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Properties for the Perron complement of three known subclasses of H-matrices

Leilei Wang, Jianzhou Liu* and Shan Chu

*Correspondence: liujz@xtu.edu.cn
Department of Mathematics and Computational Science, Xiangtan University, Xiangtan, Hunan 411105, P.R. China

Abstract
In this paper, we analyze the diagonally dominant degree for the Perron complement upon several diagonally dominant cases by using the entries and spectral radius of the original matrix. At the same time, we obtain closure properties for the Perron complement of several diagonally dominant matrices.
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1 Introduction

The Perron complement plays an important role in statistics, matrix theory, control theory, and so on (see [1–4]). It was said in [2, 5] that the Perron vector (or the stationary distribution vector) for a corresponding transition matrix of a certain Markov chain provides the long term probabilities of the chain being in a particular state. Meyer [1] introduced the concept of the Perron complement, obtained the Perron complement of a nonnegative irreducible matrix is nonnegative irreducible, and first used the closure property of a nonnegative irreducible matrix to construct a divided and conquer algorithm to compute the Perron vector for a Markov chain. Since then, a lot of work has been done on this topic. The authors in [6–8] showed closure properties of the Perron complement for some special matrices. When the Perron complement is primitive, we can design a related iterative algorithm to compute the Perron vector (see [1]). So far as we know, if the given matrix has a sharper diagonally dominant degree, then the designed iterative algorithms has faster convergent rate than the ordinary ones (see [9]). At the same time, if a given matrix has a sharper diagonally dominant degree, then we may discuss more properties about generalized nonlinear diagonal dominance in [10]. Motivated by the useful applications, we will study the diagonal dominant degree for the Perron complement of several cases based on the nonnegative and irreducible nature, which may belong to inherent properties of the Perron complement. Meanwhile, some closure properties for the Perron complement of three known subclasses of H-matrices are provided by the entries and spectral radius of the original matrix.

Let $\mathbb{C}^{n\times n}(\mathbb{R}^{n\times n})$ denote the set of all $n\times n$ complex (real) matrices, $\mathbb{N}=\{1,2,\ldots,n\}$, and $\mathbb{G}=\{(i,j): i,j\in\mathbb{N}, i\neq j\}$. For $A=(a_{ij})\in\mathbb{C}^{n\times n}$, denote

$$R_i(A)=\sum_{j\in\mathbb{N},j\neq i}|a_{ij}|,\qquad S_i(A)=\sum_{j\in\mathbb{N},j\neq i}|a_{ji}|,\quad i\in\mathbb{N}.$$

Choose

$$\mathbb{N}_r(A) = \{i : |a_{ii}| > R_i(A), i \in N\}, \qquad \mathbb{N}_c(A) = \{j : |a_{jj}| > S_j(A), j \in N\}.$$

The comparison matrix $\mu(A) = (\mu_{ij})$ of $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, defined by

$$\mu_{ij} = \begin{cases} |a_{ii}|, & i = j, \\ -|a_{ij}|, & i \neq j. \end{cases}$$

If $A = \mu(A)$ and the eigenvalues of A have positive real parts, then we call A a (nonsingular) M-matrix. We say that A is an H-matrix if $\mu(A)$ is an M-matrix. For details and numerous conditions of M-matrices, please refer to [11]. We denote by $\mathbb{H}_n$ and $\mathbb{M}_n$ the sets of all $n \times n$ H- and M-matrices, respectively.

Recall that $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ is (row) diagonally dominant and denoted by $A \in D_n$ if

$$|a_{ii}| \geq R_i(A) \tag{1.1}$$

for all $i \in \mathbb{N}$ holds. A is said to be a strictly diagonally dominant matrix and denoted by $A \in SD_n$ if the representative inequality (1.1) is strict.

$A = (a_{ij}) \in \mathbb{C}^{n\times n}$ is doubly diagonally dominant and denoted by $A \in DD_n$ if

$$|a_{ii}||a_{jj}| \geq R_i(A)R_j(A) \tag{1.2}$$

for all $(i,j) \in \mathbb{G}$ valid. Moreover, A is said to be a strictly doubly diagonally dominant matrix and denoted by $A \in SDD_n$ if the representative inequality (1.2) is strict for all $(i,j) \in \mathbb{G}$. If $A \in SDD_n$, then there exists at most one index $i_0 \in \mathbb{N}$ such that

$$|a_{i_0 i_0}| \leq R_{i_0}(A). \tag{1.3}$$

$A = (a_{ij}) \in \mathbb{C}^{n\times n}$ is γ-diagonally dominant and denoted by $A \in D_n^\gamma$ if there exists $\gamma \in [0,1]$ such that

$$|a_{ii}| \geq \gamma R_i(A) + (1-\gamma)S_i(A), \quad \text{for all } i \in \mathbb{N}. \tag{1.4}$$

If all the inequalities in (1.4) are strict, then A is called strict γ-diagonally dominant.

