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Fourth order elliptic operator-differential equations with unbounded operator boundary conditions in the Sobolev-type spaces

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Abstract

Conditions for well-posed and unique solvability of a non-homogeneous boundary value problem for a class of fourth order elliptic operator-differential equations with an unbounded operator in boundary conditions are found in this work. Note that these solvability conditions are sufficient, and they are expressed only in terms of the properties of operator coefficients of the boundary value problem. Besides, the estimates for the norms of intermediate derivative operators in a Sobolev-type space are obtained, and their close relationship with the solvability conditions is established.

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1 Introduction

Many applied problems of mathematical physics require the study of spectral problems with a polynomial appearance of a parameter in the boundary conditions [1–3], while, in corresponding inverse problems, the unknown coefficients appearing in the equation and boundary conditions are found using known spectra [4–6]. However, some non-classical problems of mathematical physics [7] reduce to the problems with an operator in the boundary conditions. Note that the well-posed and unique solvability and Fredholmness of the boundary value problems for second and third order operator-differential equations with operator boundary conditions have been widely studied both on a finite interval and on the half-axis (see, *e.g.*, [8–17] and the references therein, though far from being complete). The works dedicated to such kind of problems for fourth order operator-differential equations are relatively few. We can only mention [18, 19]. Here it should be noted that the solvability of the boundary value problems for operator-differential equations of fourth and higher orders in case where the coefficients in the boundary conditions are only complex numbers has been extensively studied in [20–34]. See the references therein.

In this work, we treat the well-posed and unique solvability of a non-homogeneous boundary value problem for a fourth order elliptic operator-differential equation with un-

bounded operator boundary conditions. The solvability of this problem allows using the obtained abstract results in the study of both new classes of boundary value problems for fourth order elliptic partial differential equations and the corresponding spectral problems.

2 Problem statement

Let H be a separable Hilbert space with scalar product (x, y), $x, y \in H$, and A be a positive definite self-adjoint operator in H ($A = A^* \geq cE$, $c > 0$, E is the identity operator). By H_γ ($\gamma \geq 0$) we will mean the scale of Hilbert spaces generated by the operator A, i.e., $H_\gamma = D(A^\gamma)$, $(x, y)_\gamma = (A^\gamma x, A^\gamma y)$, $x, y \in D(A^\gamma)$, with $H_0 = H$, $(x, y)_0 = (x, y)$, $x, y \in H$.

Denote by $L_2(\mathbb{R}_+; H)$ the Hilbert space of all vector-valued functions defined on $\mathbb{R}_+ = (0, +\infty)$ with the values in H and the finite norm

$$\|f\|_{L_2(\mathbb{R}_+; H)} = \left(\int_0^{+\infty} \|f(t)\|_H^2 \, dt \right)^{1/2}.$$

Following [35], Chapter 1, we introduce the Hilbert space

$$W_2^4(\mathbb{R}_+; H) = \left\{ u(t) : u^{(4)}(t) \in L_2(\mathbb{R}_+; H), A^4 u(t) \in L_2(\mathbb{R}_+; H) \right\}$$

equipped with the norm

$$\|u\|_{W_2^4(\mathbb{R}_+; H)} = \left(\left\| u^{(4)} \right\|_{L_2(\mathbb{R}_+; H)}^2 + \left\| A^4 u \right\|_{L_2(\mathbb{R}_+; H)}^2 \right)^{1/2}.$$

Hereinafter the derivatives $u^{(j)} \equiv \frac{d^j u}{dt^j}$ are understood in the sense of the theory of distributions in a Hilbert space [35]. Similarly we define the spaces $L_2(\mathbb{R}; H)$ and $W_2^4(\mathbb{R}; H)$, where $\mathbb{R} = (-\infty, +\infty)$.

By $L(X, Y)$ we mean the set of linear bounded operators acting from the Hilbert space X to another Hilbert space Y. Fix some operator $K \in L(H_{5/2}, H_{3/2})$. Let us define the following subspace of the space $W_2^4(\mathbb{R}_+; H)$:

$$W_{2,K}^4(\mathbb{R}_+; H) = \left\{ u(t) : u(t) \in W_2^4(\mathbb{R}_+; H), u(0) = 0, u''(0) = Ku'(0) \right\}.$$

The trace theorem of [35], Chapter 1, implies the correctness of the last definition.

Next, denote by $\sigma(\cdot)$ the spectrum of the operator $(\cdot)$.

Consider the following boundary value problem in the space H:

$$u^{(4)}(t) + A^4 u(t) + \sum_{j=1}^{4} A_j u^{(4-j)}(t) = f(t), \quad t \in \mathbb{R}_+, \tag{2.1}$$

$$u(0) = \varphi, \qquad u''(0) - Ku'(0) = \psi, \tag{2.2}$$

where $A = A^* \geq cE$, $c > 0$, A_j, $j = 1, 2, 3, 4$, are linear and, in general, unbounded operators, $K \in L(H_{5/2}, H_{3/2})$, $f(t) \in L_2(\mathbb{R}_+; H)$, $\varphi \in H_{7/2}$, $\psi \in H_{3/2}$, $u(t) \in W_2^4(\mathbb{R}_+; H)$.

