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Korobov polynomials of the third kind and of the fourth kind

Taekyun Kim[1,2] and Dae San Kim[3*]

*Correspondence: dskim@sogang.ac.kr
[3] Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea
Full list of author information is available at the end of the article

Abstract

The first degenerate version of the Bernoulli polynomials of the second kind appeared in the paper by Korobov (Math Notes 2:77–19, 1996; Proceedings of the IVinternational conference modern problems of number theory and its applications, pp 40–49, 2001). In this paper, we study two degenerate versions of the Bernoulli polynomials of the second kind which will be called Korobov polynomials of third kind and of the fourth kind. Some properties, identities, recurrence relations and connections with other polynomials are investigated by using umbral calculus.

Keywords: Korobov polynomials of the third kind and of the fourth kind, umbral calculus

Mathematics subject classification: 05A19, 05A40, 11B83

Background

The Bernoulli polynomials of the second kind $b_n(x)$ are given by the generating function

$$\frac{t}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}, \quad \text{(see Kim et al. 2014, 2015; Roman 1984).} \tag{1.1}$$

When $x = 0$, $b_n = b_n(0)$ are called Bernoulli numbers of the second kind. The degenerate version of the Bernoulli polynomials of the second kind are called Korobov polynomials of the first kind. We note here that the Carlitz degerate Bernoulli polynomials were rediscovered by Ustinov under the name of Korobov polynomial of the second kind (see Pylypiv and Maliarchuk 2014; Ustinov 2003).

The Daehee polynomials $D_n(x)$ are defined by the generating function

$$\frac{\log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!}, \quad \text{(see Kim et al. 2014, 2015).} \tag{1.2}$$

For $x = 0$, $D_n = D_n(0)$ are called Daehee numbers.

The Korobov polynomials $K_n(\lambda, x)$ of the first kind are given by the generating function

$$\frac{\lambda t}{(1+t)^\lambda - 1}(1+t)^x = \sum_{n=0}^{\infty} K_n(x \mid \lambda)\frac{t^n}{n!}, \quad \text{(see Korobov 1996; Korobov 2001).} \tag{1.3}$$

When $x=0$, $K_n(\lambda)=K_n(0\mid\lambda)$ are called Korobov numbers of the first kind.

In the following, we will review very briefly some necessary things on umbral calculus. Our basic reference is Roman (1984). Also, one is asked to look at more recent papers on umbral calculus (Nisar et al. 2015; Srivastava et al. 2014).

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all formal power series in the variable t over $\mathbb{C}$ with

$$\mathcal{F}=\left\{f(t)=\sum_{k=0}^{\infty}\frac{a_k}{k!}t^k\middle|a_k\in\mathbb{C}\right\}. \tag{1.4}$$

Let $\mathbb{P}=\mathbb{C}[x]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. For $L\in\mathbb{P}^*$, the action of the linear functional L on a polynomial $p(x)$ is denoted by $\langle L|p(x)\rangle$ with

$$\langle L+M|p(x)\rangle=\langle L|p(x)\rangle+\langle M|p(x)\rangle,\quad \langle cL|p(x)\rangle=c\langle L|p(x)\rangle,$$

where c is a complex constant (see Kim 2014; Roman 1984).

For $f(t)=\sum_{k=0}^{\infty}a_k\frac{t^k}{k!}\in\mathcal{F}$, we define a linear functional on $\mathbb{P}$ by setting

$$\left\langle f(t)|x^n\right\rangle=a_n,\quad \text{for all } n\geq 0,\quad \text{(see Kim et al. 2014; Roman 1984)}. \tag{1.5}$$

Thus, by (1.5), we easily get

$$\left\langle t^k\middle|x^n\right\rangle=n!\delta_{n,k},\quad (n,k\geq 0), \tag{1.6}$$

where $\delta_{n,k}$ is the Kronecker's symbol.

Let $f_L(t)=\sum_{k=0}^{\infty}\left\langle L|x^k\right\rangle\frac{t^n}{k!}$. Then, by (1.6), we get $\left\langle f_L(t)\middle|x^n\right\rangle=\langle L|x^n\rangle$. Additionallly, the mapping $L\mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ can be regarded as both a formal power series and a linear functional. We refer to $\mathcal{F}$ as the umbral algebra. The umbral calculus is the study of umbral algebra (see Kim 2014; Roman 1984). From (1.6), we can easily derive $\left\langle e^{yt}\middle|x^n\right\rangle=y^n$. So $\left\langle e^{yt}|p(x)\right\rangle=p(y)$. The order $o\big(f(t)\big)$ of a power series $f(t)(\neq 0)$ is the smallest nonnegative integer k for which the coefficient at t^k does not vanish. For $f(t)\in\mathcal{F}$ and $p(x)\in\mathbb{P}$, we have

$$f(t)=\sum_{k=0}^{\infty}\left\langle f(t)|x^k\right\rangle\frac{t^k}{k!},\quad p(x)=\sum_{k=0}^{\infty}\left\langle t^k\middle|p(x)\right\rangle\frac{x^k}{k!}. \tag{1.7}$$

Thus, by (1.7), we get

$$p^{(k)}(0)=\left(\frac{d}{dx}\right)^k p(x)\bigg|_{x=0}=\left\langle t^k\middle|p(x)\right\rangle=\left\langle 1|p^{(k)}(x)\right\rangle. \tag{1.8}$$

From (1.8), we note that

$$t^kp(x)=p^{(k)}(x)=\frac{d^k}{dx^k}p(x),\quad e^{yt}p(x)=p\big(x+y\big),\quad \text{(see Roman 1984)}. \tag{1.9}$$

Let $f(t),g(t)\in\mathcal{F}$ such that $o\big(f(t)\big)=1$ and $o\big(g(t)\big)=0$. Then there exists a unique sequence $s_n(x)$ $\big(\deg s_n(x)=n\big)$ of polynomials such that $\left\langle g(t)f(t)^k\middle|s_n(x)\right\rangle=n!\delta_{n,k}$, for

$n, k \geq 0$. The sequence $s_n(x)$ is called the Sheffer sequence for the pair $\big(g(t), f(t)\big)$, which is denoted by $s_n(x) \sim \big(g(t), f(t)\big)$. For $s_n(x) \sim \big(g(t), f(t)\big)$, we have

$$f(t)s_n(x) = ns_{n-1}(x), \quad (n \in \mathbb{N} \cup \{0\}), \tag{1.10}$$

and

$$\frac{1}{g\left(\bar{f}(t)\right)} e^{x\bar{f}(t)} = \sum_{k=0}^{\infty} \frac{s_k(x)}{k!} t^k, \quad \text{for all } x \in \mathbb{C}. \tag{1.11}$$

Here $\bar{f}(t)$ is the compositional inverse of $f(t)$ (see Kim and Mansour 2014; Roman 1984).

The conjugation representation for $s_n(x) \sim \big(g(t), f(t)\big)$ is given by

$$s_n(x) = \sum_{k=0}^{n} \frac{1}{k!} \left\langle g\left(\bar{f}(t)\right)^{-1} \bar{f}(t)^k \middle| x^n \right\rangle x^k, \quad (n \geq 0), \quad \text{(see Carlitz 1979; Roman 1984)}. \tag{1.12}$$

Let us consider the following two Sheffer sequences:

$$s_n(x) \sim \big(g(t), f(t)\big), \quad r_n(x) \sim (h(t), l(t)). \tag{1.13}$$

Then, we have

$$s_n(x) = \sum_{m=0}^{n} C_{n,m} r_m(x), \quad (n \geq 0), \tag{1.14}$$

where

$$C_{n,m} = \frac{1}{m!} \left\langle \frac{h\left(\bar{f}(t)\right)}{g\left(\bar{f}(t)\right)} \left(l\left(\bar{f}(t)\right)\right)^m \middle| x^n \right\rangle, \quad \text{(see Kim et al. 2014; Roman 1984)}. \tag{1.15}$$

The first degenerate version of the Bernoulli polynomials of the second kind appeared in the paper by Korobov (2001; 1996). In this paper, we study two degenerate versions of the Bernoulli polynomials of the second kind which will be called Korobov polynomials of the third kind and of the fourth kind. Some properties, identities and recurrence relations for them are investigated by using umbral calculus. In addition, some connections with other polynomials are studied for which one refers to the related papers (Dattoli et al. 2006, 2004).

Korobov polynomials of the third kind and of the fourth kind

Now, we introduce Korobov polynomials of the third kind $K_{n,3}(x \mid \lambda)$ and of the fourth kind $K_{n,4}(x \mid \lambda)$, respectively, given by the generating functions

$$\frac{\log(1+\lambda t)}{\lambda \log(1+t)} (1+t)^x = \sum_{n=0}^{\infty} K_{n,3}(x \mid \lambda) \frac{t^n}{n!}, \tag{2.1}$$

and

$$\frac{\log(1+\lambda t)}{(1+t)^{\lambda}-1}(1+t)^x=\sum_{n=0}^{\infty}K_{n,4}(x\mid\lambda)\frac{t^n}{n!}. \tag{2.2}$$

When $x=0$, $K_{n,3}(\lambda)=K_{n,3}(0,\lambda)$ and $K_{n,4}(\lambda)=K_{n,4}(0\mid\lambda)$ are called Korobov numbers of the third kind and of the fourth kind, respectively.