$A = (a_{ij}) \in \mathbb{C}^{n\times n}$ is product γ-diagonally dominant and denoted by $A \in PD_n^\gamma$ if there exists $\gamma \in [0,1]$ such that

$$|a_{ii}| \geq [R_i(A)]^\gamma [S_i(A)]^{1-\gamma}, \quad \text{for all } i \in \mathbb{N}. \tag{1.5}$$

If all the inequalities in (1.5) are strict, then A is said to be a strictly product γ-diagonally dominant matrix.

For $A \in \mathbb{C}^{n\times n}$, nonempty index sets $\alpha, \beta \subseteq \mathbb{N}$, we denote by $|\alpha|$ the cardinality of α. Let $A(\alpha,\beta)$ denote the sub-matrix of A lying in the rows indexed by α and the columns indexed by β. $A(\alpha,\alpha)$ is abbreviated to $A(\alpha)$. If $A(\alpha)$ is nonsingular, then the Schur complement of $A(\alpha)$ in A is given by

$$S(A/A(\alpha)) = A(\beta) - A(\beta,\alpha)[A(\alpha)]^{-1}A(\alpha,\beta). \tag{1.6}$$

Let A be a nonnegative irreducible matrix of order n with spectral radius $\rho(A)$, $\emptyset \neq \alpha$, and $\beta = \mathbb{N} - \alpha$. Then the Perron complement of A with respect to $A(\alpha)$, which is denoted by $P(A/A(\alpha))$ or simply $P(A/\alpha)$, is defined as

$$A(\beta) + A(\beta,\alpha)\big[\rho(A)I - A(\alpha)\big]^{-1}A(\alpha,\beta). \tag{1.7}$$

In addition, the extended Perron complement $P_t(A/A(\alpha))$ or simply $P_t(A/\alpha)$ at t is defined as

$$A(\beta) + A(\beta,\alpha)\big[tI - A(\alpha)\big]^{-1}A(\alpha,\beta). \tag{1.8}$$

2 Properties for the Perron complement of diagonally dominant matrices

In this section, we offer some properties for the Perron complement of diagonally dominant matrices by using the entries and spectral radius of the original matrix.

Lemma 1 (see [11]) *If A is an H-matrix, then $[\mu(A)]^{-1} \geq |A|^{-1}$.*

Lemma 2 (see [11]) *If $A = (a_{ij}) \in SD_n$, or $A \in SDD_n$, then $\mu(A) \in \mathbb{M}_n$, i.e., $A \in \mathbb{H}_n$.*

Proposition 1 *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is nonnegative irreducible with spectral radius $\rho(A)$, $\alpha = \{i_1, i_2, \ldots, i_k\} \subseteq \mathbb{N}_r(A)$, $\beta = \mathbb{N} - \alpha = \{j_1, j_2, \ldots, j_l\}$, $|\alpha| < n$, and denote*

$$B_{j_t} \equiv \begin{pmatrix} x & -|a_{j_t i_1}| & \cdots & -|a_{j_t i_k}| \\ -\sum_{u=1}^{l} |a_{i_1 j_u}| & & & \\ \vdots & & \mu[\rho(A)I - A(\alpha)] & \\ -\sum_{u=1}^{l} |a_{i_k j_u}| & & & \end{pmatrix}, \quad x > 0.$$

Then for $\rho(A) \geq 2|a_{ii}|$ $(i \in \alpha)$ and any $j_t \in \beta$, B_{j_t} is an M-matrix of order $k+1$ and $\det B_{j_t} > 0$ *if*

$$x > \sum_{\omega=1}^{k} |a_{j_t i_\omega}| \frac{R_{i_\omega}(A)}{\rho(A) - |a_{i_\omega i_\omega}|}. \tag{2.1}$$

Proof Denote $B_{j_t} \equiv B \equiv (b_{pq})$. Equation (2.1) means that there exists $\varepsilon > 0$ such that

$$x > \sum_{\omega=1}^{k} |a_{j_t i_\omega}| \left(\frac{R_{i_\omega}(A)}{\rho(A) - |a_{i_\omega i_\omega}|} + \varepsilon \right). \tag{2.2}$$