Definition 2.1 If for every $f(t) \in L_2(\mathbb{R}_+; H)$ and every $\varphi \in H_{7/2}$, $\psi \in H_{3/2}$ there exists a function $u(t) \in W_2^4(\mathbb{R}_+; H)$ that satisfies equation (2.1) almost everywhere in $\mathbb{R}_+$, the

boundary conditions (2.2) in the sense of convergence

$$\lim_{t \to 0^+} \left\| u(t) - \varphi \right\|_{H_{7/2}} = 0,$$

$$\lim_{t \to 0^+} \left\| u''(t) - Ku'(t) - \psi \right\|_{H_{3/2}} = 0$$

and the estimate

$$\|u\|_{W_2^4(\mathbb{R}_+;H)} \le \text{const} \left(\|f\|_{L_2(\mathbb{R}_+;H)} + \|\varphi\|_{H_{7/2}} + \|\psi\|_{H_{3/2}} \right),$$

then the boundary value problem (2.1), (2.2) is said to be regularly solvable, and $u(t)$ is called a regular solution of the boundary value problem (2.1), (2.2).

The main purpose of this work is to find the conditions for regular solvability of the boundary value problem (2.1), (2.2) under some restrictions on its operator coefficients. To achieve this purpose, we use the estimates for the norms of intermediate derivative operators by the norm of an operator generated by the principal part of the considered equation and the given boundary conditions. This is precisely what distinguishes this paper. Note that the method offered in [23, 36] and later developed in [31] to calculate the exact values of the norms of intermediate derivative operators is not directly applicable in our case. That is why we have to apply another procedure to estimate such norms, based on the classical inequalities of analysis. Being not too complicated and quite original, this procedure allows only upper estimates for the considered norms. But this is good enough for the purposes of this paper.

3 Main results

We first assume that $A_j = 0$, $j = 1, 2, 3, 4$, and $\varphi = \psi = 0$ in the boundary value problem (2.1), (2.2). Then we get a simpler boundary value problem

$$u^{(4)}(t) + A^4 u(t) = f(t), \quad t \in \mathbb{R}_+, \tag{3.1}$$

$$u(0) = 0, \qquad u''(0) = Ku'(0). \tag{3.2}$$

Denote by P_0 the operator that acts from $W_{2,K}^4(\mathbb{R}_+;H)$ to $L_2(\mathbb{R}_+;H)$ as follows:

$$P_0 u(t) = u^{(4)}(t) + A^4 u(t), \quad u(t) \in W_{2,K}^4(\mathbb{R}_+;H).$$

The following lemma is true.

Lemma 3.1 *Let* $B = A^{3/2} K A^{-5/2}$ *and* $-\sqrt{2} \notin \sigma(B)$. *Then the equation* $P_0 u(t) = 0$ *has a unique zero solution in* $W_{2,K}^4(\mathbb{R}_+;H)$.

Proof Note that the general solution of the equation $P_0 u(t) = 0$ belonging to the space $W_2^4(\mathbb{R}_+;H)$ has the following form:

$$u_0(t) = e^{\omega_1 tA} \xi_0 + e^{\omega_2 tA} \xi_1,$$

where the exponentials $e^{\omega_1 tA}$, $e^{\omega_2 tA}$ are the (C_0) semigroups generated by $\omega_1 A$, $\omega_2 A$, respectively,

$$\omega_1 = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, \qquad \omega_2 = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i,$$

and $\xi_0, \xi_1 \in H_{7/2}$. From (3.2) we have

$$\begin{cases} \xi_0 + \xi_1 = 0, \\ \omega_1^2 A^2 \xi_0 + \omega_2^2 A^2 \xi_1 = KA(\omega_1 \xi_0 + \omega_2 \xi_1). \end{cases} \tag{3.3}$$

System (3.3) yields

$$\xi_1 = -\xi_0, \tag{3.4}$$

$$\left(E + \frac{1}{\sqrt{2}}B\right)A^{7/2}\xi_0 = 0. \tag{3.5}$$

Then, by the condition $-\sqrt{2} \notin \sigma(B)$ of the lemma, it follows from equation (3.5) that $\xi_0 = 0$. Hence from (3.4) we have $\xi_1 = 0$. Consequently, $u_0(t) = 0$. The lemma is proved. □

The following theorem is true.

Theorem 3.2 *Let $B = A^{3/2}KA^{-5/2}$ and $-\sqrt{2} \notin \sigma(B)$. Then, for every $f(t) \in L_2(\mathbb{R}_+; H)$, the boundary value problem* (3.1), (3.2) *has a unique regular solution.*

Proof By virtue of Lemma 3.1, the problem

$$u^{(4)}(t) + A^4 u(t) = 0, \quad t \in \mathbb{R}_+,$$

$$u(0) = 0, \qquad u''(0) = Ku'(0)$$

has only zero solution in $W_{2,K}^4(\mathbb{R}_+; H)$.