As all $\frac{\lambda t}{(1+t)^{\lambda}-1}=\frac{t}{\frac{(1+t)^{\lambda}-1}{\lambda}}$, $\frac{\log(1+\lambda t)}{\lambda\log(1+t)}=\frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{\log(1+t)}$, $\frac{\log(1+\lambda t)}{(1+t)^{\lambda}-1}=\frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{\frac{(1+\lambda t)^{\lambda}-1}{\lambda}}$ tend to $\frac{t}{\log(1+t)}$ as $\lambda\to 0$, $\lim_{\lambda\to 0}K_n(x\mid\lambda)=\lim_{\lambda\to 0}K_{n,3}(x\mid\lambda)=\lim_{\lambda\to 0}K_{n,4}(x\mid\lambda)=b_n(x)$, $(n\geq 0)$. We observe first that $K_{n,3}(x\mid\lambda)$ and $K_{n,4}(x\mid\lambda)$ are Sheffer sequences for the respective pairs $\left(\frac{\lambda t}{\log(1+\lambda(e^t-1))},e^t-1\right)$ and $\left(\frac{e^{\lambda t}-1}{\log(1+\lambda(e^t-1))},e^t-1\right)$. That is,

$$K_{n,3}(x\mid\lambda)\sim\left(\frac{\lambda t}{\log\left(1+\lambda\left(e^t-1\right)\right)},e^t-1\right),$$

and

$$K_{n,4}(x\mid\lambda)\sim\left(\frac{e^{\lambda t}-1}{\log\left(1+\lambda\left(e^t-1\right)\right)},e^t-1\right). \tag{2.3}$$

From (1.12) and (2.2), we have

$$K_{n,3}(x\mid\lambda)=\sum_{k=0}^{n}\frac{1}{k!}\left\langle\frac{\log(1+\lambda t)}{\lambda\log(1+t)}\left(\log(1+t)\right)^k\middle|x^n\right\rangle x^k. \tag{2.4}$$

We observe that

$$\begin{aligned}
&\left\langle\frac{\log(1+\lambda t)}{\lambda\log(1+t)}\left(\log(1+x)\right)^k\middle|x^n\right\rangle\\
&=\left\langle\frac{\log(1+\lambda t)}{\lambda\log(1+t)}\middle|\left(\log(1+x)\right)^k x^n\right\rangle\\
&=\left\langle\frac{\log(1+\lambda t)}{\lambda\log(1+t)}\middle|k!\sum_{l=k}^{\infty}S_1(l,k)\frac{t^l}{l!}x^n\right\rangle\\
&=k!\sum_{l=k}^{n}\binom{n}{l}S_1(l,k)\left\langle\frac{\log(1+\lambda t)}{\lambda t}\middle|\frac{t}{\log(1+t)}x^{n-l}\right\rangle\\
&=k!\sum_{l=k}^{n}\binom{n}{l}S_1(l,k)\left\langle\frac{\log(1+\lambda t)}{\lambda t}\middle|\sum_{m=0}^{\infty}b_m\frac{t^m}{m!}x^{n-l}\right\rangle\\
&=k!\sum_{l=k}^{n}\binom{n}{l}S_1(l,k)\sum_{m=0}^{n-l}\binom{n-l}{m}b_m\left\langle\frac{\log(1+\lambda t)}{\lambda t}\middle|x^{n-l-m}\right\rangle\\
&=k!\sum_{l=k}^{n}\binom{n}{l}S_1(l,k)\sum_{m=0}^{n-l}\binom{n-l}{m}b_m\left\langle\sum_{j=0}^{\infty}D_j\lambda^j\frac{t^j}{j!}\middle|x^{n-l-m}\right\rangle\\
&=k!\sum_{l=k}^{n}\binom{n}{l}S_1(l,k)\sum_{m=0}^{n-l}\binom{n-l}{m}b_mD_{n-l-m}\lambda^{n-l-m}\\
&=k!\sum_{l=k}^{n}\sum_{m=0}^{n-l}\binom{n}{l}\binom{n-l}{m}S_1(l,k)b_mD_{n-l-m}\lambda^{n-l-m},
\end{aligned} \tag{2.5}$$

where $S_1(n,m)$ is the Stirling number of the first kind defined by

$$x(x-1)\dots(x-n+1)=(x)_n=\sum_{l=0}^{n}S_1(n,l)x^l,\quad (n\geq 0).$$

Therefore, by (2.4) and (2.5), we have

Theorem 1 *For $n\geq 0$, we have*

$$K_{n,3}(x\mid\lambda)=\sum_{k=0}^{n}\left(\sum_{l=k}^{n}\sum_{m=0}^{n-l}\binom{n}{l}\binom{n-l}{m}S_1(l,k)b_{n-l-m}D_m\lambda^m\right)x^k.$$

From (1.12) and (2.3), we have

$$K_{n,4}(x\mid\lambda)=\sum_{k=0}^{n}\frac{1}{k!}\left\langle\frac{\log(1+\lambda t)}{(1+t)^\lambda-1}(\log(1+t))^k\middle|x^n\right\rangle x^k.\tag{2.6}$$

We observe that

$$\begin{aligned}
&\left\langle\frac{\log(1+\lambda t)}{(1+t)^\lambda-1}(\log(1+t))^k\middle|x^n\right\rangle\\
&=k!\sum_{l=k}^{n}\binom{n}{l}S_1(l,k)\left\langle\frac{\log(1+\lambda t)}{\lambda t}\,\frac{\lambda t}{(1+t)^\lambda-1}\middle|x^{n-l}\right\rangle\\
&=k!\sum_{l=k}^{n}\binom{n}{l}S_1(l,k)\left\langle\frac{\log(1+\lambda t)}{\lambda t}\middle|\sum_{m=0}^{\infty}K_m(\lambda)\frac{t^m}{m!}x^{n-l}\right\rangle\\
&=k!\sum_{l=k}^{n}\binom{n}{l}S_1(l,k)\sum_{m=0}^{n-l}\binom{n-l}{m}K_m(\lambda)\left\langle\frac{\log(1+\lambda t)}{\lambda t}\middle|x^{n-l-m}\right\rangle\\
&=k!\sum_{l=k}^{n}\binom{n}{l}S_1(l,k)\sum_{m=0}^{n-l}\binom{n-l}{m}K_m(\lambda)D_{n-l-m}\lambda^{n-l-m}\\
&=k!\sum_{l=k}^{n}\sum_{m=0}^{n-l}\binom{n}{l}\binom{n-l}{m}S_1(l,k)K_{n-l-m}(\lambda)D_m\lambda^m.
\end{aligned}\tag{2.7}$$

Therefore, by (2.6) and (2.7), we obtain the following theorem.

Theorem 2 *For $n\geq 0$, we have*

$$K_{n,4}(x\mid\lambda)=\sum_{k=0}^{n}\left(\sum_{l=k}^{n}\sum_{m=0}^{n-l}\binom{n}{l}\binom{n-l}{m}S_1(l,k)K_{n-l-m}(\lambda)D_m\lambda^m\right)x^k.$$

By (1.6) and (2.1), we easily get

$$
\begin{aligned}
K_{n,3}(y \mid \lambda) &= \left\langle \frac{\log(1+\lambda t)}{\lambda \log(1+t)}(1+t)^y \middle| x^n \right\rangle \\
&= \left\langle \frac{\log(1+\lambda t)}{\lambda \log(1+t)} \middle| (1+t)^y x^n \right\rangle \\
&= \left\langle \frac{\log(1+\lambda t)}{\lambda \log(1+t)} \middle| \sum_{l=0}^{\infty} (y)_l \frac{t^l}{l!} x^n \right\rangle \\
&= \sum_{l=0}^{n} \binom{n}{l} (y)_l \left\langle \frac{\log(1+\lambda t)}{\lambda \log(1+t)} \middle| x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n} \binom{n}{l} (y)_l \sum_{m=0}^{n-l} \binom{n-l}{m} b_m D_{n-l-m} \lambda^{n-l-m} \\
&= \sum_{l=0}^{n} \binom{n}{l} (y)_l \sum_{m=0}^{n-l} \binom{n-l}{m} b_{n-l-m} D_m \lambda^m \\
&= \sum_{l=0}^{n} \left(\sum_{m=0}^{n-l} \binom{n}{l} \binom{n-l}{m} b_{n-l-m} D_m \lambda^m \right) (y)_l. \qquad (2.8)
\end{aligned}
$$

Thus, by (2.8), we get

$$
K_{n,3}(x \mid \lambda) = \sum_{l=0}^{n} \left(\sum_{m=0}^{n-l} \binom{n}{l} \binom{n-l}{m} b_{n-l-m} D_m \lambda^m \right) (x)_l, \quad (n \geq 0). \qquad (2.9)
$$