By $\rho(A) \geq 2|a_{ii}|$ $(i \in \alpha)$ and $\alpha \subseteq \mathbb{N}_r(A)$, we have $0 \leq \frac{R_{i_\omega}(A)}{\rho(A) - |a_{i_\omega i_\omega}|} < 1$ $(1 \leq \omega \leq k)$. Choose a positive matrix $D = \operatorname{diag}(d_1, d_2, \ldots, d_{k+1})$ and $K = BD = (k_{sv})$, where

$$d_v = \begin{cases} 1, & v = 1; \\ \frac{R_{i_{v-1}}(A)}{\rho(A) - |a_{i_{v-1} i_{v-1}}|} + \varepsilon, & 2 \leq v \leq k+1. \end{cases}$$

(i) For $s = 1$, (2.2) follows from

$$|k_{ss}| - \sum_{v \neq s} |k_{sv}| = |k_{11}| - \sum_{v=2}^{k+1} |k_{1v}| = x - \sum_{v=2}^{k+1} |a_{j_t i_{v-1}}| \left(\frac{R_{i_\omega}(A)}{\rho(A) - |a_{i_\omega i_\omega}|} + \varepsilon \right) > 0.$$

(ii) For $s = 2, 3, \ldots, k+1$, we obtain

$$\begin{aligned}
&|k_{ss}| - \sum_{v\neq s} |k_{sv}| \\
&= \bigl[\rho(A) - |a_{i_{s-1}i_{s-1}}|\bigr]\left(\frac{R_{i_{s-1}}(A)}{\rho(A) - |a_{i_{s-1}i_{s-1}}|} + \varepsilon\right) \\
&\quad - \sum_{u=1}^{l} |a_{i_{s-1}j_u}| - \sum_{v=2,v\neq s}^{k+1} |a_{i_{s-1}i_{v-1}}|\left(\frac{R_{i_{v-1}}(A)}{\rho(A) - |a_{i_{v-1}i_{v-1}}|} + \varepsilon\right) \\
&= R_{i_{s-1}}(A) - \sum_{u=1}^{l} |a_{i_{s-1}j_u}| - \sum_{v=1,v\neq s-1}^{k} |a_{i_{s-1}i_v}|\frac{R_{i_v}(A)}{\rho(A) - |a_{i_v i_v}|} \\
&\quad + \varepsilon\left[\rho(A) - |a_{i_{s-1}i_{s-1}}| - \sum_{v=2,v\neq s}^{k+1} |a_{i_{s-1}i_{v-1}}|\right] > 0.
\end{aligned}$$

Thus, $K \in SD_{k+1}$. Further, $B_{j_t} \in \mathbb{H}_{k+1}$. Besides, observing that $B_{j_t} = \mu(B_{j_t})$, we have $B_{j_t} \in \mathbb{M}_{k+1}$ and $\det B_{j_t} > 0$. □

Proposition 2 *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be nonnegative irreducible with spectral radius $\rho(A)$, $\alpha = \{i_1, i_2, \ldots, i_k\} \subseteq \mathbb{N}$, $\beta = \mathbb{N} - \alpha = \{j_1, j_2, \ldots, j_l\}$, $|\alpha| < n$, $P(A/\alpha) = (a'_{ts})$, and $\rho(A) \geq 2|a_{ii}|$ $(i \in \alpha)$.*

(i) *If $\alpha \subseteq \mathbb{N}_r(A)$, then for all $1 \leq t \leq l$,*

$$\begin{aligned}
&\sum_{s=1,s\neq t}^{l} \left| a_{j_t j_s} + (a_{j_t i_1}, \ldots, a_{j_t i_k})\bigl[\rho(A)I - A(\alpha)\bigr]^{-1} \begin{pmatrix} a_{i_1 j_s} \\ \vdots \\ a_{i_k j_s} \end{pmatrix} \right| \\
&\quad + \left| (a_{j_t i_1}, \ldots, a_{j_t i_k})\bigl[\rho(A)I - A(\alpha)\bigr]^{-1} \begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t} \end{pmatrix} \right| \leq R_{j_t}(A) - \omega_{j_t}.
\end{aligned} \tag{2.3}$$