Let us show that the equation $P_0 u(t) = f(t)$ has a solution $u(t) \in W_{2,K}^4(\mathbb{R}_+; H)$ for every $f(t) \in L_2(\mathbb{R}_+; H)$. For this aim, we first continue the vector function $f(t)$ by zero for $t < 0$ and denote the obtained function by $F(t)$. Let $\hat{F}(\xi)$ be the Fourier transform of the vector function $F(t)$, i.e.,

$$\hat{F}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(t)e^{-i\xi t}\, dt,$$

where the integral on the right-hand side is understood in the sense of mean convergence in H.

Applying direct and inverse Fourier transforms, we easily see that the vector function

$$v(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(\xi^4 E + A^4\right)^{-1} \left(\int_0^{+\infty} f(s)e^{-i\xi s}\, ds\right)e^{it\xi}\, d\xi, \quad t \in \mathbb{R},$$

satisfies the equation

$$v^{(4)}(t) + A^4 v(t) = F(t)$$

almost everywhere in $\mathbb{R}$. Let us show that $\upsilon(t) \in W_2^4(\mathbb{R}; H)$. Let $\hat{\upsilon}(\xi)$ be the Fourier transform of the vector function $\upsilon(t)$. By the Plancherel theorem, we have

$$
\begin{aligned}
\left\| \upsilon(t) \right\|_{W_2^4(\mathbb{R};H)}^2 &= \left\| \upsilon^{(4)}(t) \right\|_{L_2(\mathbb{R};H)}^2 + \left\| A^4 \upsilon(t) \right\|_{L_2(\mathbb{R};H)}^2 \\
&= \left\| \xi^4 \hat{\upsilon}(\xi) \right\|_{L_2(\mathbb{R};H)}^2 + \left\| A^4 \hat{\upsilon}(\xi) \right\|_{L_2(\mathbb{R};H)}^2 \\
&= \left\| \xi^4 \left(\xi^4 E + A^4 \right)^{-1} \hat{F}(\xi) \right\|_{L_2(\mathbb{R};H)}^2 \\
&\quad + \left\| A^4 \left(\xi^4 E + A^4 \right)^{-1} \hat{F}(\xi) \right\|_{L_2(\mathbb{R};H)}^2 \\
&\leq \sup_{\xi \in \mathbb{R}} \left\| \xi^4 \left(\xi^4 E + A^4 \right)^{-1} \right\|_{H \to H}^2 \left\| \hat{F}(\xi) \right\|_{L_2(\mathbb{R};H)}^2 \\
&\quad + \sup_{\xi \in \mathbb{R}} \left\| A^4 \left(\xi^4 E + A^4 \right)^{-1} \right\|_{H \to H}^2 \left\| \hat{F}(\xi) \right\|_{L_2(\mathbb{R};H)}^2 \\
&= \left(\sup_{\xi \in \mathbb{R}} \left\| \xi^4 \left(\xi^4 E + A^4 \right)^{-1} \right\|_{H \to H}^2 + \sup_{\xi \in \mathbb{R}} \left\| A^4 \left(\xi^4 E + A^4 \right)^{-1} \right\|_{H \to H}^2 \right) \\
&\quad \times \left\| F(t) \right\|_{L_2(\mathbb{R};H)}^2 .
\end{aligned}
\tag{3.6}
$$

According to the spectral theory of self-adjoint operators, for $\xi \in \mathbb{R}$, we have

$$
\left\| \xi^4 \left(\xi^4 E + A^4 \right)^{-1} \right\| \leq \sup_{\sigma \in \sigma(A)} \left| \xi^4 \left(\xi^4 + \sigma^4 \right)^{-1} \right| \leq 1,
$$

$$
\left\| A^4 \left(\xi^4 E + A^4 \right)^{-1} \right\| \leq \sup_{\sigma \in \sigma(A)} \left| \sigma^4 \left(\xi^4 + \sigma^4 \right)^{-1} \right| \leq 1.
$$

Then it follows from (3.6) that $\upsilon(t) \in W_2^4(\mathbb{R}; H)$.

Now denote by $u_1(t)$ the restriction of the function $\upsilon(t)$ to $\mathbb{R}_+$. Then $u_1(t)$ belongs to the space $W_2^4(\mathbb{R}_+; H)$, satisfies equation (3.1) almost everywhere in $\mathbb{R}_+$ and, according to the trace theorem of [35], Chapter 1, $u_1^{(j)}(0) \in H_{7/2-j}$, $j = 0, 1, 2, 3$.