From (1.6) and (2.2), we note that

$$
\begin{aligned}
K_{n,4}(y \mid \lambda) &= \left\langle \frac{\log(1+\lambda t)}{(1+t)^\lambda - 1}(1+t)^y \middle| x^n \right\rangle \\
&= \left\langle \frac{\log(1+\lambda t)}{(1+t)^\lambda - 1} \middle| (1+t)^y x^n \right\rangle \\
&= \sum_{l=0}^{n} \binom{n}{l} (y)_l \left\langle \frac{\log(1+\lambda t)}{(1+t)^\lambda - 1} \middle| x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n} \binom{n}{l} (y)_l \sum_{m=0}^{n-l} \binom{n-l}{m} K_m(\lambda) D_{n-l-m} \lambda^{n-l-m} \\
&= \sum_{l=0}^{n} \binom{n}{l} (y)_l \sum_{m=0}^{n-l} \binom{n-l}{m} K_{n-l-m}(\lambda) D_m \lambda^m \\
&= \sum_{l=0}^{n} \left(\sum_{m=0}^{n-l} \binom{n}{l} \binom{n-l}{m} K_{n-l-m}(\lambda) D_m \lambda^m \right) (y)_l. \qquad (2.10)
\end{aligned}
$$

Thus, by (2.10), we get

$$
K_{n,4}(x \mid \lambda) = \sum_{l=0}^{n} \left(\sum_{m=0}^{n-l} \binom{n}{l} \binom{n-l}{m} K_{n-l-m}(\lambda) D_m \lambda^m \right) (x)_l. \qquad (2.11)
$$

From (2.2), we note that

$$K_{n,3}(x\mid\lambda)\sim\left(\frac{\lambda t}{\log\left(1+\lambda(e^t-1)\right)},e^t-1\right)$$
$$\iff\frac{\lambda t}{\log\left(1+\lambda(e^t-1)\right)}K_{n,3}(x\mid\lambda)=(x)_n\sim\left(1,e^t-1\right). \tag{2.12}$$

By (2.12), we get

$$\begin{aligned}K_{n,3}(x\mid\lambda)&=\frac{\log\left(1+\lambda(e^t-1)\right)}{\lambda t}(x)_n\\&=\sum_{k=0}^{n}S_1(n,k)\frac{\log\left(1+\lambda(e^t-1)\right)}{\lambda t}x^k\\&=\sum_{k=0}^{n}S_1(n,k)\frac{e^t-1}{t}\frac{\log\left(1+\lambda(e^t-1)\right)}{\lambda(e^t-1)}x^k\\&=\sum_{k=0}^{n}S_1(n,k)\frac{e^t-1}{t}\sum_{l=0}^{k}D_l\lambda^l\frac{(e^t-1)^l}{l!}x^k\\&=\sum_{k=0}^{n}S_1(n,k)\frac{e^t-1}{t}\sum_{l=0}^{k}D_l\lambda^l\sum_{m=l}^{\infty}S_2(m,l)\frac{t^m}{m!}x^k\\&=\sum_{k=0}^{n}S_1(n,k)\sum_{l=0}^{k}D_l\lambda^l\sum_{m=l}^{k}\binom{k}{m}S_2(m,l)\frac{e^t-1}{t}x^{k-m}.\end{aligned} \tag{2.13}$$

We observe that

$$\begin{aligned}\frac{e^t-1}{t}x^{k-m}&=\sum_{j=1}^{\infty}\frac{t^{j-1}}{j!}x^{k-m}\\&=\sum_{j=0}^{\infty}\frac{1}{(j+1)!}t^jx^{k-m}\\&=\sum_{j=0}^{k-m}\frac{1}{(j+1)!}t^jx^{k-m}\\&=\sum_{j=0}^{k-m}\frac{1}{j+1}\binom{k-m}{j}x^{k-m-j}\\&=\sum_{j=0}^{k-m}\frac{1}{k-m-j+1}\binom{k-m}{j}x^j.\end{aligned} \tag{2.14}$$

Thus, by (2.13) and (2.14), we have

$$\begin{aligned}&K_{n,3}(x\mid\lambda)\\&=\sum_{k=0}^{n}\sum_{l=0}^{k}\sum_{m=l}^{k}\sum_{j=0}^{k-m}\frac{1}{k-m-j+1}\binom{k}{m}\binom{k-m}{j}S_1(n,k)S_2(m,l)D_l\lambda^lx^j\\&=\sum_{k=0}^{n}\sum_{l=0}^{k}\sum_{m=0}^{k-l}\sum_{j=0}^{m}\frac{1}{m-j+1}\binom{k}{m}\binom{m}{j}S_1(n,k)S_2(k-m,l)D_l\lambda^lx^j\\&=\sum_{j=0}^{n}\left(\sum_{k=j}^{n}\sum_{l=0}^{k-j}\sum_{m=j}^{k-l}\frac{1}{k+1}\binom{k+1}{m+1}\binom{m+1}{j}S_1(n,k)S_2(k-m,l)D_l\lambda^l\right)x^j,\end{aligned} \tag{2.15}$$

where $S_2(n,k)$ is the Stirling number of the second kind given by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, \quad (n \geq 0).$$

Therefore, by (2.15), we obtain the following theorem expressing $K_{n,3}(x \mid \lambda)$ in terms of the Stirling numbers of the first kind and of the second and Daehee numbers.

Theorem 3 *For $n \geq 0$, we have*

$$K_{n,3}(x \mid \lambda) = \sum_{j=0}^{n} \left(\sum_{k=j}^{n} \sum_{l=0}^{k-j} \sum_{m=j}^{k-l} \frac{1}{k+1} \binom{k+1}{m+1} \binom{m+1}{j} S_1(n,k) S_2(k-m,l) D_l \lambda^l \right) x^j.$$

From (2.3), we have

$$\frac{e^{\lambda t}-1}{\log\left(1+\lambda\left(e^t-1\right)\right)} K_{n,4}(x \mid \lambda) = (x)_n \sim \left(1, e^t-1\right), \quad (n \geq 0). \tag{2.16}$$

Thus, by (2.16), we get

$$\begin{aligned} K_{n,4}(x \mid \lambda) &= \frac{\log\left(1+\lambda\left(e^t-1\right)\right)}{e^{\lambda t}-1}(x)_n \\ &= \sum_{k=0}^{n} S_1(n,k) \frac{\log\left(1+\lambda\left(e^t-1\right)\right)}{e^{\lambda t}-1} x^k \\ &= \sum_{k=0}^{n} S_1(n,k) \frac{\lambda\left(e^t-1\right)}{e^{\lambda t}-1} \frac{\log\left(1+\lambda\left(e^t-1\right)\right)}{\lambda\left(e^t-1\right)} x^k \\ &= \sum_{k=0}^{n} S_1(n,k) \sum_{l=0}^{k} D_l \lambda^l \sum_{m=l}^{k} \binom{k}{m} S_2(m,l) \frac{e^t-1}{t} \frac{\lambda t}{e^{\lambda t}-1} x^{k-m}, \end{aligned} \tag{2.17}$$

Now, we observe that

$$\begin{aligned} \frac{e^t-1}{t} \frac{\lambda t}{e^{\lambda t}-1} x^{k-m} &= \frac{e^t-1}{t} \sum_{j=0}^{k-m} B_j \lambda^j \frac{t^j}{j!} x^{k-m} \\ &= \sum_{j=0}^{k-m} \binom{k-m}{j} B_j \lambda^j \frac{e^t-1}{t} x^{k-m-j} \\ &= \sum_{j=0}^{k-m} \binom{k-m}{j} B_j \lambda^j \sum_{i=0}^{k-m-j} \frac{1}{i+1} \binom{k-m-j}{i} x^{k-m-j-i}, \end{aligned} \tag{2.18}$$

where B_n is the n-th Bernoulli number given by the generating function

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad \left(\text{see Bayad and kim 2010; Kim et al. 2012, 2015}\right).$$

Thus, by (2.17) and (2.18), we get

$$\begin{aligned}
&K_{n,4}(x \mid \lambda)\\
&=\sum_{k=0}^{n} S_1(n,k)\sum_{l=0}^{k} D_l\lambda^l \sum_{m=0}^{k-l}\binom{k}{m} S_2(k-m,l)\\
&\quad\times \sum_{j=0}^{m}\binom{m}{j} B_{m-j}\lambda^{m-j}\sum_{i=0}^{j}\frac{1}{j-i+1}\binom{j}{i}x^i\\
&=\sum_{i=0}^{n}\left(\sum_{k=i}^{n}\sum_{l=0}^{k-i}\sum_{m=i}^{k-l}\sum_{j=i}^{m}\frac{1}{j-i+1}\binom{k}{m}\right.\\
&\quad\times\left.\binom{m}{j}\binom{j}{i}\lambda^{l+m-j}S_1(n,k)S_2(k-m,l)D_lB_{m-j}\right)x^i\\
&=\sum_{i=0}^{n}\left(\sum_{k=i}^{n}\sum_{l=0}^{k-i}\sum_{m=i}^{k-l}\sum_{j=i}^{m}\frac{1}{k+1}\binom{k+1}{m+1}\binom{m+1}{j+1}\binom{j+1}{i}\lambda^{l+m-j}\right.\\
&\quad\times\left. S_1(n,k)S_2(k-m,l)D_lB_{m-j}\right)x^i. \qquad (2.19)
\end{aligned}$$

Therefore, by (2.19), we obtain the following theorem expressing $K_{n,4}(x \mid \lambda)$ in terms of the Stirling numbers of the first kind and of the second kind, Daehee numbers and Bernoulli numbers.