(ii) *If $\alpha \subseteq \mathbb{N}_c(A)$, then for all $1 \leq t \leq l$,*

$$\begin{aligned}
&\sum_{s=1,s\neq t}^{l} \left| a_{j_s j_t} + (a_{j_s i_1}, \ldots, a_{j_s i_k})\bigl[\rho(A)I - A(\alpha)\bigr]^{-1} \begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t} \end{pmatrix} \right| \\
&\quad + \left| (a_{j_t i_1}, \ldots, a_{j_t i_k})\bigl[\rho(A)I - A(\alpha)\bigr]^{-1} \begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t} \end{pmatrix} \right| \leq S_{j_t}(A) - \omega^T_{j_t}.
\end{aligned} \tag{2.4}$$

Here

$$\omega_{j_t} = \sum_{u=1}^{k} |a_{j_t i_u}| \frac{\rho(A) - |a_{i_u i_u}| - R_{i_u}(A)}{\rho(A) - |a_{i_u i_u}|}, \qquad \omega^T_{j_t} = \sum_{u=1}^{k} |a_{i_u j_t}| \frac{\rho(A) - |a_{i_u i_u}| - S_{i_u}(A)}{\rho(A) - |a_{i_u i_u}|}.$$

Proof By $\alpha \subseteq \mathbb{N}_r(A)$, we have $A(\alpha) \in SD_k$. Further, $\rho(A) \geq 2|a_{ii}|$ $(i \in \alpha)$ yields $(\rho(A)I - A(\alpha)) \in SD_k$. By Lemma 1 and Lemma 2, we have

$$\{\mu[\rho(A)I - A(\alpha)]\}^{-1} \geq [\rho(A)I - A(\alpha)]^{-1}.$$

Thus, by the definition of the Perron complement, we obtain

$$\begin{aligned}
&\left|(a_{j_t i_1}, \ldots, a_{j_t i_k})[\rho(A)I - A(\alpha)]^{-1}\begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t}\end{pmatrix}\right| \\
&\quad + \sum_{s=1, s\neq t}^{l} \left| a_{j_t j_s} + (a_{j_t i_1}, \ldots, a_{j_t i_k})[\rho(A)I - A(\alpha)]^{-1}\begin{pmatrix} a_{i_1 j_s} \\ \vdots \\ a_{i_k j_s}\end{pmatrix}\right| \\
&\leq \sum_{s=1, s\neq t}^{l} |a_{j_t j_s}| + \sum_{s=1}^{l} \left|(a_{j_t i_1}, \ldots, a_{j_t i_k})[\rho(A)I - A(\alpha)]^{-1}\begin{pmatrix} a_{i_1 j_s} \\ \vdots \\ a_{i_k j_s}\end{pmatrix}\right| \\
&\leq \sum_{s=1, s\neq t}^{l} |a_{j_t j_s}| + \sum_{s=1}^{l} \left|(|a_{j_t i_1}|, \ldots, |a_{j_t i_k}|)\{\mu[\rho(A)I - A(\alpha)]\}^{-1}\begin{pmatrix} |a_{i_1 j_s}| \\ \vdots \\ |a_{i_k j_s}|\end{pmatrix}\right| \\
&= R_{j_t}(A) - \sum_{u=1}^{k} |a_{j_t i_u}| + (|a_{j_t i_1}|, \ldots, |a_{j_t i_k}|)\{\mu[\rho(A)I - A(\alpha)]\}^{-1}\begin{pmatrix} \sum_{s=1}^{l} |a_{i_1 j_s}| \\ \vdots \\ \sum_{s=1}^{l} |a_{i_k j_s}|\end{pmatrix} \\
&= R_{j_t}(A) - \omega_{j_t} - \frac{1}{\det\{\mu[\rho(A)I - A(\alpha)]\}} \\
&\quad \times \det\begin{pmatrix} \sum_{u=1}^{k} |a_{j_t i_u}| - \omega_{j_t} & -|a_{j_t i_1}| & \cdots & -|a_{j_t i_k}| \\ -\sum_{s=1}^{l} |a_{i_1 j_s}| & & & \\ \vdots & & \mu[\rho(A)I - A(\alpha)] & \\ -\sum_{s=1}^{l} |a_{i_k j_s}| & & & \end{pmatrix} \\
&\equiv R_{j_t}(A) - \omega_{j_t} - \frac{\det B_1}{\det\{\mu[\rho(A)I - A(\alpha)]\}}.
\end{aligned}$$

On the basis of Proposition 1, we have $\det B_1 > 0$, which implies (2.3). Moreover, (2.4) can be proved with a similar method to the above techniques. □

