm{sm}} \otimes \mathrm{Val}^{\mathrm{sm}} \to \mathrm{Val}^{\mathrm{sm}}$ of degree $-n$ such that (13) *holds for general $A, B \in \mathcal{K}^{\mathrm{sm}}$. $(\mathrm{Val}^{\mathrm{sm}}, +, *)$ is a commutative and associative algebra whose unit element is vol.*

Another characterization is in terms of mixed volumes:

Corollary 1.3 *If $k + l \geq n$ and $A_1, \ldots, A_{n-k}, B_1, \ldots, B_{n-l} \in \mathcal{K}^{\mathrm{sm}}(V)$ then*

$$V_{A_1,\ldots,A_{n-k}} * V_{B_1,\ldots,B_{n-l}} = \binom{k+l}{k}^{-1} \binom{k+l}{n} V_{A_1,\ldots,A_{n-k},B_1,\ldots,B_{n-l}}. \tag{14}$$

Remark Alesker has pointed out that Theorem 1.2 may also be stated without reference to a euclidean structure on V. As in the original treatment [2], it is formally natural to view the transform $\mathbb{D}$ as an isomorphism $\mathrm{Val}_k^{+,\mathrm{sm}}(V) \to \mathrm{Val}_{n-k}^{+,\mathrm{sm}}(V^*) \otimes \mathrm{Dens}(V)$, where $\mathrm{Dens}(V)$ is the one-dimensional space of signed Lebesgue measures on V. Note that to any $0 \neq m \in \mathrm{Dens}(V)$ there is a naturally associated $m^* \in \mathrm{Dens}(V^*)$ such that the product measure $m \times m^*$ on $V \times V^* \simeq T^*V$ equals the canonical Liouville measure on the cotangent bundle of V. Thus, even in the absence of a euclidean structure it makes sense to consider the valuations μ_A as elements of $\mathrm{Val}(V) \otimes \mathrm{Dens}(V^*)$: given $A, B \in \mathcal{K}(V)$, put

$$\mu_A(B) := m(A + B) \cdot m^*$$

for m, m^* as above, which is clearly independent of choices. In this language Theorem 1.2 becomes

Theorem 1.4 *There exists a continuous convolution product $*$ on $\mathrm{Val}^{\mathrm{sm}}(V) \otimes \mathrm{Dens}(V^*)$, uniquely characterized by the condition that $\mu_A * \mu_B = \mu_{A+B}$ for any $A, B \in \mathcal{K}^{\mathrm{sm}}(V)$. If $\phi, \psi \in \mathrm{Val}^{+,\mathrm{sm}}(V) \otimes \mathrm{Dens}(V^*)$ then*

$$\phi * \psi = \mathbb{D}(\mathbb{D}^{-1}\phi \cdot \mathbb{D}^{-1}\psi).$$

However, in the present paper we will continue to use the language of Theorem 1.2.

1.3.2 Applications

Alesker has shown that if $G \subset O(V)$ is a compact subgroup acting transitively on the sphere of V, then the vector space Val^G of continuous, translation-invariant, G-invariant valuations is finite dimensional. From this it is easy to show (cf. [13])

Proposition 1.5 *Let $\varphi_1, \ldots, \varphi_N$ be a basis for Val^G and let $\psi \in \mathrm{Val}^G$. Then there exist constants c_{ij}^{ψ}, $1 \leq i, j \leq N$ such that whenever $A, B \in \mathcal{K}(\mathbb{R}^n)$*

$$\int_{\overline{G}} \psi(A \cap \bar{g}B)\, d\bar{g} = \sum_{i,j} c_{ij}^{\psi} \varphi_i(A) \varphi_j(B). \tag{15}$$

As before, $\overline{G} := G \ltimes V$ and $d\bar{g}$ is the Haar measure. The same argument from finite-dimensionality applies also to yield the following "additive kinematic formula."

Proposition 1.6 *There exist constants d_{ij}^{ψ} such that*

$$\int_G \psi(A + gB)\, dg = \sum_{i,j} d_{ij}^{\psi} \varphi_i(A)\varphi_j(B) \tag{16}$$

for all $A, B \in \mathcal{K}$.

We encode these facts by defining the maps $k_G, a_G \colon \mathrm{Val}^G \to \mathrm{Val}^G \otimes \mathrm{Val}^G$

$$k_G(\psi) := \sum_{i,j} c_{ij}^{\psi} \varphi_i \otimes \varphi_j, \tag{17}$$

$$a_G(\psi) := \sum_{i,j} d_{ij}^{\psi} \varphi_i \otimes \varphi_j, \tag{18}$$

where the $c_{ij}^{\psi}, d_{ij}^{\psi}$ are the constants from (15) and (16). Our first application of Theorem 1.2 relates these two formulas via $\mathbb{D}$:

Theorem 1.7 *Let G be as above, and suppose that $\mathrm{Val}^G \subset \mathrm{Val}^+$. Then*

$$a_G = (\mathbb{D} \otimes \mathbb{D}) \circ k_G \circ \mathbb{D}. \tag{19}$$

In other words the structure constants for the two comultiplications k_G, a_G are identical provided the bases are changed appropriately. In the case $G = SO(n)$ the structure constants had been computed by Hadwiger (cf. (52) below and [17]).

As a second application we recall the definitions of the "Lefschetz operators" used in the two different forms of Alesker's Hard Lefschetz Theorem for $\mathrm{Val}^{+,\mathrm{sm}}$:

$$\Lambda \colon \mathrm{Val}_* \to \mathrm{Val}_{*-1},$$
$$L \colon \mathrm{Val}_* \to \mathrm{Val}_{*+1}$$

given by

$$\Lambda(\phi)(A) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0+} \phi(A + tB),$$
$$L(\phi) := \mu_1 \cdot \phi,$$

where B is the unit ball of V.