Next, we seek the solution of the boundary value problem (3.1), (3.2) in the following form:

$$
u(t) = u_1(t) + e^{\omega_1 t A} \eta_0 + e^{\omega_2 t A} \eta_1,
$$

where

$$
\omega_1 = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i, \qquad \omega_2 = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i,
$$

and $\eta_0, \eta_1 \in H_{7/2}$ are subject to be determined from (3.2). Then we get the following system:

$$
\begin{cases}
u_1(0) + \eta_0 + \eta_1 = 0, \\
u_1''(0) + \omega_1^2 A^2 \eta_0 + \omega_2^2 A^2 \eta_1 = K(u_1'(0) + \omega_1 A \eta_0 + \omega_2 A \eta_1).
\end{cases}
\tag{3.7}
$$

Considering the relation

$$
\eta_1 = -\eta_0 - u_1(0)
$$

in the second equation of (3.7), and taking into account the condition $-\sqrt{2} \notin \sigma(B)$, we uniquely determine

$$\eta_0 = A^{-7/2} \left(E + \frac{1}{\sqrt{2}} B \right)^{-1} A^{7/2} \varsigma \in H_{7/2},$$

where

$$\varsigma = \frac{1}{2i} A^{-2} \big[\omega_2 K A u_1(0) - K u_1'(0) + u_1''(0) - i A^2 u_1(0) \big] \in H_{7/2}.$$

Thus, $u(t)$ belongs to the space $W_2^4(\mathbb{R}_+; H)$, satisfies equation (3.1) almost everywhere in $\mathbb{R}_+$ and conditions (3.2).

On the other hand, the operator $P_0 : W_{2,K}^4(\mathbb{R}_+; H) \to L_2(\mathbb{R}_+; H)$ is bounded:

$$\|P_0 u\|_{L_2(\mathbb{R}_+;H)}^2 = \|u^{(4)} + A^4 u\|_{L_2(\mathbb{R}_+;H)}^2 \leq 2\|u\|_{W_2^4(\mathbb{R}_+;H)}^2. \tag{3.8}$$

Therefore, by the Banach inverse operator theorem, there exists the inverse operator $P_0^{-1} : L_2(\mathbb{R}_+; H) \to W_{2,K}^4(\mathbb{R}_+; H)$, and this operator is bounded. It follows

$$\|u\|_{W_2^4(\mathbb{R}_+;H)} \leq \mathrm{const}\, \|f\|_{L_2(\mathbb{R}_+;H)}.$$

The theorem is proved. □

In the sequel, we will need the following auxiliary statement.

Lemma 3.3 *Let $B = A^{3/2} K A^{-5/2}$ and $\mathrm{Re}\, B \geq 0$. Then the inequality*

$$\|P_0 u\|_{L_2(\mathbb{R}_+;H)}^2 \geq \|u\|_{W_2^4(\mathbb{R}_+;H)}^2 + 2\|A^2 u''\|_{L_2(\mathbb{R}_+;H)}^2 \tag{3.9}$$

is true for every $u(t) \in W_{2,K}^4(\mathbb{R}_+; H)$.

Proof Integrating by parts, for $u(t) \in W_{2,K}^4(\mathbb{R}_+; H)$, we obtain

$$\mathrm{Re}\big(u^{(4)}, A^4 u\big)_{L_2(\mathbb{R}_+;H)} = \mathrm{Re}\big(B A^{5/2} u'(0), A^{5/2} u'(0)\big) + \|A^2 u''\|_{L_2(\mathbb{R}_+;H)}^2. \tag{3.10}$$

Due to (3.10), we have

$$\begin{aligned}
\|P_0 u\|_{L_2(\mathbb{R}_+;H)}^2 &= \|u^{(4)}\|_{L_2(\mathbb{R}_+;H)}^2 + \|A^4 u\|_{L_2(\mathbb{R}_+;H)}^2 + 2\mathrm{Re}\big(u^{(4)}, A^4 u\big)_{L_2(\mathbb{R}_+;H)} \\
&= \|u\|_{W_2^4(\mathbb{R}_+;H)}^2 + 2\mathrm{Re}\big(B A^{5/2} u'(0), A^{5/2} u'(0)\big) + 2\|A^2 u''\|_{L_2(\mathbb{R}_+;H)}^2. \tag{3.11}
\end{aligned}$$

As $\mathrm{Re}\, B \geq 0$, equality (3.11) implies the validity of inequality (3.9). The lemma is proved. □

Note that Theorem 3.2 combined with Lemma 3.1 implies that the operator P_0, under the condition $-\sqrt{2} \notin \sigma(B)$ with $B = A^{3/2} K A^{-5/2}$, maps the space $W_{2,K}^4(\mathbb{R}_+; H)$ isomorphically onto $L_2(\mathbb{R}_+; H)$. Consequently, the norm $\|P_0 u\|_{L_2(\mathbb{R}_+;H)}$ is equivalent in $W_{2,K}^4(\mathbb{R}_+; H)$ to the initial norm $\|u\|_{W_2^4(\mathbb{R}_+;H)}$. And, as the intermediate derivative operators

$$A^j \frac{d^{4-j}}{dt^{4-j}} : W_{2,K}^4(\mathbb{R}_+; H) \to L_2(\mathbb{R}_+; H), \quad j = 1, 2, 3, 4,$$

are continuous (see [35]), the norms of these operators can be estimated through the norm $\|P_0 u\|_{L_2(\mathbb{R}_+;H)}$. The need for these estimates arises when one tries to establish solvability conditions for the boundary value problem (2.1), (2.2) using only the properties of its operator coefficients.