Theorem 1 *If $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is nonnegative irreducible with spectral radius $\rho(A)$, $\alpha = \{i_1, i_2, \ldots, i_k\} \subseteq \mathbb{N}_r(A)$, $\beta = \mathbb{N} - \alpha = \{j_1, j_2, \ldots, j_l\}$, $|\alpha| < n$, $P(A/\alpha) = (a'_{ts})$, then for $\rho(A) \geq 2|a_{ii}|$ $(i \in \alpha)$ and $1 \leq t \leq l$,*

$$|a'_{tt}| - R_t(P(A/\alpha)) \geq |a_{j_t j_t}| - R_{j_t}(A) + \omega_{j_t} \geq |a_{j_t j_t}| - R_{j_t}(A) \tag{2.5}$$

and

$$|a'_{tt}| + R_t(P(A/\alpha)) \leq |a_{j_t j_t}| + R_{j_t}(A) - \omega_{j_t} \leq |a_{j_t j_t}| + R_{j_t}(A). \tag{2.6}$$

Proof By $\alpha \subseteq \mathbb{N}_r(A)$ and Proposition 2, we have

$$\begin{aligned}
\left|a'_{tt}\right| - R_t\big(P(A/\alpha)\big) &= \left|a'_{tt}\right| - \sum_{s=1,s\neq t}^{l} \left|a'_{ts}\right| \\
&= \left| a_{j_tj_t} + (a_{j_ti_1},\ldots,a_{j_ti_k})\big[\rho(A)I - A(\alpha)\big]^{-1} \begin{pmatrix} a_{i_1j_t} \\ \vdots \\ a_{i_kj_t} \end{pmatrix} \right| \\
&\quad - \sum_{s=1,s\neq t}^{l} \left| a_{j_tj_s} + (a_{j_ti_1},\ldots,a_{j_ti_k})\big[\rho(A)I - A(\alpha)\big]^{-1} \begin{pmatrix} a_{i_1j_s} \\ \vdots \\ a_{i_kj_s} \end{pmatrix} \right| \\
&\geq |a_{j_tj_t}| - \left| (a_{j_ti_1},\ldots,a_{j_ti_k})\big[\rho(A)I - A(\alpha)\big]^{-1} \begin{pmatrix} a_{i_1j_t} \\ \vdots \\ a_{i_kj_t} \end{pmatrix} \right| \\
&\quad - \sum_{s=1,s\neq t}^{l} \left| a_{j_tj_s} + (a_{j_ti_1},\ldots,a_{j_ti_k})\big[\rho(A)I - A(\alpha)\big]^{-1} \begin{pmatrix} a_{i_1j_s} \\ \vdots \\ a_{i_kj_s} \end{pmatrix} \right| \\
&\geq |a_{j_tj_t}| - \big(R_{j_t}(A) - \omega_{j_t}\big),
\end{aligned}$$

which implies (2.5).

Similarly, we can also verify (2.6) immediately. □

When $A \in SD_n$, (2.5) and Theorem 2.1 of [1] show the following fact.

Corollary 1 *If $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is nonnegative irreducible and strictly diagonally dominant with spectral radius $\rho(A)$, $\emptyset \neq \alpha \subseteq \mathbb{N}$, $|\alpha| < n$, $\beta = \mathbb{N} - \alpha$, then for $\rho(A) \geq 2|a_{ii}|$ $(i \in \alpha)$,*

$$P(A/\alpha) = A(\beta) + A(\beta,\alpha)\big[\rho(A)I - A(\alpha)\big]^{-1}A(\alpha,\beta)$$

is nonnegative irreducible and strictly diagonally dominant.

3 Properties for the Perron complement of strictly γ- and product γ-diagonally dominant matrices

In this section, we obtain several properties for the Perron complement of strictly γ- and product γ-diagonally dominant matrices through using the entries and spectral radius of the original matrix.

Lemma 3 (see [12]) *If $a > b$, $c > b$, $b > 0$, and $0 \leq r \leq 1$, then*

$$a^r c^{1-r} \geq (a-b)^r (c-b)^{1-r} + b.$$

Theorem 2 *If $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is nonnegative irreducible and with spectral radius $\rho(A)$, $\mathbb{N}_r(A) \cap \mathbb{N}_c(A) \neq \emptyset$, $\alpha = \{i_1, i_2, \ldots, i_k\} \subseteq (\mathbb{N}_r(A) \cap \mathbb{N}_c(A))$, $|\alpha| < n$, $\beta = \mathbb{N} - \alpha = \{j_1, j_2, \ldots, j_l\}$,* *and* $P(A/\alpha) = (a'_{ts})$, *then for* $\rho(A) \ge 2|a_{ii}|$ $(i \in \alpha)$, $1 \le t \le l$, *and* $0 \le \gamma \le 1$,