Corollary 1.8 *For all $\phi \in \mathrm{Val}^{\mathrm{sm}}$,*

$$\Lambda(\phi) = 2\mu_{n-1} * \phi$$

Corollary 1.9

$$\Lambda|_{\mathrm{Val}^{+,\mathrm{sm}}} = 2\mathbb{D} \circ L \circ \mathbb{D}.$$

It follows that the two forms of the "hard Lefschetz theorem" for even valuations, established by Alesker in [2] and [3], are equivalent.

1.4 Generalities about kinematic formulas

We take this opportunity to clarify the relation between the product in Val^G and the kinematic operator k_G, sketched in [13].

Let G be a subgroup of $O(V)$, acting transitively on the unit sphere of V. We endow G with the unique Haar measure of volume 1. The semidirect product $\bar{G} := G \ltimes V$ is endowed with the product of Haar and Lebesgue measure. We denote by $\mathrm{Val}^G(V)$ the space of $\bar{G}$-invariant valuations on V, i.e. the space of valuations that are both translation invariant and G-invariant.

The best known case is $G = SO(V)$, in which case the intrinsic volumes $\mu_0, \ldots, \mu_n$ span $\mathrm{Val}^{SO(V)}$. We write ω_n for the volume of the n-dimensional unit ball and (following [15]) denote by

$$\begin{bmatrix} n \\ k \end{bmatrix} := \binom{n}{k} \frac{\omega_n}{\omega_k \omega_{n-k}}$$

the *flag coefficient*. The kinematic formula for $G = SO(V)$ is (cf. [15])

$$k_{SO(V)}(\mu_k) = \sum_{i+j=n+k} \begin{bmatrix} n+k \\ k \end{bmatrix} \begin{bmatrix} n+k \\ i \end{bmatrix}^{-1} \mu_i \otimes \mu_j. \tag{20}$$

More generally, the map $k_G \colon \mathrm{Val}^G \to \mathrm{Val}^G \otimes \mathrm{Val}^G$ from (17) is a cocommutative, coassociative coproduct. It is compatible with Alesker's product in the sense that

$$k_G(\psi \cdot \phi) = (\psi \otimes \chi) \cdot k_G(\phi) \tag{21}$$

(cf. [13]). Denote by

$$m_G \colon \mathrm{Val}^G \otimes \mathrm{Val}^G \to \mathrm{Val}^G$$

the multiplication map, and put

$$m_G^* \colon \mathrm{Val}^{G*} \to \mathrm{Val}^{G*} \otimes \mathrm{Val}^{G*}$$

for its adjoint. Recall that $\mathrm{Val}_n = \mathrm{Val}_n^G$ is spanned by a choice of a Lebesgue measure vol on V, where $\dim V = n$.

Let $\mathrm{vol}^* \in \mathrm{Val}^{G*}$ be the element such that $\langle \mathrm{vol}, \mathrm{vol}^* \rangle = 1$ and $\mathrm{vol}^* \perp \mathrm{Val}_k^G$ for $k < n$. Recall ([4]) that the graded algebra $\mathrm{Val}^{\mathrm{sm}}$ satisfies Poincaré duality. Restricting to Val^G, this means that the map $p \in \mathbf{Hom}(\mathrm{Val}^G, \mathrm{Val}^{G*})$ determined by

$$\langle a, p(b) \rangle \equiv \langle ab, \mathrm{vol}^* \rangle$$

is an isomorphism of vector spaces. Thus, Val^G is structurally similar to the cohomology algebra of a compact oriented manifold M. The following result states that in this picture the kinematic formula is analogous to the linear injection $H^*(M) \to H^*(M) \otimes H^*(M)$ induced by the diagonal map and Poincaré duality.

Theorem 1.10

$$(p \otimes p) \circ k_G = m_G^* \circ p. \tag{22}$$

Proof   If $a, b, c \in \mathrm{Val}^G$ then

$$\langle (p \otimes p) \circ k_G(a), b \otimes c \rangle = \langle k_G(a) \cdot (b \otimes c), \mathrm{vol}^* \otimes \mathrm{vol}^* \rangle, \tag{23}$$
$$\langle m_G^* \circ p(a), b \otimes c \rangle = \langle abc, \mathrm{vol}^* \rangle, \tag{24}$$

so it is enough to show that the right hand sides of (23) and (24) agree.

In the case $a = \chi = 1$ this is the content of [13], Theorem 2.6: if $\{d_i\}$ is any basis for Val^G and $\{d_i^*\} \subset \mathrm{Val}^{G^*}$ is the dual basis, then

$$k_G(1) = \sum_i d_i \otimes p^{-1}(d_i^*).$$

If this basis includes $c = d_0$, then $\langle p^{-1}(d_i^*) \cdot c, \mathrm{vol}^* \rangle = \langle d_i^*, c \rangle = \delta_0^i$, so

$$\begin{aligned}
\langle k_G(1) \cdot b \otimes c, \mathrm{vol}^* \otimes \mathrm{vol}^* \rangle &= \langle (c \otimes p^{-1}(c^*)) \cdot (b \otimes c), \mathrm{vol}^* \otimes \mathrm{vol}^* \rangle \\
&= \langle (bc) \otimes \mathrm{vol}, \mathrm{vol}^* \otimes \mathrm{vol}^* \rangle \\
&= \langle bc, \mathrm{vol}^* \rangle.
\end{aligned}$$

Now the theorem follows from (21). $\qquad\square$

2 Product and convolution

In this section we prove the main Theorem 1.2. The first part, characterizing the convolution for even valuations in terms of the Minkowski sum, will be proved in Sect. 2.1. The second part, extending the resulting product to all valuations regardless of parity, is proved in 2.2.