Theorem 4 *For $n \geq 0$, we have*

$$\begin{aligned}
&K_{n,4}(x \mid \lambda)\\
&=\sum_{i=0}^{n}\left(\sum_{k=i}^{n}\sum_{l=0}^{k-i}\sum_{m=i}^{k-l}\sum_{j=i}^{m}\frac{1}{k+1}\binom{k+1}{m+1}\binom{m+1}{j+1}\binom{j+1}{i}\right.\\
&\quad\times\left.\lambda^{l+m-j}S_1(n,k)S_2(k-m,l)D_lB_{m-j}\right)x^i.
\end{aligned}$$

From (2.8), we have

$$\begin{aligned}
K_{n,3}(y \mid \lambda) &= \sum_{l=0}^{n}\binom{n}{l}(y)_l\left\langle \frac{\log(1+\lambda t)}{\lambda\log(1+t)}\middle| x^{n-l}\right\rangle\\
&=\sum_{l=0}^{n}\binom{n}{l}(y)_l\left\langle \sum_{i=0}^{\infty}K_{i,3}(\lambda)\frac{t^i}{i!}\middle| x^{n-l}\right\rangle\\
&=\sum_{l=0}^{n}\binom{n}{l}(y)_l K_{n-l,3}(\lambda). \qquad (2.20)
\end{aligned}$$

Thus, by (2.20), we get

$$K_{n,3}(x \mid \lambda) = \sum_{l=0}^{n}\binom{n}{l}K_{n-l,3}(\lambda)(x)_l. \qquad (2.21)$$

From (2.10), we have

$$\begin{aligned} K_{n,4}(y \mid \lambda) &= \sum_{l=0}^{n} \binom{n}{l} (y)_l \left\langle \frac{\log(1+\lambda t)}{(1+t)^{\lambda}-1} \middle| x^{n-l} \right\rangle \\ &= \sum_{l=0}^{n} \binom{n}{l} (y)_l \left\langle \sum_{i=0}^{\infty} K_{i,4}(\lambda) \frac{t^i}{i!} \middle| x^{n-l} \right\rangle \\ &= \sum_{l=0}^{n} \binom{n}{l} (y)_l K_{n-l,4}(\lambda). \end{aligned} \tag{2.22}$$

By (2.22), we get

$$K_{n,4}(x \mid \lambda) = \sum_{l=0}^{n} \binom{n}{l} K_{n-l,4}(\lambda)(x)_l, \quad (n \geq 0). \tag{2.23}$$

From (2.19), we have

$$\begin{aligned} K_{n,3}(y \mid \lambda) &= \left\langle \frac{\log(1+\lambda t)}{\lambda \log(1+t)} (1+t)^y \middle| x^n \right\rangle \\ &= \left\langle \frac{\log(1+\lambda t)}{\lambda t} \middle| \frac{t}{\log(1+t)} (1+t)^y x^n \right\rangle \\ &= \sum_{l=0}^{n} \binom{n}{l} b_l(y) \left\langle \frac{\log(1+\lambda t)}{\lambda t} \middle| x^{n-l} \right\rangle \\ &= \sum_{l=0}^{n} \binom{n}{l} b_l(y) \left\langle \sum_{m=0}^{\infty} D_m \lambda^m \frac{t^m}{m!} \middle| x^{n-l} \right\rangle \\ &= \sum_{l=0}^{n} \binom{n}{l} b_l(y) D_{n-l} \lambda^{n-l}. \end{aligned} \tag{2.24}$$

Thus, by (2.24), we get

$$K_{n,3}(x \mid \lambda) = \sum_{l=0}^{n} \binom{n}{l} D_{n-l} \lambda^{n-l} b_l(x), \quad (n \geq 0). \tag{2.25}$$

From (2.10), we can also derive the following equation:

$$\begin{aligned} K_{n,4}(x \mid \lambda) &= \left\langle \frac{\log(1+\lambda t)}{(1+t)^{\lambda}-1} (1+t)^y \middle| x^n \right\rangle \\ &= \left\langle \frac{\log(1+\lambda t)}{\lambda t} \middle| \frac{\lambda t}{(1+t)^{\lambda}-1} (1+t)^y x^n \right\rangle \\ &= \left\langle \frac{\log(1+\lambda t)}{\lambda t} \middle| \sum_{l=0}^{\infty} K_l(y \mid \lambda) \frac{t^l}{l!} x^n \right\rangle \\ &= \sum_{l=0}^{n} \binom{n}{l} K_l(y \mid \lambda) \left\langle \frac{\log(1+\lambda t)}{\lambda t} \middle| x^{n-l} \right\rangle \\ &= \sum_{l=0}^{n} \binom{n}{l} K_l(y \mid \lambda) \left\langle \sum_{m=0}^{\infty} D_m \lambda^m \frac{t^m}{m!} \middle| x^{n-l} \right\rangle \\ &= \sum_{l=0}^{n} \binom{n}{l} K_l(y \mid \lambda) D_{n-l} \lambda^{n-l}. \end{aligned} \tag{2.26}$$

Thus, by (2.26), we get

$$K_{n,4}(x \mid \lambda) = \sum_{l=0}^{n} \binom{n}{l} D_{n-l}\lambda^{n-l}K_l(x \mid \lambda). \tag{2.27}$$

Therefore, by (2.21), (2.23), (2.25) and (2.27), we obtain the following theorem expressing $K_{n,3}(x \mid \lambda)$ and $K_{n,4}(x \mid \lambda)$ both in terms of falling factorial polynomials. Also, we express $K_{n,3}(x \mid \lambda)$ and $K_{n,4}(x \mid \lambda)$ respectively by Bernoulli polynomials of the second kind and Korobov polynomials of the first kind.

Theorem 5 *For $n \geq 0$, we have*

$$K_{n,3}(x \mid \lambda) = \sum_{l=0}^{n} \binom{n}{l} K_{n-l,3}(\lambda)(x)_l = \sum_{l=0}^{n} \binom{n}{l} D_{n-l}\lambda^{n-l}b_l(x),$$

and

$$K_{n,4}(x \mid \lambda) = \sum_{l=0}^{n} \binom{n}{l} K_{n-l,4}(\lambda)(x)_l = \sum_{l=0}^{n} \binom{n}{l} D_{n-l}\lambda^{n-l}K_l(x \mid \lambda).$$

It is easy to see that

$$x^n \sim (1, t), \quad \frac{\lambda t}{\log\left(1+\lambda\left(e^t-1\right)\right)}K_{n,3}(x \mid \lambda) \sim \left(1, e^t-1\right). \tag{2.28}$$

For $n \geq 1$, we have

$$\begin{aligned}\frac{\lambda t}{\log\left(1+\lambda\left(e^t-1\right)\right)}K_{n,3}(x \mid \lambda) &= x\left(\frac{t}{e^t-1}\right)^n x^{-1}x^n \\ &= xB_{n-1}^{(n)}(x) \\ &= \sum_{k=0}^{n-1}\binom{n-1}{k}B_k^{(n)}x^{n-k} \\ &= \sum_{k=1}^{n}\binom{n-1}{k-1}B_{n-k}^{(n)}x^k.\end{aligned} \tag{2.29}$$

Thus, by (2.29), we get

$$\begin{aligned}&K_{n,3}(x \mid \lambda) \\ &= \sum_{k=1}^{n}\binom{n-1}{k-1}B_{n-k}^{(n)}\frac{\log\left(1+\lambda\left(e^t-1\right)\right)}{\lambda t}x^k \\ &= \sum_{k=1}^{n}\binom{n-1}{k-1}B_{n-k}^{(n)}\sum_{l=0}^{k}\sum_{m=0}^{k-l}\sum_{j=0}^{m}\frac{1}{k+1}\binom{k+1}{m+1}\binom{m+1}{j}S_2(k-m,l)D_l\lambda^l x^j \\ &= \sum_{k=0}^{n}\sum_{l=0}^{k}\sum_{m=0}^{k-l}\sum_{j=0}^{m}\frac{1}{k+1}\binom{n-1}{k-1}\binom{k+1}{m+1}\binom{m+1}{j}S_2(k-m,l)\lambda^l D_l B_{n-k}^{(n)}x^j \\ &= \sum_{j=0}^{n}\left(\sum_{k=j}^{n}\sum_{l=0}^{k-j}\sum_{m=j}^{k-l}\frac{1}{k+1}\binom{n-1}{k-1}\binom{k+1}{m+1}\binom{m+1}{j}S_2(k-m,l)\lambda^l D_l B_{n-k}^{(n)}\right)x^j.\end{aligned} \tag{2.30}$$

From (2.3), we note that

$$x^n \sim (1,t), \quad \frac{e^{\lambda t}-1}{\log\left(1+\lambda\left(e^t-1\right)\right)}K_{n,4}(x\mid\lambda) \sim \left(1, e^t-1\right). \tag{2.31}$$

For $n \geq 1$, by (2.31), we get

$$\begin{aligned}\frac{e^{\lambda t}-1}{\log\left(1+\lambda\left(e^t-1\right)\right)}K_{n,4}(x\mid\lambda) &= x\left(\frac{t}{e^t-1}\right)^n x^{-1}x^n = xB_{n-1}^{(n)}(x)\\ &= \sum_{k=1}^{n}\binom{n-1}{k-1}B_{n-k}^{(n)}x^k. \end{aligned}\tag{2.32}$$