Theorem 3.4 *Let $B = A^{3/2}KA^{-5/2}$ and $\operatorname{Re} B \geq 0$. Then the estimates*

$$\left\|A^j u^{(4-j)}\right\|_{L_2(\mathbb{R}_+;H)} \leq c_j \|P_0 u\|_{L_2(\mathbb{R}_+;H)}, \quad j = 0, 1, 2, 3, 4, \tag{3.12}$$

are true for every $u(t) \in W_{2,K}^4(\mathbb{R}_+;H)$ with

$$c_0 = c_1 = c_4 = 1, \qquad c_2 = \frac{1}{2}, \qquad c_3 = \frac{1}{\sqrt{2}}.$$

Proof We first multiply both sides of equation (3.1) scalarly by $A^4 u(t)$ in the space $L_2(\mathbb{R}_+;H)$ and then integrate by parts. Then, taking into account (3.10) and the condition $\operatorname{Re} B \geq 0$, we have

$$\begin{aligned}
&\operatorname{Re}\left(P_0 u, A^4 u\right)_{L_2(\mathbb{R}_+;H)} \\
&= \operatorname{Re}\left(u^{(4)} + A^4 u, A^4 u\right)_{L_2(\mathbb{R}_+;H)} \\
&= \left\|A^4 u\right\|_{L_2(\mathbb{R}_+;H)}^2 + \operatorname{Re}\left(BA^{5/2}u'(0), A^{5/2}u'(0)\right) + \left\|A^2 u''\right\|_{L_2(\mathbb{R}_+;H)}^2 \\
&\geq \left\|A^4 u\right\|_{L_2(\mathbb{R}_+;H)}^2 + \left\|A^2 u''\right\|_{L_2(\mathbb{R}_+;H)}^2.
\end{aligned} \tag{3.13}$$

Applying the Cauchy-Schwarz inequality and then the Young inequality to the left-hand side of (3.13), we get

$$\begin{aligned}
&\left\|A^4 u\right\|_{L_2(\mathbb{R}_+;H)}^2 + \left\|A^2 u''\right\|_{L_2(\mathbb{R}_+;H)}^2 \\
&\leq \|P_0 u\|_{L_2(\mathbb{R}_+;H)} \left\|A^4 u\right\|_{L_2(\mathbb{R}_+;H)} \\
&\leq \frac{\delta}{2}\|P_0 u\|_{L_2(\mathbb{R}_+;H)}^2 + \frac{1}{2\delta}\left\|A^4 u\right\|_{L_2(\mathbb{R}_+;H)}^2, \quad \delta > 0.
\end{aligned} \tag{3.14}$$

Assume $\delta = \frac{1}{2}$ in (3.14). Then we obtain

$$\left\|A^2 u''\right\|_{L_2(\mathbb{R}_+;H)}^2 \leq \frac{1}{4}\|P_0 u\|_{L_2(\mathbb{R}_+;H)}^2$$

or

$$\left\|A^2 u''\right\|_{L_2(\mathbb{R}_+;H)} \leq \frac{1}{2}\|P_0 u\|_{L_2(\mathbb{R}_+;H)}. \tag{3.15}$$

On the other hand, from (3.14) we have

$$\left\|A^4 u\right\|_{L_2(\mathbb{R}_+;H)}^2 \leq \|P_0 u\|_{L_2(\mathbb{R}_+;H)} \left\|A^4 u\right\|_{L_2(\mathbb{R}_+;H)}.$$

Therefore,

$$\left\|A^4 u\right\|_{L_2(\mathbb{R}_+;H)} \leq \|P_0 u\|_{L_2(\mathbb{R}_+;H)}. \tag{3.16}$$

Note that the validity of (3.16) can be also obtained from inequality (3.9). Besides, inequality (3.9) implies the validity of the following one:

$$\left\| u^{(4)} \right\|_{L_2(\mathbb{R}_+;H)} \le \| P_0 u \|_{L_2(\mathbb{R}_+;H)}. \tag{3.17}$$

Now let us estimate the norm $\| A^3 u' \|_{L_2(\mathbb{R}_+;H)}$. Integrating by parts, taking into account $u(t) \in W_{2,K}^4(\mathbb{R}_+;H)$, and then applying the Cauchy-Schwarz inequality and inequalities (3.15), (3.16), we obtain

$$
\begin{aligned}
\left\| A^3 u' \right\|_{L_2(\mathbb{R}_+;H)}^2 &= \int_0^{+\infty} \left(A^3 u', A^3 u' \right) dt \\
&= \left(A^3 u, A^3 u' \right)\big|_0^{+\infty} - \int_0^{+\infty} \left(A^4 u, A^2 u'' \right) dt \\
&= -\int_0^{+\infty} \left(A^4 u, A^2 u'' \right) dt \le \left\| A^4 u \right\|_{L_2(\mathbb{R}_+;H)} \left\| A^2 u'' \right\|_{L_2(\mathbb{R}_+;H)} \\
&\le \frac{1}{2} \| P_0 u \|_{L_2(\mathbb{R}_+;H)}^2.
\end{aligned}
$$