$$\begin{aligned}
&|a'_{tt}| - \gamma R_t(P(A/\alpha)) - (1-\gamma) S_t(P(A/\alpha)) \\
&\quad \ge |a_{j_t j_t}| - \gamma R_{j_t}(A) - (1-\gamma) S_{j_t}(A) + \gamma \omega_{j_t} + (1-\gamma)\omega_{j_t}^T \\
&\quad \ge |a_{j_t j_t}| - \gamma R_{j_t}(A) - (1-\gamma) S_{j_t}(A)
\end{aligned}$$

and

$$\begin{aligned}
&|a'_{tt}| + \gamma R_t(P(A/\alpha)) + (1-\gamma) S_t(P(A/\alpha)) \\
&\quad \le |a_{j_t j_t}| + \gamma R_{j_t}(A) + (1-\gamma) S_{j_t}(A) - \gamma \omega_{j_t} - (1-\gamma)\omega_{j_t}^T \\
&\quad \le |a_{j_t j_t}| + \gamma R_{j_t}(A) + (1-\gamma) S_{j_t}(A).
\end{aligned}$$

Proof In light of $\alpha \subseteq (\mathbb{N}_r(A) \cap \mathbb{N}_c(A))$ and Proposition 2, we have

$$\begin{aligned}
&|a'_{tt}| - \gamma R_t(P(A/\alpha)) - (1-\gamma) S_t(P(A/\alpha)) \\
&\quad = |a'_{tt}| - \gamma \sum_{s=1, s\neq t}^{l} |a'_{ts}| - (1-\gamma) \sum_{s=1, s\neq t}^{l} |a'_{st}| \\
&\quad = \left| a_{j_t j_t} + (a_{j_t i_1}, \ldots, a_{j_t i_k}) \left[\rho(A) I - A(\alpha)\right]^{-1} \begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t} \end{pmatrix} \right| \\
&\qquad - \gamma \sum_{s=1, s\neq t}^{l} \left| a_{j_t j_s} + (a_{j_t i_1}, \ldots, a_{j_t i_k}) \left[\rho(A) I - A(\alpha)\right]^{-1} \begin{pmatrix} a_{i_1 j_s} \\ \vdots \\ a_{i_k j_s} \end{pmatrix} \right| \\
&\qquad - (1-\gamma) \sum_{s=1, s\neq t}^{l} \left| a_{j_s j_t} + (a_{j_s i_1}, \ldots, a_{j_s i_k}) \left[\rho(A) I - A(\alpha)\right]^{-1} \begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t} \end{pmatrix} \right| \\
&\quad \ge |a_{j_t j_t}| - \left| (a_{j_t i_1}, \ldots, a_{j_t i_k}) \left[\rho(A) I - A(\alpha)\right]^{-1} \begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t} \end{pmatrix} \right| \\
&\qquad - \gamma \sum_{s=1, s\neq t}^{l} \left| a_{j_t j_s} + (a_{j_t i_1}, \ldots, a_{j_t i_k}) \left[\rho(A) I - A(\alpha)\right]^{-1} \begin{pmatrix} a_{i_1 j_s} \\ \vdots \\ a_{i_k j_s} \end{pmatrix} \right| \\
&\qquad - (1-\gamma) \sum_{s=1, s\neq t}^{l} \left| a_{j_s j_t} + (a_{j_s i_1}, \ldots, a_{j_s i_k}) \left[\rho(A) I - A(\alpha)\right]^{-1} \begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t} \end{pmatrix} \right| \\
&\quad = |a_{j_t j_t}| - \gamma \left| (a_{j_t i_1}, \ldots, a_{j_t i_k}) \left[\rho(A) I - A(\alpha)\right]^{-1} \begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t} \end{pmatrix} \right|
\end{aligned}$$

