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# Some notes on the paper "The equivalence of cone metric spaces and metric spaces"

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## Abstract

In this article, we shall show that the metrics defined by Feng and Mao, and Du are equivalent. We also provide some examples for one of the metrics.

## 1 Introduction and preliminary

Let $E$ be a topological vector space (t.v.s.) with zero vector $\theta$. A nonempty subset $K$ of $E$ is called a convex cone if $K + K \subseteq K$ and $\lambda K \subseteq K$ for each $\lambda \geq 0$. A convex cone $K$ is said to be pointed if $K \cap - K = \{\theta\}$. For a given cone $K \subseteq E$, we can define a partial ordering $\preccurlyeq$ with respect to $K$ by

$$x \preccurlyeq y \Leftrightarrow y - x \in K.$$

$x < y$ will stand for $x \preccurlyeq y$ and $x \neq y$ while $x \prec\prec y$ stands for $y - x \in K^\circ$, where $K^\circ$ denotes the interior of $K$. In the following, we shall always assume that $Y$ is a locally convex Hausdorff t.v.s. with zero vector $\theta$, $K$ is a proper, closed, and convex pointed cone in $Y$ with $K^\circ \neq \emptyset$, $e \in K^\circ$ and $\preccurlyeq$ a partial ordering with respect to $K$. The non-linear scalarization function $\xi_e : Y \to \mathbb{R}$ is defined by

$$\xi_e(y) = \inf\{r \in \mathbb{R} : y \in re - K\}$$

for all $y \in Y$.

We will use $P$ instead of $K$ when $E$ is a real Banach spaces.

**Lemma 1.1** [1]*For each $r \in R$ and $y \in Y$, the following statements are satisfied:*

*(i)* $\xi_e(y) \leq r \Leftrightarrow y \in re - K$.

*(ii)* $\xi_e(y) > r \Leftrightarrow y \notin re - K$.

*(iii)* $\xi_e(y) \geq r \Leftrightarrow y \notin re - K^\circ$.

*(iv)* $\xi_e(y) < r \Leftrightarrow y \in re - K^\circ$.

*(v)* $\xi_e(.)$ *is positively homogeneous and continuous on* $Y$.

*(vi)* $y_1 \in y_2 + K \Rightarrow \xi_e(y_2) \leq \xi_e(y_1)$

*(vii)* $\xi_e(y_1 + y_2) \leq \xi_e(y_1) + \xi_e(y_2)$ *for all* $y_1, y_2 \in Y$.

**Definition 1.2** [1]*Let $X$ be a nonempty set. A vector-valued function $d : X \times X \to Y$ is said to be a TVS-cone metric, if the following conditions hold:*

*(C1)* $\theta \preccurlyeq d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = \theta$ *iff* $x = y$

*(C2)* $d(x, y) = d(y, x)$ *for all* $x, y \in X$

*(C3)*$d(x, y) \preccurlyeq (x, z) + d(z, y)$ *for all* $x, y, z \in X$.

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*The pair (X, d) is then called a TVS-cone metric space.*

Huang and Zhang [2] discuss the case in which $Y$ is a real Banach space and call a vector-valued function $d : X \times X \to Y$ a cone metric if $d$ satisfies (C1)-(C3). Clearly, a cone metric space, in the sense of Huang and Zhang, is a special case of a TVS-cone metric space.

In the following, some conclusions are listed.

**Lemma 1.3** [3]*Let (X, D) be a cone metric space. Then*

$$d(x, y) = \inf_{\{u \in P | D(x,y) \preccurlyeq u\}} ||u||, \quad x, y \in X$$

*is a metric on X.*

**Theorem 1.4** [3]*The metric space (X, d) is complete if and only if the cone metric space (X, D) is complete .*

**Theorem 1.5** [1]*Let (X, D) be a TVS-cone metric space. Then* $d_2 : X \times X \to [0, \infty)$ *defined by* $d_2(x, y) = \xi_e(D(x, y))$ *is a metric.*

## 2 Main results

We first show that the metrics introduced the Lemma 1.3 and the Theorem 1.5 are equivalent. Then, we provide some examples involving the metric defined in Lemma 1.3.

**Theorem 2.1** *For every cone metric* $D : X \times X \to E$ *there exists a metric* $d : X \times X \to \mathbb{R}^+$ *which is equivalent to D on X.*

**Proof**. Define $d(x, y) = \inf \{||u||: D(x, y) \preccurlyeq u\}$. By the Lemma 1.3 $d$ is a metric. We shall now show that each sequence $\{x_n\} \subseteq X$ which converges to a point $x \in X$ in the $(X, d)$ metric also converges to $x$ in the $(X, D)$ metric, and conversely. We have

$$\forall n, \ m \in \mathbb{N} \quad \exists u_{nm} \text{ such that } \quad ||u_{nm}|| < d(x_n, x) + \frac{1}{m}, \quad D(x_n, x) \preccurlyeq u_{nm}.$$

Put $v_n := u_{nn}$ then $||v_n|| < d(x_n, x) + \frac{1}{n}$ and $D(x_n, x) \preccurlyeq v_n$. Now if $x_n \to x$ in $(X, d)$ then $d(x_n, x) \to 0$ and so $v_n \to 0$ too, therefore for all $c \gg 0$ there exists $N \in \mathbb{N}$ such that $v_n \ll c$ for all $n \geq N$. This implies that $D(x_n, x) \ll c$ for all $n \geq N$. Namely $x_n \to x$ in $(X, D)$.