2.1 Convolution for even valuations

Lemma 2.1 *Let $A \in \mathcal{K}^{\mathrm{sm}}(V)$ be symmetric about the origin. Then the Klain function of the degree k component $\mu_{A,k}$ of μ_A is given by*

$$\mathrm{Kl}_{\mu_{A,k}}(L) = \mu_{n-k}(\pi_{L^\perp} A), \quad L \in \mathrm{Gr}_k(V).$$

Proof Let $L \in \mathrm{Gr}_k(V)$ and let $B_L(r)$ be a ball in L of radius r. Then

$$\mu_A(B_L(r)) = \mathrm{vol}(B_L(r) + A) \tag{25}$$
$$= \mathrm{vol}(B_L(r) + \pi_{L^\perp} A) + o(r^k) \tag{26}$$
$$= \mu_k(B_L(r)) \mu_{n-k}(\pi_{L^\perp} A) + o(r^k) \tag{27}$$

as $r \to \infty$. Since $\mu_k(B_L(r)) = O(r^k)$,

$$\mathrm{Kl}_{\mu_{A,k}}(L) = \lim_{r \to \infty} \frac{\mu_A(B_L(r))}{\mu_k(B_L(r))} = \mu_{n-k}(\pi_{L^\perp} A).$$

$\qquad\square$

From this it is easy to deduce

Corollary 2.2 *Let $W \subset V$ be an m-dimensional subspace, $\mathbb{D}_W \colon \mathrm{Val}_*^+(W) \to \mathrm{Val}_{m-*}^+(W)$ the duality operator in W, and $r_W \colon \mathrm{Val}(V) \to \mathrm{Val}(W)$ the restriction map. Then*

$$r_W(\mathbb{D}\mu_A) = \mathbb{D}_W(\mu_{\pi_W A}). \tag{28}$$

Put $\cos \colon \mathrm{Gr}_k \times \mathrm{Gr}_k \to \mathbb{R}_{\geq 0}$ for the function $\cos(P, Q) := \sin(P, Q^\perp)$. Alternatively, $\mathrm{vol}_k(\pi_Q(A)) = \cos(P, Q) \mathrm{vol}_k(A)$ for $A \subset P$, or vice versa. In particular $\cos(P, Q) = \cos(Q, P)$.

Lemma 2.3 *Let $\phi, \psi \in \mathrm{Val}_k^{+,\mathrm{sm}}$. Then*

$$\langle p(\psi), \mathbb{D}\phi \rangle = \int_{\mathrm{Gr}_k(V)} \mathrm{Kl}_\psi(P) \, \mathrm{d}m_\phi(P)$$

$$= \int \int \cos(P, Q) \, \mathrm{d}m_\psi(Q) \, \mathrm{d}m_\phi(P). \tag{29}$$

In particular

$$\psi \cdot \mathbb{D}\phi = \mathbb{D}\psi \cdot \phi. \tag{30}$$

Proof This is immediate from (9) and (11). □

Corollary 2.4

$$(\mathbb{D} \otimes \mathbb{D})(k_G(\chi)) = k_G(\chi).$$

Proof Observing that $\mathbb{D}^* \circ p = p \circ \mathbb{D}$, Theorem 1.10 implies that

$$(\mathbb{D} \otimes \mathbb{D})(k_G(\chi)) = (\mathbb{D} \otimes \mathbb{D}) \circ (p^{-1} \otimes p^{-1}) \circ m_G^*(\mathrm{vol}^*)$$

$$= (p^{-1} \otimes p^{-1}) \circ (\mathbb{D}^* \otimes \mathbb{D}^*) \circ m_G^*(\mathrm{vol}^*). \tag{31}$$

Meanwhile, given any $\phi, \psi \in \mathrm{Val}^G(V)$,

$$\langle \phi \otimes \psi, (\mathbb{D}^* \otimes \mathbb{D}^*) \circ m_G^*(\mathrm{vol}^*) \rangle = \langle m_G \circ (\mathbb{D} \otimes \mathbb{D})(\phi \otimes \psi), \mathrm{vol}^* \rangle$$

$$= \langle \mathbb{D}\phi \cdot \mathbb{D}\psi, \mathrm{vol}^* \rangle$$

$$= \langle \phi \cdot \psi, \mathrm{vol}^* \rangle \quad \text{(by (30))}$$

$$= \langle \phi \otimes \psi, m_G^* \mathrm{vol}^* \rangle.$$

Thus $(\mathbb{D}^* \otimes \mathbb{D}^*) \circ m_G^*(\mathrm{vol}^*) = m_G^*(\mathrm{vol}^*)$, so (31) becomes

$$(\mathbb{D} \otimes \mathbb{D})(k_G(\chi)) = (p^{-1} \otimes p^{-1}) \circ m_G^*(\mathrm{vol}^*) = k_G(\chi).$$

□

Proof (Proof of the first part of Theorem. 1.2)