Consequently,

$$\left\| A^3 u' \right\|_{L_2(\mathbb{R}_+;H)} \le \frac{1}{\sqrt{2}} \| P_0 u \|_{L_2(\mathbb{R}_+;H)}.$$

Finally, we proceed to estimate the norm $\| A u''' \|_{L_2(\mathbb{R}_+;H)}$. It was shown in [25] that the inequality

$$\left\| A u''' \right\|_{L_2(\mathbb{R}_+;H)}^2 \le 2 \left\| A^2 u'' \right\|_{L_2(\mathbb{R}_+;H)} \left\| u^{(4)} \right\|_{L_2(\mathbb{R}_+;H)} \tag{3.18}$$

is true for $u(t) \in W_2^4(\mathbb{R}_+;H)$. Considering inequalities (3.15) and (3.17) in (3.18), we have

$$\left\| A u''' \right\|_{L_2(\mathbb{R}_+;H)}^2 \le \| P_0 u \|_{L_2(\mathbb{R}_+;H)}^2$$

or

$$\left\| A u''' \right\|_{L_2(\mathbb{R}_+;H)} \le \| P_0 u \|_{L_2(\mathbb{R}_+;H)}.$$

The theorem is proved. □

Now we consider another case of problem (2.1), (2.2) with $A_j \ne 0$, $j = 1, 2, 3, 4$, while φ and ψ are equal to zero as before:

$$u^{(4)}(t) + A^4 u(t) + \sum_{j=1}^{4} A_j u^{(4-j)}(t) = f(t), \quad t \in \mathbb{R}_+, \tag{3.19}$$

$$u(0) = 0, \qquad u''(0) = K u'(0). \tag{3.20}$$

Denote by P the operator that acts from $W_{2,K}^4(\mathbb{R}_+;H)$ to $L_2(\mathbb{R}_+;H)$ as follows:

$$Pu(t) = u^{(4)}(t) + A^4 u(t) + \sum_{j=1}^{4} A_j u^{(4-j)}(t), \quad u(t) \in W_{2,K}^4(\mathbb{R}_+;H).$$

The following lemma is true.

Lemma 3.5 *Let $A_j A^{-j} \in L(H,H)$, $j = 1,2,3,4$. Then the operator P is a bounded operator from $W_{2,K}^4(\mathbb{R}_+;H)$ to $L_2(\mathbb{R}_+;H)$.*

Proof Taking into account the conditions of the lemma and inequality (3.8), for every $u(t) \in W_{2,K}^4(\mathbb{R}_+;H)$, we have

$$\|Pu\|_{L_2(\mathbb{R}_+;H)} \leq \|P_0 u\|_{L_2(\mathbb{R}_+;H)} + \left\|\sum_{j=1}^4 A_j u^{(4-j)}\right\|_{L_2(\mathbb{R}_+;H)}$$

$$\leq \sqrt{2}\|u\|_{W_2^4(\mathbb{R}_+;H)} + \sum_{j=1}^4 \left\|A_j u^{(4-j)}\right\|_{L_2(\mathbb{R}_+;H)}$$

$$\leq \sqrt{2}\|u\|_{W_2^4(\mathbb{R}_+;H)} + \sum_{j=1}^4 \left\|A_j A^{-j}\right\|_{H\to H} \left\|A^j u^{(4-j)}\right\|_{L_2(\mathbb{R}_+;H)}. \qquad (3.21)$$

Then, by the theorem on intermediate derivatives [35], Chapter 1, from (3.21) we obtain

$$\|Pu\|_{L_2(\mathbb{R}_+;H)} \leq \text{const}\,\|u\|_{W_2^4(\mathbb{R}_+;H)}.$$

The lemma is proved. $\qquad\qquad\qquad\square$

The following theorem on solvability of the boundary value problem (3.19), (3.20) is true, stated only in terms of the properties of its operator coefficients.

Theorem 3.6 *Let $B = A^{3/2} K A^{-5/2}$, $\mathrm{Re}\, B \geq 0$, $A_j A^{-j} \in L(H,H)$, $j = 1,2,3,4$, and the inequality*

$$\alpha = \sum_{j=1}^4 c_j \left\|A_j A^{-j}\right\|_{H\to H} < 1$$

be fulfilled, where the numbers c_j, $j = 1,2,3,4$, are defined in Theorem 3.4, i.e.,

$$c_1 = c_4 = 1, \qquad c_2 = \frac{1}{2}, \qquad c_3 = \frac{1}{\sqrt{2}}.$$

Then the boundary value problem (3.19), (3.20) *has a unique regular solution for every $f(t) \in L_2(\mathbb{R}_+;H)$.*

Proof First we rewrite the boundary value problem (3.19), (3.20) in the form of operator equation

$$P_0 u(t) + (P - P_0)u(t) = f(t), \qquad (3.22)$$

where $f(t) \in L_2(\mathbb{R}_+;H)$, $u(t) \in W_{2,K}^4(\mathbb{R}_+;H)$.