Conversely, for every real $\varepsilon > 0$, choose $c \in E$ with $c \gg 0$ and $||c|| < \varepsilon$. Then there exists $N \in \mathbb{N}$ such that $D(x_n, x) \ll c$ for all $n \geq N$. This means that for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) \leq ||c|| < \varepsilon$ for all $n \geq N$. Therefore $d(x_n, x) \to 0$ as $n \to \infty$ so $x_n \to x$ in $(X, d)$.

□

**Theorem 2.2** *If* $d_1(x, y) = \inf \{||u||: D(x, y) \preccurlyeq u\}$ *and* $d_2(x, y) = \xi_e(D(x, y))$ *where D is a cone metric on X. Then* $d_1$ *is equivalent with* $d_2$.

**Proof**. Let $x_n \xrightarrow{d_1} x$ then $d_1(x_n, x) \xrightarrow{\mathbb{R}} 0$ so by Theorem 2.1 in $x_n \xrightarrow{D} x$ so

$$\forall \varepsilon > 0, \quad \forall e \succ\succ 0 \quad \exists N \quad \forall n (n \geq N \Rightarrow D(x_n, x) \prec\prec \varepsilon e),$$

and or $\varepsilon e - D(x_n, x) \in K^\circ$ for all $n \geq N$. So $D(x_n, x) \in e - K^\circ$ for $n \geq N$. Now by [[1], Lemma 1.1 (iv)] $\xi_e(D(x_n, x)) < \varepsilon$ for all $n \geq N$. Namely $d_2(x_n, x) < \varepsilon$ for all $n \geq N$ therefore $d_2(x_n, x) \xrightarrow{\mathbb{R}} 0$ or $x_n \xrightarrow{d_2} x$.

Conversely, $x_n \underset{\rightarrow}{d_2} x$ hence $d_2(x_n, x) \underset{\rightarrow}{\mathbb{R}} 0$ so $\xi_e(D(x_n, x)) \underset{\rightarrow}{\mathbb{R}} 0$, therefore

$$\forall \varepsilon > 0 \quad \exists N \quad \forall n (n \geq N \Rightarrow \xi_e(D(x_n, x)) < \varepsilon).$$

So $D(x_n, x) \in \varepsilon e - K^\circ$ for $n \geq N$ by [[1], Lemma 1.1 (iv)]. Hence, $D(x_n, x) = \varepsilon e - k$ for some $k \in K^\circ$, so $D(x_n, x) \prec\prec \varepsilon e$ for $n \geq N$ this implies that $x_n \overset{D}{\rightarrow} x$ and again by Theorem 2.1 $x_n \overset{d_1}{\rightarrow} x$. □

In the following examples, we use the metric of Lemma 1.3.

**Example 2.3** *Let* $0 \neq a \in P \subseteq \mathbb{R}^n$ *with* $||a|| = 1$ *and for every* $x, y \in \mathbb{R}^n$ *define*

$$D(x, y) = \begin{cases} a, & x \neq y; \\ 0, & x = y. \end{cases}$$

*Then D is a cone metric on* $\mathbb{R}^n$ *and its equivalent metric d is*

$$d(x, y) = \begin{cases} 1, & x \neq y; \\ 0, & x = y, \end{cases}$$

*which is a discrete metric.*

**Example 2.4** *Let* $a, b \geq 0$ *and consider the cone metric* $D : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^2$ *with*

$$D(x, y) = (ad_1(x, y), bd_2(x, y))$$

*where* $d_1$, $d_2$ *are metrics on* $\mathbb{R}$. *Then its equivalent metric is*

$$d(x, y) = \sqrt{a^2 + b^2}||(d_1(x, y), d_2(x, y))||.$$

*In particular if* $d_1(x, y) := |x - y|$ *and* $d_2(x, y) := \alpha|x - y|$, *where* $\alpha \geq 0$ *then D is the same famous cone metric which has been introduced in [*[2]*, Example 1] and its equivalent metric is*

$$d(x, y) = \sqrt{1 + \alpha^2}|x - y|.$$

**Example 2.5** *For* $q > 0$, $b > 1$, $E = l^q$, $P = \{\{x_n\}_{n \geq 1} : x_n \geq 0$, *for all* $n\}$ *and* $(X, \rho)$ *a metric space, define* $D : X \times X \rightarrow E$ *which is the same cone metric as [*[4]*, Example 1.3] by*

$$D(x, y) = \left\{ \left( \frac{\rho(x, y)}{b^n} \right)^{\frac{1}{q}} \right\}_{n \geq 1}.$$

*Then its equivalent metric on* × *is*

$$d(x, y) = \left\| \left\{ \left( \frac{\rho(x, y)}{b^n} \right)^{\frac{1}{q}} \right\}_{n \geq 1} \right\|_{l^q} = \left( \sum_{n=1}^{\infty} \frac{\rho(x, y)}{b^n} \right)^{\frac{1}{q}} = \left( \frac{\rho(x, y)}{b - 1} \right)^{\frac{1}{q}}.$$

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**Authors' contributions**
All authors have read and approved the final manuscript.

**Competing interests**
The authors declare that they have no competing interests.

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