Let $A, B \in \mathcal{K}^{\mathrm{sm}}(V)$ be centrally symmetric. Let $\mu_A = \sum_{k=0}^n \mu_{A,k}$ and $\mu_B = \sum_{k=0}^n \mu_{B,k}$ be the decompositions of μ_A and μ_B by degree of homogeneity, and let $m_{B,k}$ be a smooth Crofton measure for $\mu_{B,k}$. Then

$$\mathbb{D}\mu_{A,n-k} \cdot \mathbb{D}\mu_{B,k} = \int_{\mathrm{Gr}_k(V)} \mathrm{Kl}_{\mathbb{D}\mu_{A,n-k}}(L) \, \mathrm{d}m_{B,k}(L) \, \mathrm{vol} \text{ by (29)}$$

$$= \int_{\mathrm{Gr}_k(V)} \mathrm{vol}_k(\pi_L(A)) \, \mathrm{d}m_{B,k}(L) \, \mathrm{vol} \text{ by Lemma 2.1}$$

$$= \mu_{B,k}(A) \, \mathrm{vol} \, .$$

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Summing over $k = 0, \ldots, n$,

$$(\mathbb{D}\mu_A \cdot \mathbb{D}\mu_B)_n = \sum_{k=0}^{n} \mu_{B,k}(A) \, \mathrm{vol} = \mu_B(A) \, \mathrm{vol} = \mathrm{vol}(A + B) \, \mathrm{vol} = (\mathbb{D}\mu_{A+B})_n. \quad (32)$$

Now let $W \subset V$ be an m-dimensional subspace. Corollary 2.2 implies that

$$r_W(\mathbb{D}\mu_A \cdot \mathbb{D}\mu_B) = r_W(\mathbb{D}\mu_A) \cdot r_W(\mathbb{D}\mu_A)$$
$$= \mathbb{D}_W \mu_{\pi_W A} \cdot \mathbb{D}_W \mu_{\pi_W B}$$

Applying (32) for the subspace W and using (28), we conclude that the degree m component of this valuation is

$$(r_W(\mathbb{D}\mu_A \cdot \mathbb{D}\mu_B))_m = (\mathbb{D}_W \mu_{\pi_W(A+B)})_m$$
$$= (r_W(\mathbb{D}\mu_{A+B}))_m. \quad (33)$$

Since this holds for all $W \in \mathrm{Gr}_m(V)$, we deduce that the Klain functions of $(\mathbb{D}\mu_A \cdot \mathbb{D}\mu_B)_m$ and $(\mathbb{D}\mu_{A+B})_m$ coincide for all $m = 0, \ldots, n$. Therefore, $\mathbb{D}\mu_A \cdot \mathbb{D}\mu_B = \mathbb{D}\mu_{A+B}$. □

2.2 Convolution for odd valuations

As we have mentioned above, the convolution extends in a natural way to include also the (smooth) odd valuations. Let us fix an orientation of V.

2.2.1 Representation of valuations by means of currents

Recall that a differential form on the sphere bundle SV is said to be *vertical* if it annihilates the contact distribution of SV. For $0 \leq k \leq n - 1$, put Ω_k^V for the space of smooth, translation-invariant differential forms β of bidegree $(k, n - k - 1)$ on SV such that $d\beta$ is vertical. Put $\Omega^V := \bigoplus_{k=0}^{n-1} \Omega_k^V$.

The next statement is a special case of Theorem 5.2.1 of [5] and of the main theorem of [11]. Let $N(K) \in \mathbb{I}_{n-1}(SV)$ be the normal cycle of K [19,10].

Lemma 2.5 *If $k \neq n$ then the map $v: \Omega_k^V \to \mathrm{Val}_k^{\mathrm{sm}}(V)$, given by*

$$v(\beta)(K) := \int_{N(K)} \beta \quad (34)$$

is surjective, with kernel

$$\ker v = \{\beta \in \Omega^V : \beta \text{ is exact}\}. \quad (35)$$

Definition Let $\pi_1: SV \to V$ and $\pi_2: SV \to S(V)$ denote the canonical projections, where $S(V)$ is the unit sphere of V. Let $*_V$ be the Hodge star on the space $\Omega^*(V)$ of differential forms on V, and let $*_1$ be the linear operator on $\Omega^*(SV)$ which is uniquely defined by

$$*_1 \, (\pi_1^* \gamma_1 \wedge \pi_2^* \gamma_2) = (-1)^{\binom{n - \deg \gamma_1}{2}} \pi_1^*(*_V \gamma_1) \wedge \pi_2^* \gamma_2, \quad (36)$$
$$\gamma_1 \in \Omega^*(V), \qquad \gamma_2 \in \Omega^*(S(V)).$$

We define $*_1$ on $\Omega^*(TV)$ in a similar way.

Note that $d*_1 = (-1)^n *_1 d$ on translation invariant forms and that $*_1$ is, up to a sign, an involution.

If $\beta \in \Omega_k^V, \gamma \in \Omega_l^V$ put

$$\beta \,\hat{*}\, \gamma := (2n - k - l)^{-1} *_1^{-1} ((n - k) *_1 \beta \wedge *_1 d\gamma + (n - l) *_1 \gamma \wedge *_1 d\beta)$$
$$\equiv *_1^{-1}(*_1\beta \wedge *_1 d\gamma) \quad \text{mod ker } \nu. \tag{37}$$

Proposition 2.6 *Extending by bilinearity, $\hat{*}$ is a continuous, commutative, associative product on Ω^V of degree $-n$, which descends to a continuous, commutative, associative product $*$ on $\mathrm{Val}^{\mathrm{sm}}$ of degree $-n$ by taking*

$$\nu(\beta) * \nu(\gamma) := \nu(\beta \,\hat{*}\, \gamma), \tag{38}$$
$$\mathrm{vol} * \phi := \phi. \tag{39}$$

The reader will observe that the definition in (38), (39) is a clear abuse of notation, since for even valuations we defined $*$ differently in the last section. However, we will see shortly (Proposition. 2.7 below) that the two definitions agree in this case, so we hope the reader will tolerate this momentary formal ambiguity. Note also that $*$, unlike $\hat{*}$, does not depend on the orientation of V.