$$
\begin{aligned}
&- \gamma \sum_{s=1,s\neq t}^{l} \left| a_{j_t j_s} + (a_{j_t i_1}, \ldots, a_{j_t i_k})\big[\rho(A)I - A(\alpha)\big]^{-1} \begin{pmatrix} a_{i_1 j_s} \\ \vdots \\ a_{i_k j_s} \end{pmatrix} \right| \\
&- (1-\gamma) \left| (a_{j_t i_1}, \ldots, a_{j_t i_k})\big[\rho(A)I - A(\alpha)\big]^{-1} \begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t} \end{pmatrix} \right| \\
&- (1-\gamma) \sum_{s=1,s\neq t}^{l} \left| a_{j_s j_t} + (a_{j_s i_1}, \ldots, a_{j_s i_k})\big[\rho(A)I - A(\alpha)\big]^{-1} \begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t} \end{pmatrix} \right| \\
\geq{}& |a_{j_t j_t}| - \gamma\big(R_{j_t}(A) - \omega_{j_t}\big) - (1-\gamma)\big(S_{j_t}(A) - \omega_{j_t}^T\big) \\
={}& |a_{j_t j_t}| - \gamma R_{j_t}(A) - (1-\gamma)S_{j_t}(A) + \gamma\omega_{j_t} + (1-\gamma)\omega_{j_t}^T,
\end{aligned}
$$

which yields the first type of inequalities. Similarly, the rest of the inequalities of Theorem 2 can be verified. □

By Theorem 2 and Theorem 2.1 of [1], we get the following corollary.

Corollary 2 *If $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is nonnegative irreducible and strictly γ-diagonally dominant with spectral radius $\rho(A)$, $\mathbb{N}_r(A) \cap \mathbb{N}_c(A) \neq \emptyset$, $\alpha \subseteq \mathbb{N}_r(A) \cap \mathbb{N}_c(A)$, $|\alpha| < n$, $\beta = \mathbb{N} - \alpha$, then for $\rho(A) \geq 2|a_{ii}|$ $(i \in \alpha)$,*

$$
P(A/\alpha) = A(\beta) + A(\beta,\alpha)\big[\rho(A)I - A(\alpha)\big]^{-1}A(\alpha,\beta)
$$

is nonnegative irreducible and strictly γ-diagonally dominant, and so is $P_t(A/\alpha)$.

Theorem 3 *If $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is nonnegative irreducible with spectral radius $\rho(A)$, $\mathbb{N}_r(A) \cap \mathbb{N}_c(A) \neq \emptyset$, $\alpha = \{i_1, i_2, \ldots, i_k\} \subseteq \mathbb{N}_r(A) \cap \mathbb{N}_c(A)$, $|\alpha| < n$, $\beta = \mathbb{N} - \alpha = \{j_1, j_2, \ldots, j_l\}$, $P(A/\alpha) = (a'_{ts})$, then for $\rho(A) \geq 2|a_{ii}|$ $(i \in \alpha)$, $1 \leq t \leq l$, and $0 \leq \gamma \leq 1$,*

$$
\begin{aligned}
&\big|a'_{tt}\big| - R_t^{\gamma}\big(P(A/\alpha)\big)S_t^{1-\gamma}\big(P(A/\alpha)\big) \\
&\quad \geq |a_{j_t j_t}| - \big[R_{j_t}(A) - \omega_{j_t}\big]^{\gamma}\big[S_{j_t}(A) - \omega_{j_t}^T\big]^{1-\gamma} \geq |a_{j_t j_t}| - R_{j_t}^{\gamma}(A)S_{j_t}^{1-\gamma}(A)
\end{aligned}
$$

and

$$
\begin{aligned}
&\big|a'_{tt}\big| + R_t^{\gamma}\big(P(A/\alpha)\big)S_t^{1-\gamma}\big(P(A/\alpha)\big) \\
&\quad \leq |a_{j_t j_t}| + \big[R_{j_t}(A) - \omega_{j_t}\big]^{\gamma}\big[S_{j_t}(A) - \omega_{j_t}^T\big]^{1-\gamma} \leq |a_{j_t j_t}| + R_{j_t}^{\gamma}(A)S_{j_t}^{1-\gamma}(A).
\end{aligned}
$$

Proof By the definition of the Perron complement, we obtain

$$
\begin{aligned}
&\big|a'_{tt}\big| - \big[R_t\big(P(A/\alpha)\big)\big]^{\gamma}\big[S_t\big(P(A/\alpha)\big)\big]^{1-\gamma} \\
&\quad = \big|a'_{tt}\big| - \left[\sum_{s=1,s\neq t}^{l} \big|a'_{ts}\big|\right]^{\gamma}\left[\sum_{s=1,s\neq t}^{l} \big|a'_{st}\big|\right]^{1-\gamma}
\end{aligned}
$$