Thus, by (2.32), we have

$$\begin{aligned}&K_{n,4}(x\mid\lambda)\\ &= \sum_{k=1}^{n}\binom{n-1}{k-1}B_{n-k}^{(n)}\frac{\log\left(1+\lambda\left(e^t-1\right)\right)}{e^{\lambda t}-1}x^k\\ &= \sum_{k=1}^{n}\binom{n-1}{k-1}B_{n-k}^{(n)}\sum_{l=0}^{k}\sum_{m=0}^{k-l}\sum_{j=0}^{m}\sum_{i=0}^{j}\frac{1}{j-i+1}\binom{k}{m}\\ &\times\binom{m}{j}\binom{j}{i}\lambda^{l+m-j}S_2(k-m,l)D_lB_{m-j}x^i\\ &= \sum_{i=0}^{n}\left(\sum_{k=i}^{n}\sum_{l=0}^{k-i}\sum_{m=i}^{k-l}\sum_{j=i}^{m}\frac{1}{j-i+1}\binom{n-1}{k-1}\binom{k}{m}\binom{m}{j}\binom{j}{i}\right.\\ &\left.\times\ \lambda^{l+m-j}S_2(k-m,l)D_lB_{m-j}B_{n-k}^{(n)}\right)x^i\\ &= \sum_{i=0}^{n}\left(\sum_{k=i}^{n}\sum_{l=0}^{k-i}\sum_{m=i}^{k-l}\sum_{j=i}^{m}\frac{1}{k+1}\binom{n-1}{k-1}\binom{k+1}{m+1}\binom{m+1}{j+1}\right.\\ &\left.\times\binom{j+1}{i}\lambda^{l+m-j}S_2(k-m,l)D_lB_{m-j}B_{n-k}^{(n)}\right)x^i. \end{aligned}\tag{2.33}$$

Therefore, by (2.30) and (2.33), we obtain the following theorem.

Theorem 6 *For $n \geq 0$, we have*

$$\begin{aligned}&K_{n,3}(x\mid\lambda)\\ &= \sum_{j=0}^{n}\left(\sum_{k=j}^{n}\sum_{l=0}^{k-j}\sum_{m=j}^{k-l}\frac{1}{k+1}\binom{n-1}{k-1}\binom{k+1}{m+1}\binom{m+1}{j}S_2(k-m,l)\lambda^lD_lB_{n-k}^{(n)}\right)x^j\end{aligned}$$

and

$$\begin{aligned}&K_{n,4}(x\mid\lambda)\\ &= \sum_{i=0}^{n}\left(\sum_{k=i}^{n}\sum_{l=0}^{k-i}\sum_{m=i}^{k-l}\sum_{j=i}^{m}\frac{1}{k+1}\binom{n-1}{k-1}\binom{k+1}{m+1}\binom{m+1}{j+1}\right.\\ &\left.\times\binom{j+1}{i}\lambda^{l+m-j}S_2(k-m,l)D_lB_{m-j}B_{n-k}^{(n)}\right)x^i\end{aligned}$$

For $s_n(x) \sim \big(g(t), f(t)\big)$, we note that Sheffer identity is given by

$$s_n\big(x+y\big) = \sum_{j=0}^{n} \binom{n}{j} s_j(x) p_{n-j}\big(y\big), \quad \text{where } p_n(x) = g(t)s_n(x). \tag{2.34}$$

By (2.2) and (2.34), we get

$$K_{n,3}\big(x+y \mid \lambda\big) = \sum_{j=0}^{n} \binom{n}{j} K_{j,3}(x \mid \lambda)\big(y\big)_{n-j}, \tag{2.35}$$

where $p_n(x) = \frac{\lambda t}{\log(1+\lambda(e^t-1))} K_{n,3}(x \mid \lambda) = (x)_n, \quad (n \geq 0).$

From (2.3) and (2.34), we have

$$K_{n,4}\big(x+y \mid \lambda\big) = \sum_{j=0}^{n} \binom{n}{j} K_{j,4}(x \mid \lambda)\big(y\big)_{n-j}, \tag{2.36}$$

where $p_n(x) = \frac{e^{\lambda t}-1}{\log(1+\lambda(e^t-1))} K_{n,4}(x \mid \lambda) = (x)_n.$

By (1.10), we see that

$$\big(e^t - 1\big) K_{n,3}(x \mid \lambda) = nK_{n-1,3}(x \mid \lambda), \tag{2.37}$$

and

$$\begin{aligned}\big(e^t - 1\big) K_{n,3}(x \mid \lambda) &= e^t K_{n,3}(x \mid \lambda) - K_{n,3}(x \mid \lambda) \\ &= K_{n,3}(x+1 \mid \lambda) - K_{n,3}(x \mid \lambda).\end{aligned} \tag{2.38}$$

From (2.37) and (2.38), we have

$$nK_{n-1,3}(x \mid \lambda) = K_{n,3}(x+1 \mid \lambda) - K_{n,3}(x \mid \lambda). \tag{2.39}$$

By (1.10) and (2.3), we get

$$\big(e^t - 1\big) K_{n,4}(x \mid \lambda) = nK_{n-1,4}(x \mid \lambda). \tag{2.40}$$

Thus, by (2.40), we have

$$K_{n,4}(x+1 \mid \lambda) - K_{n,4}(x \mid \lambda) = nK_{n-1,4}(x \mid \lambda). \tag{2.41}$$

Therefore, by (2.35), (2.36), (2.39) and (2.41), we obtain the following theorem.

Theorem 7 *For $n \geq 0$, we have*

$$K_{n,3}\big(x+y \mid \lambda\big) = \sum_{j=0}^{n} \binom{n}{j} K_{j,3}(x \mid \lambda)\big(y\big)_{n-j},$$

$$K_{n,4}\big(x+y \mid \lambda\big) = \sum_{j=0}^{n} \binom{n}{j} K_{j,4}(x \mid \lambda)\big(y\big)_{n-j},$$

$$nK_{n-1,3}(x \mid \lambda) = K_{n,3}(x+1 \mid \lambda) - K_{n,3}(x \mid \lambda),$$

and

$$nK_{n-1,4}(x \mid \lambda) = K_{n,4}(x+1 \mid \lambda) - K_{n,4}(x \mid \lambda).$$

For $s_n(x) \sim \big(g(t), f(t)\big)$, we note that

$$\frac{d}{dx}s_n(x) = \sum_{l=0}^{n-1}\binom{n}{l}\left\langle \bar{f}(t) \middle| x^{n-l}\right\rangle s_l(x). \tag{2.42}$$

For $K_{n,3}(x \mid \lambda) \sim \left(\frac{\lambda t}{\log(1+\lambda(e^t-1))}, e^t - 1\right)$, by (2.42), we get

$$\begin{aligned}\left\langle \bar{f}(t) \middle| x^{n-l}\right\rangle &= \left\langle \log(1+t) \middle| x^{n-l}\right\rangle \\ &= \left\langle \sum_{m=1}^{\infty}(-1)^{m-1}(m-1)!\frac{t^m}{m!} \middle| x^{n-l}\right\rangle \\ &= (-1)^{n-l-1}(n-l-1)!.\end{aligned} \tag{2.43}$$

Thus, by (2.42) and (2.43), we have

$$\begin{aligned}\frac{d}{dx}K_{n,3}(x \mid \lambda) &= \sum_{l=0}^{n-1}\binom{n}{l}(-1)^{n-l-1}(n-l-1)!K_{l,3}(x \mid \lambda) \\ &= n!\sum_{l=0}^{n-1}\frac{(-1)^{n-l-1}}{l!(n-l)}K_{l,3}(x \mid \lambda).\end{aligned} \tag{2.44}$$

By the same method as (2.44), we get

$$\frac{d}{dx}K_{n,4}(x \mid \lambda) = n!\sum_{l=0}^{n-1}\frac{(-1)^{n-l-1}}{l!(n-l)}K_{l,4}(x \mid \lambda). \tag{2.45}$$

Therefore, by (2.44) and (2.45), we obtain the following theorem.

Theorem 8 *For $n \geq 1$, we have*

$$\frac{d}{dx}K_{n,3}(x \mid \lambda) = n!\sum_{l=0}^{n-1}\frac{(-1)^{n-l-1}}{l!(n-l)}K_{l,3}(x \mid \lambda),$$

and

$$\frac{d}{dx}K_{n,4}(x \mid \lambda) = n!\sum_{l=0}^{n-1}\frac{(-1)^{n-l-1}}{l!(n-l)}K_{l,4}(x \mid \lambda).$$

Let $n \geq 1$. Then, by (1.6), (2.1) and (2.3), we get

$$\begin{aligned}&K_{n,3}(y \mid \lambda) \\ &= \left\langle \frac{\log(1+\lambda t)}{\lambda\log(1+t)}(1+t)^y \middle| x^n\right\rangle \\ &= \left\langle \partial_t\left(\frac{\log(1+\lambda t)}{\lambda\log(1+t)}(1+t)^y\right) \middle| x^{n-1}\right\rangle \\ &= \left\langle \frac{\log(1+\lambda t)}{\lambda\log(1+t)}\partial_t(1+t)^y \middle| x^{n-1}\right\rangle + \left\langle \left(\partial_t\frac{\log(1+\lambda t)}{\log(1+t)}\right)(1+t)^y \middle| x^{n-1}\right\rangle.\end{aligned} \tag{2.46}$$