Proof Commutativity and associativity may be verified via straightforward computations, taking into account that $\deg *_1 \beta$ is always odd for $\beta \in \Omega^V$.

To see that $*$ in (38) is well defined it is enough to show that if $\beta \in \Omega^V$ is exact then so is $\beta \,\hat{*}\, \gamma$ for all $\gamma \in \Omega^V$. However this follows at once from (37).

To prove continuity, let $\phi_1^j \to \phi_1, \phi_2^j \to \phi_2$ in $\mathrm{Val}^{\mathrm{sm}}$. We may assume that the degrees of all of these valuations are $< n$. Since the map $\Omega^V \to \mathrm{Val}^{\mathrm{sm}}_{<n}$ is surjective, and is obviously continuous with respect to the C^∞ topology on Ω^V, the open mapping theorem implies that we can choose sequences $\beta_1^j, \beta_2^j \in \Omega^V$ representing ϕ_1^j, ϕ_2^j and converging to β_1, β_2 in the C^∞ topology. Then $\beta_1^j \,\hat{*}\, \beta_2^j \to \beta_1 \,\hat{*}\, \beta_2$ in C^∞, and thus $\phi_1^j * \phi_2^j \to \phi_1 * \phi_2$. $\qquad\square$

It remains to show

Proposition 2.7 *If $*$ is defined as in (38), (39) then*

$$\mu_A * \mu_B = \mu_{A+B} \tag{40}$$

for all $A, B \in \mathcal{K}^{\mathrm{sm}}$.

Proof It will be convenient to use a variation on the representation (34) of valuations as integrals over the normal cycle N. Thinking of N as analogous to the manifold of unit normal vectors of a submanifold, we use instead the analogue of the bundle of unit balls:

$$N_1(K) := \mathbf{N}(K) \, \llcorner \, (V \times B(0, 1)), \quad K \in \mathcal{K}, \tag{41}$$

where $\mathbf{N}$ is the image under the identification $TV \simeq T^*V$, induced by the euclidean metric, of the conormal cycle $\mathbf{N}^*$ of [9].

Let us put $r(x, y) := |y|$ and $p: TV \setminus V \to SV, (x, y) \mapsto (x, \frac{y}{r})$. Given $\beta \in \Omega_k^V$, let

$$\tilde{\beta} := \begin{cases} d\left(r^{n-k} p^* \beta\right), & \text{on } TV \setminus V, \\ 0, & \text{on } V. \end{cases}$$

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Since $\tilde{\beta}$ is the differential of a Lipschitz form it defines a flat cochain in the sense of [18] (cf. also [12], 4.1.19), and the usual form of Stokes' theorem applies to give

$$\int_{N_1(K)} \tilde{\beta} = \int_{\partial N_1(K)} r^{n-k} p^* \beta = \int_{N(K)} \beta = \nu(\beta)(K) \tag{42}$$

for all $K \in \mathcal{K}$. Furthermore the volume valuation is given by

$$\mathrm{vol}(K) = \int_{N_1(K)} \pi_1^*(d\,\mathrm{vol}_V).$$

Put $\tilde{\Omega}_k^V := \{\tilde{\beta} : \beta \in \Omega_k^V\}$ for $k \neq n$, and $\tilde{\Omega}_n^V := \langle \pi_1^* \mathrm{vol}_V \rangle$; put $\tilde{\Omega}^V := \bigoplus_{k=0}^n \tilde{\Omega}_k^V$. Put $\tilde{\nu}(\tilde{\beta})$ for the left-hand side of (42). Thus $\tilde{\nu} \colon \tilde{\Omega}^V \to \mathrm{Val}^{\mathrm{sm}}$ is surjective. For $\phi, \psi \in \tilde{\Omega}^V$ we put

$$\phi \,\tilde{*}\, \psi := *_1^{-1}(*_1\phi \wedge *_1\psi).$$

Unwinding the definitions, if $\beta, \gamma \in \Omega^V$ then

$$\tilde{\beta} \,\tilde{*}\, \tilde{\gamma} = \widetilde{\beta \,\hat{*}\, \gamma}. \tag{43}$$

In particular, by (38), (39)

$$\tilde{\nu}(\phi) * \tilde{\nu}(\psi) = \tilde{\nu}(\phi \,\tilde{*}\, \psi) \tag{44}$$

for all $\phi, \psi \in \tilde{\Omega}^V$.

Lemma 2.8 *Let θ be a translation-invariant form of bidegree $(k, n - k)$ on TV. Then $\theta \in \tilde{\Omega}_k^V$ iff θ is closed, homogeneous of degree $n - k$, and $\theta|_{SV}$ is vertical.*

Proof of lemma   Suppose $k < n$ and that θ has these properties. By homogeneity,

$$\theta = d(r^{n-k}) \wedge p^* \beta + r^{n-k} p^* \gamma$$

for some forms β, γ on SV, where β has bidegree $(k, n - k - 1)$. If $d\theta = 0$ then $\gamma = d\beta$, whence $\theta = d(r^{n-k} p^* \beta)$. The restriction of θ to SV is then equal to $d\beta$, which is therefore vertical, from which we conclude that $\beta \in \Omega_k^V$ and $\theta = \tilde{\beta}$.