Note that the conditions $B = A^{3/2}KA^{-5/2}$, $\mathrm{Re}\,B \geq 0$ provide the existence of the bounded inverse operator P_0^{-1} from $L_2(\mathbb{R}_+; H)$ to $W_{2,K}^4(\mathbb{R}_+; H)$. Replacing $u(t) = P_0^{-1}v(t)$, where $v(t) \in L_2(\mathbb{R}_+; H)$, we arrive at the following equation in the space $L_2(\mathbb{R}_+; H)$:

$$v(t) + (P - P_0)P_0^{-1}v(t) = f(t).$$

In this case, taking into account estimates (3.12), for every $v(t) \in L_2(\mathbb{R}_+; H)$, we obtain

$$
\begin{aligned}
\left\| (P - P_0)P_0^{-1}v \right\|_{L_2(\mathbb{R}_+;H)} &= \left\| (P - P_0)u \right\|_{L_2(\mathbb{R}_+;H)} \\
&\leq \sum_{j=1}^{4} \left\| A_j A^{-j} \right\|_{H \to H} \left\| A^j u^{(4-j)} \right\|_{L_2(\mathbb{R}_+;H)} \\
&\leq \sum_{j=1}^{4} c_j \left\| A_j A^{-j} \right\|_{H \to H} \| P_0 u \|_{L_2(\mathbb{R}_+;H)} \\
&= \alpha \| v \|_{L_2(\mathbb{R}_+;H)}.
\end{aligned}
$$

As $\alpha < 1$, the operator $E + (P - P_0)P_0^{-1}$ has an inverse in the space $L_2(\mathbb{R}_+; H)$. Consequently, equation (3.22) is uniquely solvable in the space $W_{2,K}^4(\mathbb{R}_+; H)$, and

$$u(t) = P_0^{-1}\bigl(E + (P - P_0)P_0^{-1}\bigr)^{-1}f(t).$$

It follows

$$
\begin{aligned}
&\| u \|_{W_2^4(\mathbb{R}_+;H)} \\
&\quad \leq \left\| P_0^{-1} \right\|_{L_2(\mathbb{R}_+;H) \to W_2^4(\mathbb{R}_+;H)} \left\| \bigl(E + (P - P_0)P_0^{-1}\bigr)^{-1} \right\|_{L_2(\mathbb{R}_+;H) \to L_2(\mathbb{R}_+;H)} \| f \|_{L_2(\mathbb{R}_+;H)} \\
&\quad \leq \mathrm{const}\, \| f \|_{L_2(\mathbb{R}_+;H)}.
\end{aligned}
$$

The theorem is proved. $\qquad\square$

Remark 3.7 In Theorem 3.6, the condition $\mathrm{Re}\,B \geq 0$ with $B = A^{3/2}KA^{-5/2}$ allows to omit the condition $-\sqrt{2} \notin \sigma(B)$.

Finally, from Theorem 3.6 we can obtain the conditions for the regular solvability of the boundary value problem (2.1), (2.2).

Theorem 3.8 *Let all the conditions of Theorem 3.6 be fulfilled. Then the boundary value problem (2.1), (2.2) is regularly solvable.*

Proof Obviously, in case $\varphi = \psi = 0$ the regular solvability of the boundary value problem (2.1), (2.2) follows from Theorem 3.6.

In case $A_j = 0$, $j = 1, 2, 3, 4$, and $f(t) = 0$, we have the problem

$$u^{(4)}(t) + A^4 u(t) = 0, \quad t \in \mathbb{R}_+, \tag{3.23}$$

$$u(0) = \varphi, \qquad u''(0) - Ku'(0) = \psi, \tag{3.24}$$

where $\varphi \in H_{7/2}$, $\psi \in H_{3/2}$. The solution of problem (3.23), (3.24) will be sought in the following form:

$$u_0(t) = e^{\omega_1 tA} \zeta_0 + e^{\omega_2 tA} \zeta_1, \tag{3.25}$$

where

$$\omega_1 = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, \qquad \omega_2 = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i,$$

and ζ_0, ζ_1 are the unknown vectors to be determined from conditions (3.24):

$$\begin{cases} \zeta_0 + \zeta_1 = \varphi, \\ \omega_1^2 A^2 \zeta_0 + \omega_2^2 A^2 \zeta_1 - KA(\omega_1 \zeta_0 + \omega_2 \zeta_1) = \psi. \end{cases} \tag{3.26}$$

Considering the relation

$$\zeta_1 = \varphi - \zeta_0$$

in the second equation of system (3.26) and taking into account the condition $-\sqrt{2} \notin \sigma(B)$, we uniquely determine

$$\zeta_0 = \frac{i}{2} A^{-7/2} \left(E + \frac{1}{\sqrt{2}} B \right)^{-1} A^{3/2} \left(\psi - \omega_2^2 A^2 \varphi + \omega_2 KA\varphi \right).$$

Then

$$\zeta_1 = \varphi - \frac{i}{2} A^{-7/2} \left(E + \frac{1}{\sqrt{2}} B \right)^{-1} A^{3/2} \left(\psi - \omega_2^2 A^2 \varphi + \omega_2 KA\varphi \right).$$