We observe that

$$\left\langle \frac{\log(1+\lambda t)}{\lambda\log(1+t)}\partial_t(1+t)^y \middle| x^{n-1}\right\rangle = y\left\langle \frac{\log(1+\lambda t)}{\lambda\log(1+t)}(1+t)^{y-1}\middle| x^{n-1}\right\rangle$$
$$= yK_{n-1,3}(y-1\mid\lambda), \tag{2.47}$$

and

$$\partial_t\left(\frac{\log(1+\lambda t)}{\lambda\log(1+t)}\right) = \frac{\frac{\lambda}{1+\lambda t}\cdot\lambda\log(1+t)-\log(1+\lambda t)\frac{\lambda}{1+t}}{\left(\lambda\log(1+t)\right)^2}$$
$$= \frac{t}{\log(1+t)}\frac{1}{t}\left\{\frac{1}{1+\lambda t}-\frac{\log(1+\lambda t)}{\lambda\log(1+t)}(1+t)^{-1}\right\}. \tag{2.48}$$

By (2.48), we get

$$\left\langle \left(\partial_t\left(\frac{\log(1+\lambda t)}{\log(1+t)}\right)\right)(1+t)^y\middle| x^{n-1}\right\rangle$$
$$= \left\langle \frac{t}{\log(1+t)}\frac{1}{t}\left\{\frac{1}{1+\lambda t}-\frac{\log(1+\lambda t)}{\lambda\log(1+t)}(1+t)^{-1}\right\}(1+t)^y\middle| x^{n-1}\right\rangle$$
$$= \frac{1}{n}\left\langle \left\{\frac{1}{1+\lambda t}-\frac{\log(1+\lambda t)}{\lambda\log(1+t)}(1+t)^{-1}\right\}(1+t)^y\middle| \frac{t}{\log(1+t)}x^n\right\rangle$$
$$= \frac{1}{n}\left\langle \left\{\frac{1}{1+\lambda t}-\frac{\log(1+\lambda t)}{\lambda\log(1+t)}(1+t)^{-1}\right\}(1+t)^y\middle| \sum_{l=0}^{\infty}b_l\frac{t^l}{l!}x^n\right\rangle$$
$$= \frac{1}{n}\sum_{l=0}^{n}\binom{n}{l}b_l\left\{\left\langle\frac{1}{1+\lambda t}(1+t)^y\middle| x^{n-l}\right\rangle-\left\langle\frac{\log(1+\lambda t)}{\lambda\log(1+t)}(1+t)^{y-1}\middle| x^{n-l}\right\rangle\right\}$$
$$= \frac{1}{n}\sum_{l=0}^{n}\binom{n}{l}b_l\left\{\left\langle(1+t)^y\middle|\sum_{m=0}^{\infty}(-\lambda t)^m x^{n-l}\right\rangle-K_{n-l,3}(y-1\mid\lambda)\right\}$$
$$= \frac{1}{n}\sum_{l=0}^{n}\binom{n}{l}b_l\left\{\sum_{m=0}^{n-l}(-\lambda)^m(n-l)_m\left\langle(1+t)^y\middle| x^{n-l-m}\right\rangle-K_{n-l,3}(y-1\mid\lambda)\right\}$$
$$= \frac{1}{n}\sum_{l=0}^{n}\binom{n}{l}b_l\left\{\sum_{m=0}^{n-l}(-\lambda)^m(n-l)_m(y)_{n-l-m}-K_{n-l,3}(y-1\mid\lambda)\right\}. \tag{2.49}$$

For $n\geq 1$, from (2.46), (2.47) and (2.49), we have

$$K_{n,3}(x\mid\lambda)-xK_{n-1,3}(x-1\mid\lambda)$$
$$= \frac{1}{n}\sum_{l=0}^{n}\binom{n}{l}b_l\left\{\sum_{m=0}^{n-l}(-\lambda)^m(n-l)_m(x)_{n-l-m}-K_{n-l,3}(x-1\mid\lambda)\right\}. \tag{2.50}$$

Therefore, by (2.50), we obtain the following theorem.

Theorem 9 *For $n \geq 1$, we have*

$$\begin{aligned} & K_{n,3}(x \mid \lambda) - xK_{n-1,3}(x-1 \mid \lambda) \\ & = \frac{1}{n}\sum_{l=0}^{n}\binom{n}{l} b_l \left\{ \sum_{m=0}^{n-l} (-\lambda)^m (n-l)_m (x)_{n-l-m} - K_{n-l,3}(x-1 \mid \lambda) \right\}. \end{aligned}$$

Remark We note that

$$\begin{aligned} b_n(x) - xb_{n-1}(x-1) &= \lim_{\lambda \to 0} \big(K_{n,3}(x \mid \lambda) - xK_{n-1,3}(x-1 \mid \lambda)\big) \\ &= \frac{1}{n}\sum_{l=0}^{n}\binom{n}{l} b_l \big((x)_{n-l} - b_{n-l}(x-1)\big). \end{aligned}$$

From (1.6) and (2.2), we have

$$\begin{aligned} K_{n,4}(y \mid \lambda) &= \left\langle \frac{\log(1+\lambda t)}{(1+t)^\lambda - 1}(1+t)^y \middle| x^n \right\rangle \\ &= \left\langle \partial_t \left(\frac{\log(1+\lambda t)}{(1+t)^\lambda - 1}(1+t)^y \right) \middle| x^{n-1} \right\rangle \\ &= \left\langle \frac{\log(1+\lambda t)}{(1+t)^\lambda - 1} \partial_t (1+t)^y \middle| x^{n-1} \right\rangle \\ &\quad + \left\langle \left(\partial_t \left(\frac{\log(1+\lambda t)}{(1+t)^\lambda - 1} \right) \right) (1+t)^y \middle| x^{n-1} \right\rangle, \end{aligned} \tag{2.51}$$

where $n \geq 1$.

We note that

$$\begin{aligned} & \left\langle \frac{\log(1+\lambda t)}{(1+t)^\lambda - 1} \big(\partial_t (1+t)^y\big) \middle| x^{n-1} \right\rangle \\ & = y \left\langle \frac{\log(1+\lambda t)}{(1+t)^\lambda - 1} (1+t)^{y-1} \middle| x^{n-1} \right\rangle \\ & = yK_{n-1,4}(y-1 \mid \lambda). \end{aligned} \tag{2.52}$$

Now, we observe that

$$\begin{aligned} & \partial_t \left(\frac{\log(1+\lambda t)}{(1+t)^\lambda - 1} \right) \\ & = \frac{\frac{\lambda}{1+\lambda t}\Big((1+t)^\lambda - 1\Big) - \log(1+\lambda t)\lambda(1+t)^{\lambda-1}}{\Big((1+t)^\lambda - 1\Big)^2} \\ & = \frac{\lambda t}{(1+t)^\lambda - 1} \frac{1}{t} \left\{ \frac{1}{1+\lambda t} - \frac{\log(1+\lambda t)}{(1+t)^\lambda - 1}(1+t)^{\lambda-1} \right\}. \end{aligned} \tag{2.53}$$

Thus, by (2.53), we get

$$\begin{aligned}
&\left\langle \left(\partial_t \frac{\log(1+\lambda t)}{(1+t)^{\lambda}-1}\right)(1+t)^y \middle| x^{n-1}\right\rangle \\
&= \left\langle \frac{\lambda t}{(1+t)^{\lambda}-1}\frac{1}{t}\left\{\frac{1}{1+\lambda t}-\frac{\log(1+\lambda t)}{(1+t)^{\lambda}-1}(1+t)^{\lambda-1}\right\}(1+t)^y \middle| x^{n-1}\right\rangle \\
&= \frac{1}{n}\left\langle \frac{\lambda t}{(1+t)^{\lambda}-1}\left\{\frac{1}{1+\lambda t}-\frac{\log(1+\lambda t)}{(1+t)^{\lambda}-1}(1+t)^{\lambda-1}\right\}(1+t)^y \middle| x^{n}\right\rangle \\
&= \frac{1}{n}\left\langle \left\{\frac{1}{1+\lambda t}-\frac{\log(1+\lambda t)}{(1+t)^{\lambda}-1}(1+t)^{\lambda-1}\right\}(1+t)^y \middle| \frac{\lambda t}{(1+t)^{\lambda}-1}x^{n}\right\rangle \\
&= \frac{1}{n}\left\langle \left\{\frac{1}{1+\lambda t}-\frac{\log(1+\lambda t)}{(1+t)^{\lambda}-1}(1+t)^{\lambda-1}\right\}(1+t)^y \middle| \sum_{l=0}^{\infty}K_l(\lambda)\frac{t^l}{l!}x^{n}\right\rangle \\
&= \frac{1}{n}\sum_{l=0}^{n}\binom{n}{l}K_l(\lambda)\left\langle \frac{1}{1+\lambda t}(1+t)^y \middle| x^{n-l}\right\rangle \\
&\quad - \left\langle \frac{\log(1+\lambda t)}{(1+t)^{\lambda}-1}(1+t)^{y+\lambda-1} \middle| x^{n-l}\right\rangle \\
&= \frac{1}{n}\sum_{l=0}^{n}\binom{n}{l}K_l(\lambda)\left\{\left\langle (1+t)^y \middle| \sum_{m=0}^{\infty}(-\lambda t)^m x^{n-l}\right\rangle - K_{n-1,4}(y+\lambda-1\mid\lambda)\right\} \\
&= \frac{1}{n}\sum_{l=0}^{n}\binom{n}{l}K_l(\lambda)\left\{\sum_{m=0}^{n-l}(-\lambda)^m(n-l)_m\left\langle (1+t)^y \middle| x^{n-l-m}\right\rangle\right. \\
&\quad \left. -K_{n-1,4}(y+\lambda-1\mid\lambda)\right\} \\
&= \frac{1}{n}\sum_{l=0}^{n}\binom{n}{l}K_l(\lambda)\left\{\sum_{m=0}^{n-l}(-\lambda)^m(n-l)_m(y)_{n-l-m} - K_{n-l,4}(y+\lambda-1\mid\lambda)\right\} \\
&= \frac{1}{n}\sum_{l=0}^{n}\binom{n}{l}K_l(\lambda)\left\{\sum_{m=0}^{n-l}(-\lambda)^m(n-l)_m(y)_{n-l-m} - K_{n-l,4}(y+\lambda-1\mid\lambda)\right\}. \qquad (2.54)
\end{aligned}$$