The converse, and the case $k = n$, are trivial. □

Returning to the proof of (40), given $A \in \mathcal{K}^{\mathrm{sm}}$ we put $h_A \colon V \to \mathbb{R}$ for the support function $h_A(y) := \max_{x \in A} \langle y, x \rangle$ of A, put

$$\eta = \eta(y) := \begin{cases} r\nabla h_A(y), & y \neq 0, \\ 0, & y = 0 \end{cases}$$

and define the Lipschitz map $G_A \colon TV \to V$ by

$$G_A(x, y) := x + \eta(y).$$

We claim that for any $K \in \mathcal{K}$

$$G_{A*} N_1(K) = [\![K + A]\!]. \tag{45}$$

Since $G_{A*} \colon \mathbb{I}_*(TV) \to \mathbb{I}_*(V)$ and $N_1 \colon \mathcal{K} \to \mathbb{I}_n(TV)$ are continuous (cf. [9,13]), it is enough to prove this in the case that $K \in \mathcal{K}^{\mathrm{sm}}$. Furthermore, since the left-hand side represents an integral current of top dimension with compact support, it is enough to

show that $G_{A*}N(K) = [\![\partial(K + A)]\!]$. In this case, $N(K)$ is given by integration over $\{(x, n_x) : x \in \partial K\}$, where n_x is the unit outward normal to K at x. Thus, $G_{A*}N(K)$ is given by integration over the smooth hypersurface $H := \{x + \nabla h_A(n_x) : x \in \partial K\}$. It is easy to verify that n_x is normal to H at $G_A(x, n_x)$, and that

$$
\begin{aligned}
\langle n_x, G_A(x, n_x) \rangle &= \langle n_x, x \rangle + \langle n_x, \nabla h_A(n_x) \rangle \\
&= h_K(n_x) + h_A(n_x) \\
&= h_{K+A}(n_x).
\end{aligned}
$$

This is sufficient to characterize H as $\partial(K + A)$, as claimed. Therefore,

$$
\text{vol}(K + A) = \int_{G_{A*}N_1(K)} d\,\text{vol}_V = \int_{N_1(K)} G_A^*(d\,\text{vol}_V). \tag{46}
$$

We claim that $\theta_A := G_A^*(d\,\text{vol}_V) \in \tilde{\Omega}^V$. Appealing to Lemma 2.8, it is clear that in the decomposition of θ_A by bidegree the components are all closed and of the correct homogeneity. It remains only to show that $\theta_A|_{SV}$ is vertical, i.e. that $\alpha \wedge \theta_A|_{SV} = 0$, where $\alpha(x, y) = \sum_{i=1}^n y_i \, dx_i$ is the canonical 1-form.

Since h_A is homogeneous of degree 1 it follows that $\sum_i y_i d\eta_i|_{SV} = 0$, and we establish the claim by computing

$$
\begin{aligned}
\alpha \wedge \theta_A|_{SV} &= \left(\sum_i y_i dx_i \right) \wedge G_A^*(d\,\text{vol}_V)|_{SV} \\
&= \left(\sum_i y_i (dx_i + d\eta_i) \right) \wedge G_A^*(d\,\text{vol}_V)|_{SV} \\
&= \left(\sum_i y_i (G_A^* dx_i) \right) \wedge G_A^*(d\,\text{vol}_V)|_{SV} \\
&= \sum_i y_i G_A^* (dx_i \wedge d\,\text{vol}_V)|_{SV} \\
&= 0.
\end{aligned}
$$

Thus, $\mu_A = \tilde{\nu}(\theta_A)$, and since

$$
\theta_A := \bigwedge_{i=1}^n G_A^* dx_i = \bigwedge_{i=1}^n (dx_i + d\eta_i)
$$

a straightforward computation using (36) reveals that

$$
*_1 \theta_A = \bigwedge_{i=1}^n (1 + dx_i \wedge d\eta_i). \tag{47}
$$

Therefore, if $B \in \mathcal{K}^{\mathrm{sm}}$ and $\zeta := r\nabla h_B(y)$ then

$$
\begin{aligned}
_1(\theta_A \, \tilde{} \, \theta_B) &= *_1\theta_A \wedge *_1\theta_B \\
&= \bigwedge_{i=1}^{n} (1 + \mathrm{d}x_i \wedge \mathrm{d}\eta_i) \wedge (1 + \mathrm{d}x_i \wedge \mathrm{d}\zeta_i) \\
&= \bigwedge_{i=1}^{n} (1 + \mathrm{d}x_i \wedge (\mathrm{d}\eta_i + \mathrm{d}\zeta_i)) \\
&= *_1\theta_{A+B}
\end{aligned}
$$

since $\eta + \zeta = r(\nabla h_A + \nabla h_B) = r\nabla h_{A+B}$. Therefore, $\theta_A \, \tilde{*} \, \theta_B = \theta_{A+B}$, which with (44) completes the proof. $\qquad\square$

Let us denote by $\tilde{\sigma} \colon \mathrm{Val}^{\mathrm{sm}} \to \mathrm{Val}^{\mathrm{sm}}$ the involution defined by $\tilde{\sigma}\phi(K) := \phi(-K)$ ($\tilde{\sigma}$ agrees up to a sign with the Euler–Verdier involution σ introduced in [6]). Note that even valuations are fixed by $\tilde{\sigma}$ and that $\tilde{\sigma}\mu_A = \mu_{-A}$. Relation (30) may be rewritten as $\phi \cdot \psi = \mathbb{D}(\phi * \tilde{\sigma}\psi)$ for ϕ, ψ even and of complementary degrees. In this form, this formula holds for all smooth valuations:

Corollary 2.9 *For $\phi \in \mathrm{Val}_k^{\mathrm{sm}}(V), \psi \in \mathrm{Val}_{n-k}^{\mathrm{sm}}(V)$,*

$$
\phi \cdot \psi = \mathbb{D}(\phi * \tilde{\sigma}\psi). \tag{48}
$$

Proof An equivalent formulation of (48) is that for arbitrary $\phi, \psi \in \mathrm{Val}^{\mathrm{sm}}(V)$, the highest degree part of $\phi \cdot \psi$ equals the lowest degree part of $\phi * \tilde{\sigma}\psi$. By continuity and linearity it suffices to show this for $\phi = \mu_A$ and $\psi = \mu_B$ with $A, B \in \mathcal{K}^{\mathrm{sm}}$. The lowest degree component of $\mu_A * \tilde{\sigma}\mu_B = \mu_{A-B}$ obviously equals $\mathrm{vol}(A - B)\chi$. On the other hand, by (6) the highest degree component of $\mu_A \cdot \mu_B$ is $\mathrm{vol}(A - B)\,\mathrm{vol}$. $\qquad\square$

2.3 An alternative formulation

Proof of Corollary 1.3 By definition of the mixed volume,

$$
V_{A_1,\dots,A_{n-k}} = \frac{k!}{n!} \left. \frac{\partial^{n-k}}{\partial t_1 \dots \partial t_{n-k}} \right|_{t=0} \mu_{\sum_{i=1}^{n-k} t_i A_i},
$$

$$
V_{B_1,\dots,B_{n-l}} = \frac{l!}{n!} \left. \frac{\partial^{n-l}}{\partial s_1 \dots \partial s_{n-l}} \right|_{s=0} \mu_{\sum_{j=1}^{n-l} s_j B_j},
$$

from which we deduce (using (13) and continuity) that

$$
\begin{aligned}
V_{A_1,\dots,A_{n-k}} * V_{B_1,\dots,B_{n-l}} &= \frac{k!l!}{n!^2} \left. \frac{\partial^{n-k}}{\partial t_1 \dots \partial t_{n-k}} \right|_{t=0} \left. \frac{\partial^{n-l}}{\partial s_1 \dots \partial s_{n-l}} \right|_{s=0} \\
&\quad \mu_{\sum_{i=1}^{n-k} t_i A_i + \sum_{j=1}^{n-l} s_j B_j} \\
&= \binom{k+l}{k}^{-1} \binom{k+l}{n} V_{A_1,\dots,A_{n-k},B_1,\dots,B_{n-l}}.
\end{aligned}
$$

$\qquad\square$

3 Proofs of Theorem 1.7 and Corollary 1.8

Proof of Theorem 1.7 We claim first that

$$a_G(\psi * \phi) = (\psi \otimes \mathrm{vol}) * a_G(\phi) = (\mathrm{vol} \otimes \psi) * a_G(\phi), \qquad (49)$$

for $\psi, \phi \in \mathrm{Val}^G(V)$. The valuations μ_L^G defined by

$$\mu_L^G(K) = \int_G \mathrm{vol}(K + gL)\,\mathrm{d}g,$$

$L \in \mathcal{K}^{\mathrm{sm}}(V)$, span $\mathrm{Val}^G(V)$. It thus suffices to show (49) for $\psi = \mu_L^G$, in which case we compute

$$
\begin{aligned}
a_G(\mu_L^G * \phi)(A, B) &= \int_G \mu_L^G * \phi(A + gB)\,\mathrm{d}g \\
&= \int_G \int_G \phi(A + gB + hL)\,\mathrm{d}h\,\mathrm{d}g \\
&= \int_G \int_G \phi(A + gB + hL)\,\mathrm{d}g\,\mathrm{d}h \\
&= \int_G \sum_{i,j=1}^N \mathrm{d}_{i,j}^\phi \varphi_i(A + hL)\varphi_j(B)\,\mathrm{d}h \\
&= \sum_{i,j=1}^N \mathrm{d}_{i,j}^\phi (\mu_L^G * \varphi_i)(A)(\mathrm{vol} * \varphi_j)(B) \\
&= (\mu_L^G \otimes \mathrm{vol}) * a_G(\phi)(A, B),
\end{aligned}
$$

which is the first equation of (49), and the second follows similarly.

To prove (19) we compute

$$
\begin{aligned}
a_G(\mathrm{vol})(A, B) &= \int_G \mathrm{vol}(A + gB)\mathrm{d}g = \int_{\bar{G}} \chi(A \cap \mathrm{d}g)\mathrm{d}g = k_G(\chi)(A, B) \\
&= (\mathbb{D} \otimes \mathbb{D})(k_G(\chi))(A, B)
\end{aligned}
$$

by Corollary 2.4. In other words, both sides of (19) give the same result when evaluated on the valuation vol. Now (49) and (21) yield

$$
\begin{aligned}
a_G(\phi) &= a_G(\phi * \mathrm{vol}) \\
&= (\phi \otimes \mathrm{vol}) * a_G(\mathrm{vol}) \\
&= (\mathbb{D} \otimes \mathbb{D})((\mathbb{D}\phi \otimes \chi) \cdot k_G(\chi)) \\
&= (\mathbb{D} \otimes \mathbb{D})(k_G(\mathbb{D}\phi))
\end{aligned}
$$