$$
\begin{aligned}
&= \left| a_{j_tj_t} + (a_{j_ti_1},\ldots,a_{j_ti_k})\bigl[\rho(A)I - A(\alpha)\bigr]^{-1}\begin{pmatrix} a_{i_1j_t} \\ \vdots \\ a_{i_kj_t}\end{pmatrix}\right| \\
&\quad - \left[\sum_{s=1,s\neq t}^{l}\left| a_{j_tj_s} + (a_{j_ti_1},\ldots,a_{j_ti_k})\bigl[\rho(A)I - A(\alpha)\bigr]^{-1}\begin{pmatrix} a_{i_1j_s} \\ \vdots \\ a_{i_kj_s}\end{pmatrix}\right|\right]^{\gamma} \\
&\quad \times \left[\sum_{s=1,s\neq t}^{l}\left| a_{j_sj_t} + (a_{j_si_1},\ldots,a_{j_si_k})\bigl[\rho(A)I - A(\alpha)\bigr]^{-1}\begin{pmatrix} a_{i_1j_t} \\ \vdots \\ a_{i_kj_t}\end{pmatrix}\right|\right]^{1-\gamma} \\
&\geq |a_{j_tj_t}| - \left| (a_{j_ti_1},\ldots,a_{j_ti_k})\bigl[\rho(A)I - A(\alpha)\bigr]^{-1}\begin{pmatrix} a_{i_1j_t} \\ \vdots \\ a_{i_kj_t}\end{pmatrix}\right| \\
&\quad - \left[\sum_{s=1,s\neq t}^{l}\left| a_{j_tj_s} + (a_{j_ti_1},\ldots,a_{j_ti_k})\bigl[\rho(A)I - A(\alpha)\bigr]^{-1}\begin{pmatrix} a_{i_1j_s} \\ \vdots \\ a_{i_kj_s}\end{pmatrix}\right|\right]^{\gamma} \\
&\quad \times \left[\sum_{s=1,s\neq t}^{l}\left| a_{j_sj_t} + (a_{j_si_1},\ldots,a_{j_si_k})\bigl[\rho(A)I - A(\alpha)\bigr]^{-1}\begin{pmatrix} a_{i_1j_t} \\ \vdots \\ a_{i_kj_t}\end{pmatrix}\right|\right]^{1-\gamma}.
\end{aligned}
$$

Denote

$$
h = \left| (a_{j_ti_1},\ldots,a_{j_ti_k})\bigl[\rho(A)I - A(\alpha)\bigr]^{-1}\begin{pmatrix} a_{i_1j_t} \\ \vdots \\ a_{i_kj_t}\end{pmatrix}\right|.
$$

In light of Lemma 3, we get

$$
\begin{aligned}
&\bigl|a'_{tt}\bigr| - \bigl[R_t\bigl(P(A/\alpha)\bigr)\bigr]^{\gamma}\bigl[S_t\bigl(P(A/\alpha)\bigr)\bigr]^{1-\gamma} \\
&\quad \geq |a_{j_tj_t}| - h - \bigl[R_{j_t}(A) - \omega_{j_t} - h\bigr]^{\gamma}\bigl[S_{j_t}(A) - \omega_{j_t}^{T} - h\bigr]^{1-\gamma} \\
&\quad \geq |a_{j_tj_t}| - h - \bigl[\bigl(R_{j_t}(A) - \omega_{j_t}\bigr)^{\gamma}\bigl(S_{j_t}(A) - \omega_{j_t}^{T}\bigr) - h\bigr] \\
&\quad = |a_{j_tj_t}| - \bigl[R_{j_t}(A) - \omega_{j_t}\bigr]^{\gamma}\bigl[S_{j_t}(A) - \omega_{j_t}^{T}\bigr]^{1-\gamma}.
\end{aligned}
$$

Hence, we can get the first type of inequalities of Theorem 3. Similarly, we can immediately verify the other one. □

By Theorem 3 and Theorem 2.1 of [1], we have the following corollary.

Corollary 3 *If $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is nonnegative irreducible and strictly product γ-diagonally dominant with spectral radius $\rho(A)$, $\mathbb{N}_r(A) \cap \mathbb{N}_c(A) \neq \emptyset$, $\alpha \subseteq \mathbb{N}_r(A) \cap \mathbb{N}_c(A)$, $|\alpha| < n$, and*

$\beta = \mathbb{N} - \alpha$, then for $\rho(A) \geq 2|a_{ii}|$ $(i \in \alpha)$,

$$P(A/\alpha) = A(\beta) + A(\beta,\alpha)\left[\rho(A)I - A(\alpha)\right]^{-1}A(\alpha,\beta)$$

is nonnegative irreducible and strictly product γ-diagonally dominant, and so is $P_t(A/\alpha)$.

Competing interests
The authors declare that they have no competing interests.

Authors? contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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