From (2.51), (2.52) and (2.54), we have

$$\begin{aligned}
&K_{n,4}(x\mid\lambda) - xK_{n-1,4}(x-1\mid\lambda) \\
&= \frac{1}{n}\sum_{l=0}^{n}\binom{n}{l}K_l(\lambda)\left\{\sum_{m=0}^{n-l}(-\lambda)^m(n-l)_m(x)_{n-l-m} - K_{n-l,4}(x+\lambda-1\mid\lambda)\right\}. \qquad (2.55)
\end{aligned}$$

Therefore, by (2.55), we obtain the following theorem.

Theorem 10 *For $n \geq 1$, we have*

$$\begin{aligned}
&K_{n,4}(x\mid\lambda) - xK_{n-1,4}(x-1\mid\lambda) \\
&= \frac{1}{n}\sum_{l=0}^{n}\binom{n}{l}K_l(\lambda)\left\{\sum_{m=0}^{n-l}(-\lambda)^m(n-l)_m(x)_{n-l-m} - K_{n-l,4}(x+\lambda-1\mid\lambda)\right\}.
\end{aligned}$$

Let us consider the following two Sheffer sequences:

$$\begin{aligned}
&K_{n,3}(x\mid\lambda) \sim \left(\frac{\lambda t}{\log\left(1+\lambda\left(e^t-1\right)\right)}, e^t-1\right), \\
&(x)^{(n)} = x(x+1)\cdots(x+(n-1)) \sim \left(1, 1-e^{-t}\right). \qquad (2.56)
\end{aligned}$$

From (1.14) and (1.15), we have

$$K_{n,3}(x \mid \lambda) = \sum_{m=0}^{n} C_{n,m}(x)^{(m)}, \tag{2.57}$$

where

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!}\left\langle \frac{\log(1+\lambda t)}{\lambda \log(1+t)}\left(1-\frac{1}{1+t}\right)^m \middle| x^n \right\rangle \\
&= \frac{1}{m!}\left\langle \frac{\log(1+\lambda t)}{\lambda \log(1+t)} \middle| \sum_{l=0}^{\infty}(-1)^l (m+l-1)_l \frac{t^{m+l}}{l!} x^n \right\rangle \\
&= \frac{1}{m!}\sum_{l=0}^{n-m}(-1)^l (m+l-1)_l \frac{(n)_{m+l}}{l!}\left\langle \frac{\log(1+\lambda t)}{\lambda t} \middle| \frac{t}{\log(1+t)} x^{n-m-l} \right\rangle \\
&= \frac{1}{m!}\sum_{l=0}^{n-m}(-1)^l (m+l-1)_l \frac{(n)_{m+l}}{l!}\left\langle \frac{\log(1+\lambda t)}{\lambda t} \middle| \sum_{k=0}^{\infty} b_k \frac{t^k}{k!} x^{n-m-l} \right\rangle \\
&= \sum_{l=0}^{n-m}(-1)^l l! \binom{m+l-1}{l}\binom{n}{m+l}\binom{m+l}{m} \\
&\quad \times \sum_{k=0}^{n-m-l}\binom{n-m-l}{k} b_k \left\langle \frac{\log(1+\lambda t)}{\lambda t} \middle| x^{n-m-l-k} \right\rangle \\
&= \sum_{l=0}^{n-m}\sum_{k=0}^{n-m-l}(-1)^l l! \binom{m+l-1}{l}\binom{n}{m+l}\binom{m+l}{m} \\
&\quad \times \binom{n-m-l}{k} b_k D_{n-m-l-k} \lambda^{n-m-l-k} \\
&= \sum_{l=0}^{n-m}\sum_{k=0}^{n-m-l}(-1)^l l! \binom{m+l-1}{l}\binom{n}{m+l}\binom{m+l}{m} \\
&\quad \times \binom{n-m-l}{k} b_{n-m-l-k} D_k \lambda^k
\end{aligned} \tag{2.58}$$

Therefore, by (2.57) and (2.58), we obtain the following theorem.

Theorem 11 *For $n \geq 0$, we have*

$$\begin{aligned}
K_{n,3}(x \mid \lambda) &= \sum_{m=0}^{n}\left\{ \sum_{l=0}^{n-m}\sum_{k=0}^{n-m-l}(-1)^l l! \binom{m+l-1}{l}\binom{n}{m+l} \right. \\
&\quad \left. \times \binom{m+l}{m}\binom{n-m-l}{k} b_{n-m-l-k} D_k \lambda^k \right\} (x)^{(m)}.
\end{aligned}$$

For $K_{n,4}(x \mid \lambda) \sim \left(\frac{e^{\lambda t}-1}{\log(1+\lambda(e^t-1))}, e^t-1\right)$, $(x)^{(n)} \sim \left(1, 1-e^{-t}\right)$, we have

$$K_{n,4}(x \mid \lambda) = \sum_{m=0}^{n} C_{n,m}(x)^{(m)}, \tag{2.59}$$

where

$$
\begin{aligned}
&C_{n,m}\\
&=\frac{1}{m!}\left\langle \frac{\log(1+\lambda t)}{(1+t)^{\lambda}-1}\left(1-\frac{1}{1+t}\right)^{m}\middle| x^{n}\right\rangle\\
&=\frac{1}{m!}\left\langle \frac{\log(1+\lambda t)}{(1+t)^{\lambda}-1}\middle|\sum_{l=0}^{\infty}(-1)^{l}(m+l-1)_{l}\frac{t^{m+l}}{l!}x^{n}\right\rangle\\
&=\frac{1}{m!}\sum_{l=0}^{n-m}(-1)^{l}(m+l-1)_{l}\frac{(n)_{m+l}}{l!}\left\langle \frac{\log(1+\lambda t)}{\lambda t}\middle|\frac{\lambda t}{(1+t)^{\lambda}-1}x^{n-m-l}\right\rangle\\
&=\frac{1}{m!}\sum_{l=0}^{n-m}(-1)^{l}(m+l-1)_{l}\frac{(n)_{m+l}}{l!}\left\langle \frac{\log(1+\lambda t)}{\lambda t}\middle|\sum_{k=0}^{\infty}K_{k}(\lambda)\frac{t^{k}}{k!}x^{n-m-l}\right\rangle\\
&=\frac{1}{m!}\sum_{l=0}^{n-m}(-1)^{l}(m+l-1)_{l}\frac{(n)_{m+l}}{l!}\sum_{k=0}^{n-m-l}\binom{n-m-l}{k}K_{k}(\lambda)\\
&\quad\times\left\langle \frac{\log(1+\lambda t)}{\lambda t}\middle| x^{n-m-l-k}\right\rangle\\
&=\frac{1}{m!}\sum_{l=0}^{n-m}(-1)^{l}(m+l-1)_{l}\frac{(n)_{m+l}}{l!}\\
&\quad\times\sum_{k=0}^{n-m-l}\binom{n-m-l}{k}K_{k}(\lambda)D_{n-m-l-k}\lambda^{n-m-l-k}\\
&=\sum_{l=0}^{n-m}\sum_{k=0}^{n-m-l}(-1)^{l}l!\binom{m+l-1}{l}\binom{n}{m+l}\binom{m+l}{m}\\
&\quad\times\binom{n-m-l}{k}K_{n-m-l-k}(\lambda)D_{k}\lambda^{k}.
\end{aligned}
\tag{2.60}
$$

Therefore, by (2.59) and (2.60), we obtain the following theorem.

Theorem 12 *For $n \geq 0$, we have*

$$
\begin{aligned}
&K_{n,4}(x\mid\lambda)\\
&=\sum_{m=0}^{n}\left\{\sum_{l=0}^{n-m}\sum_{k=0}^{n-m-l}(-1)^{l}l!\binom{m+l-1}{l}\right.\\
&\quad\left.\times\binom{n}{m+l}\binom{m+l}{m}\binom{n-m-l}{k}K_{n-m-l-k}(\lambda)D_{k}\lambda^{k}\right\}(x)^{(m)}.
\end{aligned}
$$

Let us consider the following two Sheffer sequences:

$$
K_{n,3}(x\mid\lambda)\sim\left(\frac{\lambda t}{\log\left(1+\lambda\left(e^{t}-1\right)\right)},e^{t}-1\right),\quad B_{n}^{(s)}(x)\sim\left(\left(\frac{e^{t}-1}{t}\right)^{s},t\right).
$$

Note that

$$
\sum_{n=0}^{\infty}B_{n}^{(s)}(x)=\left(\frac{t}{e^{t}-1}\right)^{s}e^{xt},\quad \text{(see Kim et al. 2015; Sen et al. 2013; Ustinov et al. 2002),}
$$

where $B_n^{(s)}(x)$ are called the higher-order Bernoulli polynomials.