as claimed. $\qquad\square$

Remarks (1) Let $c_G : \mathrm{Val}^G \otimes \mathrm{Val}^G \rightarrow \mathrm{Val}^G$ denote the convolution map, determined by

$$c_G(\varphi \otimes \psi) := \varphi * \psi.$$

Using the fact that the relation (30) may be stated

$$p \circ \mathbb{D} = \mathbb{D}^* \circ p, \qquad (50)$$

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it is easy to deduce that, analogously to Theorem 1.10,

$$(p \otimes p) \circ a_G = c_G^* \circ p. \tag{51}$$

(2) Theorem 1.7 says that the coefficients for the additive kinematic formulas in a given basis are the same as the coefficients of the ordinary kinematic formulas for the Alesker dual basis. For instance, if $G = SO(V)$ then

$$
\begin{aligned}
a_{SO(V)}(\mu_k) &= (\mathbb{D} \otimes \mathbb{D}) k_{SO(V)}(\mu_{n-k}) \\
&= \begin{bmatrix} 2n-k \\ n-k \end{bmatrix} \sum_{i+j=2n-k} \begin{bmatrix} 2n-k \\ i \end{bmatrix}^{-1} \mathbb{D}(\mu_i) \otimes \mathbb{D}(\mu_j) \\
&= \begin{bmatrix} 2n-k \\ n-k \end{bmatrix} \sum_{i+j=k} \begin{bmatrix} 2n-k \\ n-i \end{bmatrix}^{-1} \mu_i \otimes \mu_j
\end{aligned}
$$

yielding the following formula of Hadwiger (cf. [17]):

$$\int_{SO(V)} \mu_k(A + gB)\mathrm{d}g = \begin{bmatrix} 2n-k \\ n-k \end{bmatrix} \sum_{i+j=k} \begin{bmatrix} 2n-k \\ n-i \end{bmatrix}^{-1} \mu_i(A)\mu_j(B) \tag{52}$$

for $A, B \in \mathcal{K}(V)$.

Proof of Corollary 1.8 Let B be the unit ball and $r > 0$. By Steiner's formula,

$$\mu_{rB}(K) = \mathrm{vol}(K + rB) = \sum_{i=0}^{n} \mu_{n-i}(K)\omega_i r^i,$$

from which we deduce that $\frac{\mathrm{d}}{\mathrm{d}r}\big|_{r=0} \mu_{rB} = 2\mu_{n-1}$. For $\phi \in \mathrm{Val}^{\mathrm{sm}}(V)$ it follows that

$$\Lambda\phi = \frac{\mathrm{d}}{\mathrm{d}r}\bigg|_{r=0} \phi(\cdot + rB) = \frac{\mathrm{d}}{\mathrm{d}r}\bigg|_{r=0} \mu_{rB} * \phi = 2\mu_{n-1} * \phi. \tag{53}$$

References

1. Alesker, S.: Description of translation invariant valuations on convex sets with solution of P. McMullen's conjecture. Geom. Func. Anal. **11**, 244–272 (2001)
2. Alesker, S.: Hard Lefschetz theorem for valuations, complex integral geometry, and unitarily invariant valuations. J. Differential Geom. **63**, 63–95 (2003)
3. Alesker, S.: Hard Lefschetz theorem for valuations and related questions of integral geometry. In: Milman, V.D. (ed.) *Geometric aspects of functional analysis*, pp. 9–20, LNM 1850. Springer, Berlin, Heidelberg Newyork (2004)
4. Alesker, S.: The multiplicative structure on polynomial valuations. Geom. Funct. Anal. **14**, 1–26 (2004)
5. Alesker, S.: Theory of valuations on manifolds I. Linear spaces. To appear in *Israel J. Math.*
6. Alesker, S.: Theory of valuations on manifolds II. Adv. Math. **207**, 420–454 (2006)
7. Alesker, S.: Theory of valuations on manifolds IV. New properties of the multiplicative structure. To appear in *GAFA seminar notes*

8. Alesker, S.: Valuations on manifolds: a survey. Preprint 2006.
9. Alesker, S., Fu, J.H.G.: Theory of valuations on manifolds III. Multiplicative structure in the general case. To appear in *Trans. Amer. Math. Soc.*
10. Bernig, A.: Valuations with Crofton formula and Finsler geometry. To appear in *Adv. Math.*
11. Bernig, A., Bröcker, L.: Valuations on manifolds and Rumin cohomology. To appear in *J. Differential Geom.*
12. Federer, H.: Geometric Measure Theory. Springer-Verlag, New York (1969)
13. Fu, J.H.G.: Structure of the unitary valuation algebra. J. Differential Geom. **72**, 509–533 (2006)
14. Klain, D.: Even valuations on convex bodies. Trans. Amer. Math. Soc. **352**, 71–93 (2000)
15. Klain, D., Rota, G.-C.: Introduction to Geometric Probability. Lezione Lincee Cambridge University Press, Cambridge (1997)
16. McMullen, P.: Valuations and Euler-type relations on certain classes of convex polytopes. Proc. London Math. Soc. **35**, 113–135 (1977)
17. Schneider, R, Wieacker, J.A: Integral geometry. In: Gruber, P.M., Wills, J.M. (eds.) Handbook of Convex Geometry vol B., pp. 113. Amsterdam, North Holland (1993)
18. Whitney, H.: Geometric integration theory. Princeton University Press, princeton (1957)
19. Zähle, M.: Integral and current representation of Federer's curvature measures. Arch. Math. **46**, 557–567 (1986)