It is clear that $\zeta_0, \zeta_1 \in H_{7/2}$. Consequently, from the representation (3.25) we have [35], Chapter 1

$$\|u_0\|_{W_2^4(\mathbb{R}_+;H)} \le \mathrm{const} \left(\|\zeta_0\|_{H_{7/2}} + \|\zeta_1\|_{H_{7/2}} \right)$$
$$\le \mathrm{const} \left(\|\varphi\|_{H_{7/2}} + \|\psi\|_{H_{3/2}} \right). \tag{3.27}$$

Now we can directly proceed to the study of boundary value problem (2.1), (2.2). We will seek the solution of the boundary value problem (2.1), (2.2) in the form $u(t) = v(t) + u_0(t)$, where $u_0(t)$ is a regular solution of problem (3.23), (3.24). Then the function $v(t)$ is the solution of the boundary value problem

$$v^{(4)}(t) + A^4 v(t) + \sum_{j=1}^{4} A_j v^{(4-j)}(t) = g(t), \quad t \in \mathbb{R}_+, \tag{3.28}$$

$$v(0) = 0, \qquad v''(0) = Kv'(0), \tag{3.29}$$

where

$$g(t) \equiv -\sum_{j=1}^{4} A_j u_0^{(4-j)}(t) + f(t).$$

Estimating the norm of the function $g(t)$ in $L_2(\mathbb{R}_+; H)$, we obtain

$$\|g\|_{L_2(\mathbb{R}_+;H)} \leq \left\| \sum_{j=1}^{4} A_j u_0^{(4-j)} \right\|_{L_2(\mathbb{R}_+;H)} + \|f\|_{L_2(\mathbb{R}_+;H)}$$

$$\leq \sum_{j=1}^{4} \left\| A_j A^{-j} \right\| \left\| A^j u_0^{(4-j)} \right\|_{L_2(\mathbb{R}_+;H)} + \|f\|_{L_2(\mathbb{R}_+;H)}$$

$$\leq \text{const} \left(\|\varphi\|_{H_{7/2}} + \|\psi\|_{H_{3/2}} + \|f\|_{L_2(\mathbb{R}_+;H)} \right).$$

Consequently, by Theorem 3.6 and estimate (3.27), we have

$$\|u\|_{W_2^4(\mathbb{R}_+;H)} \leq \|v\|_{W_2^4(\mathbb{R}_+;H)} + \|u_0\|_{W_2^4(\mathbb{R}_+;H)}$$

$$\leq \text{const} \|g\|_{L_2(\mathbb{R}_+;H)} + \|u_0\|_{W_2^4(\mathbb{R}_+;H)}$$

$$\leq \text{const} \left(\|\varphi\|_{H_{7/2}} + \|\psi\|_{H_{3/2}} + \|f\|_{L_2(\mathbb{R}_+;H)} \right).$$

The theorem is proved. □

Note that the case where the operator $\operatorname{Re} B$ is not non-negative requires special consideration.

Appendix

Now we apply the abstract results of Theorem 3.6 to a problem for partial differential equations.

On the half-strip $\mathbb{R}_+ \times (0, \pi)$, consider the following boundary value problem for fourth order elliptic equations:

$$\frac{\partial^4 u(t,x)}{\partial t^4} + \frac{\partial^4 u(t,x)}{\partial x^4} + \sum_{j=1}^{4} p_j(x) \frac{\partial^4 u(t,x)}{\partial t^{4-j} \partial x^j} = f(t,x), \tag{A.1}$$

$$u(0,x) = 0, \qquad \frac{\partial^2 u(0,x)}{\partial t^2} = a \frac{\partial^2 u(0,x)}{\partial t \partial x}, \tag{A.2}$$

$$\frac{\partial^{2k} u(t,0)}{\partial x^{2k}} = \frac{\partial^{2k} u(t,\pi)}{\partial x^{2k}} = 0, \quad k = 0, 1, \tag{A.3}$$

where a is a complex number, $p_j(x)$, $j = 1, 2, 3, 4$, are bounded functions on the interval $(0, \pi)$, $f(t,x) \in L_2(\mathbb{R}_+; L_2(0, \pi))$. Note that this problem can be reduced to the boundary value problem (3.19), (3.20). Here $H = L_2(0, \pi)$, $A_j = p_j(x) \frac{\partial^j}{\partial x^j}$, $j = 1, 2, 3, 4$. The operator A^4 is defined in $L_2(0, \pi)$ by the equality $A^4 u = \frac{d^4 u}{dx^4}$ with the conditions $u(0) = u(\pi) = u''(0) = u''(\pi) = 0$, and the operator K is defined in $L_2(0, \pi)$ by the equality $Ku = a \frac{du}{dx}$.

Taking into account Theorem 3.6, we obtain that under the conditions $\operatorname{Re} a \geq 0$ and $\sum_{j=1}^{4} c_j \sup_{0 \leq x \leq \pi} |p_j(x)| < 1$, the boundary value problem (A.1)-(A.3) has a unique solution in the space $W_2^4(\mathbb{R}_+; L_2(0, \pi))$.

Authors' contributions
The authors contributed equally to this work. All authors read and approved the final manuscript.

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