From (1.14) and (1.15), we have

$$K_{n,3}(x \mid \lambda) = \sum_{m=0}^{n} C_{n,m} B_m^{(s)}(x), \tag{2.61}$$

where

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!}\left\langle \left(\frac{t}{\log(1+t)}\right)^s \frac{\log(1+\lambda t)}{\lambda\log(1+t)} \left(\log(1+t)\right)^m \middle| x^n \right\rangle \\
&= \left\langle \left(\frac{t}{\log(1+t)}\right)^s \frac{\log(1+\lambda t)}{\lambda\log(1+t)} \middle| \frac{1}{m!}\left(\log(1+t)\right)^m x^n \right\rangle \\
&= \left\langle \left(\frac{t}{\log(1+t)}\right)^s \frac{\log(1+\lambda t)}{\lambda\log(1+t)} \middle| \sum_{l=m}^{\infty} S_1(l,m)\frac{t^l}{l!} x^n \right\rangle \\
&= \sum_{l=m}^{n} \binom{n}{l} S_1(l,m) \left\langle \left(\frac{t}{\log(1+t)}\right)^s \middle| \frac{\log(1+\lambda t)}{\lambda\log(1+t)} x^{n-l} \right\rangle \\
&= \sum_{l=m}^{n} \binom{n}{l} S_1(l,m) \left\langle \left(\frac{t}{\log(1+t)}\right)^s \middle| \sum_{k=0}^{\infty} K_{k,3}(\lambda)\frac{t^k}{k!} x^{n-l} \right\rangle \\
&= \sum_{l=m}^{n} \binom{n}{l} S_1(l,m) \sum_{k=0}^{n-l} \binom{n-l}{k} K_{k,3}(\lambda) \left\langle \left(\frac{t}{\log(1+t)}\right)^s \middle| x^{n-l-k} \right\rangle \\
&= \sum_{l=m}^{n} \binom{n}{l} S_1(l,m) \sum_{k=0}^{n-l} \binom{n-l}{k} K_{k,3}(\lambda) b_{n-l-k}^{(s)} \\
&= \sum_{l=m}^{n} \sum_{k=0}^{n-l} \binom{n}{l} \binom{n-l}{k} S_1(l,m) K_{k,3}(\lambda) b_{n-l-k}^{(s)}.
\end{aligned} \tag{2.62}$$

Here, the Bernoulli numbers of the second kind of order s are defined by the generating function

$$\left(\frac{t}{\log(1+t)}\right)^s = \sum_{j=0}^{\infty} b_j^{(s)} \frac{t^j}{j!}, \quad \text{(see Kim et al. 2015; Roman 1984).}$$

Therefore, by (2.61) and (2.62), we obtain the following theorem.

Theorem 13 *For* $n \geq 0$, *we have*

$$K_{n,3}(x \mid \lambda) = \sum_{m=0}^{n} \left(\sum_{l=m}^{n} \sum_{k=0}^{n-l} \binom{n}{l} \binom{n-l}{k} S_1(l,m) K_{k,3}(\lambda) b_{n-l-k}^{(s)} \right) B_m^{(s)}(x).$$

Remark In a similar manner, one shows that, for $n \geq 0$,

$$K_{n,4}(x \mid \lambda) = \sum_{m=0}^{n} \left(\sum_{l=m}^{n} \sum_{k=0}^{n-l} \binom{n}{l} \binom{n-l}{k} S_1(l,m) K_{k,4}(\lambda) b_{n-l-k}^{(s)} \right) B_m^{(s)}(x).$$

For $\mu \in \mathbb{C}$ with $\mu \neq 1$, $s \in \mathbb{N}$, the Frobenius-Euler polynomials of order s are defined by the generating function

$$\left(\frac{1-\mu}{e^t-\mu}\right)^s e^{xt} = \sum_{n=0}^{\infty} H_n^{(s)}(x \mid \mu)\frac{t^n}{n!}, \quad \text{(see [1,16]).}$$

For $K_{n,3}(x \mid \lambda) \sim \left(\frac{\lambda t}{\log(1+\lambda(e^t-1))}, e^t-1\right)$, $H_n^{(s)}(x \mid \mu) \sim \left(\left(\frac{e^t-\mu}{1-\mu}\right)^s, t\right)$, we have

$$K_{n,3}(x \mid \lambda) = \sum_{m=0}^{n} C_{n,m} H_m^{(s)}(x \mid \mu), \tag{2.63}$$

where

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!}\left\langle \left(\frac{1-\mu+t}{1-\mu}\right)^s \frac{\log(1+\lambda t)}{\lambda\log(1+t)}(\log(1+t))^m \middle| x^n \right\rangle \\
&= \left\langle \left(\frac{1-\mu+t}{1-\mu}\right)^s \frac{\log(1+\lambda t)}{\lambda\log(1+t)} \middle| \frac{1}{m!}(\log(1+t))^m x^n \right\rangle \\
&= \left\langle \left(\frac{1-\mu+t}{1-\mu}\right)^s \frac{\log(1+\lambda t)}{\lambda\log(1+t)} \middle| \sum_{l=m}^{\infty} S_1(l,m)\frac{t^l}{l!} x^n \right\rangle \\
&= \sum_{l=m}^{n}\binom{n}{l} S_1(l,m)\left\langle \left(\frac{1-\mu+t}{1-\mu}\right)^s \middle| \frac{\log(1+\lambda t)}{\lambda\log(1+t)} x^{n-l} \right\rangle \\
&= \sum_{l=m}^{n}\binom{n}{l} S_1(l,m)\left\langle \left(\frac{1-\mu+t}{1-\mu}\right)^s \middle| \sum_{k=0}^{\infty} K_{k,3}(\lambda)\frac{t^k}{k!} x^{n-l} \right\rangle \\
&= \sum_{l=m}^{n}\binom{n}{l} S_1(l,m)\sum_{k=0}^{n-l}\binom{n-l}{k} K_{k,3}(\lambda)\left\langle \left(\frac{1-\mu+t}{1-\mu}\right)^s \middle| x^{n-l-k} \right\rangle \\
&= \frac{1}{(1-\mu)^s}\sum_{l=m}^{n}\binom{n}{l} S_1(l,m)\sum_{k=0}^{n-l}\binom{n-l}{k} K_{k,3}(\lambda) \\
&\quad \times \sum_{j=0}^{s}\binom{s}{j}(1-\mu)^{s-j}\left\langle t^j \middle| x^{n-l-k} \right\rangle \\
&= \frac{1}{(1-\mu)^s}\sum_{l=m}^{n}\binom{n}{l} S_1(l,m)\sum_{k=0}^{n-l}\binom{n-l}{k} K_{n-l-k,3}(\lambda) \\
&\quad \times \sum_{j=0}^{s}\binom{s}{j}(1-\mu)^{s-j}\left\langle t^j \middle| x^{k} \right\rangle \\
&= \frac{1}{(1-\mu)^s}\sum_{l=m}^{n}\binom{n}{l} S_1(l,m)\sum_{k=0}^{n-l}\binom{n-l}{k} K_{n-l-k,3}(\lambda) k!\binom{s}{k}(1-\mu)^{s-k} \\
&= \sum_{l=m}^{n}\sum_{k=0}^{n-l}\binom{n}{l}\binom{n-l}{k}\binom{s}{k}\frac{k!}{(1-\mu)^k} S_1(l,m) K_{n-l-k,3}(\lambda).
\end{aligned} \tag{2.64}$$

Therefore, by (2.63) and (2.64), we obtain the following theorem.

Theorem 14 *For $n \geq 0$, we have*

$$K_{n,3}(x \mid \lambda) = \sum_{m=0}^{n} \left(\sum_{l=m}^{n} \sum_{k=0}^{n-l} \binom{n}{l} \binom{n-l}{k} \binom{s}{k} \frac{k!}{(1-\mu)^k} S_1(l,m) K_{n-l-k,3}(\lambda) \right) H_m^{(s)}(x \mid \mu).$$

Remark Proceeding similarly to the above, one can show that, for $n \geq 0$,

$$K_{n,4}(x \mid \lambda) = \sum_{m=0}^{n} \left(\sum_{l=m}^{n} \sum_{k=0}^{n-l} \binom{n}{l} \binom{n-l}{k} \binom{s}{k} \frac{k!}{(1-\mu)^k} S_1(l,m) K_{n-l-k,4}(\lambda) \right) H_m^{(s)}(x \mid \mu).$$

Conclusion

The first degenerate version of the Bernoulli polynomials of the second kind appeared in the paper by Korobov (1996, 2001). Here, we study two degenerate versions of the Bernoulli polynomials of the second kind which will be called Korobov polynomials of third kind and of the fourth kind. Some properties, identities, recurrence relations and connections with other polynomials are investigated by using umbral calculus.

Authors' contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

Author details

[1] Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin 300387, China. [2] Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea. [3] Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea.

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Competing interests

The authors declare that they have no competing interests.

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