Magnetization plateau in the $S = \frac{1}{2}$ spin ladder with alternating rung exchange

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Abstract
We have studied the ground-state phase diagram of a spin ladder with alternating rung exchange $J_\perp^n = J_\perp[1 + (-1)^n\delta]$ in a magnetic filed, in the limit where the rung coupling is dominant. In this limit the model is mapped onto an XXZ Heisenberg chain in uniform and staggered longitudinal magnetic fields, where the amplitude of the staggered field is $\sim\delta$. We have shown that the magnetization curve of the system exhibits a plateau at magnetization equal to half of the saturation value. The width of a plateau scales as δ^ν, where $\nu = 4/5$ in the case of a ladder with isotropic antiferromagnetic legs and $\nu = 2$ in the case of a ladder with isotropic ferromagnetic legs. We have calculated four critical fields ($H_{c1}^\pm$ and $H_{c2}^\pm$) corresponding to transitions between different magnetic phases of the system. We have shown that these transitions belong to the universality class of the commensurate–incommensurate transition. The possibility for the realization of a new type of spin-Peierls instability, characterized by the spontaneous appearance of an alternating rung exchange in a uniform magnetic field and at a system magnetization equal to the half of its saturation value, is briefly discussed.

1. Introduction

A theoretical understanding of the magnetic properties of quantum spin systems, in particular of spin $S = 1/2$ *isotropic antiferromagnetic* two-leg ladders, has attracted a lot of interest for a number of reasons. On the one hand, there has been remarkable progress in recent years in the fabrication of such ladder compounds [1]. On the other hand, spin-ladder models pose interesting theoretical problems, since antiferromagnetic two-leg ladder systems have a gap in the excitation spectrum and, in the presence of a magnetic field, they reveal an extremely rich behaviour, dominated by quantum effects. These quantum phase transitions have been investigated intensively both theoretically [2–17] and experimentally [18–23].

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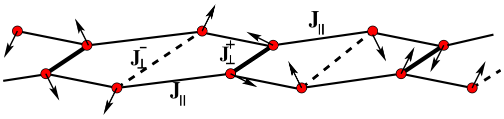

Figure 1. The ladder with alternating rung exchange.
(This figure is in colour only in the electronic version)

In this paper we study the ground-state magnetic phase diagram of the spin $S = 1/2$ two-leg ladder with alternating rung exchange (see figure 1) given by the Hamiltonian

$$\mathcal{H} = J_\parallel \sum_{n,\alpha} \mathbf{S}_{n,\alpha} \cdot \mathbf{S}_{n+1,\alpha} - H \sum_{n,\alpha} S^z_{n,\alpha} + J_\perp \sum_{n} \left[1 + (-1)^n \delta\right] \mathbf{S}_{n,1} \cdot \mathbf{S}_{n,2}, \quad (1)$$

where $\mathbf{S}_{n,\alpha}$ is a spin $S = 1/2$ operator of rung n ($n = 1, \ldots, N$) and leg α ($\alpha = 1, 2$). The interleg coupling is antiferromagnetic, $J_\perp^\pm = J_\perp(1 \pm \delta) > 0$.

Our interest in this model comes from its rich magnetic phase diagram exhibiting complex quantum nature, especially in the case of applied magnetic field. The model describes a new mechanism for magnetization plateau formation. Although up to now no materials have been available that realize this model, current progress in the technology of manufacturing low-dimensional materials with desired parameters makes the possibility of realizing a ladder system with rung-exchange modulation very plausible.

Moreover, as we show in this paper, the model describes an unconventional nontrivial spin-Peierls transition: in the case of an applied magnetic field, at a magnetization equal to half its saturation value, the Luttinger-liquid phase of a standard ladder with equal rungs becomes unstable towards a doubling of the lattice unit via the spontaneous staggered modulation of the interrung distance (see figure 1), accompanied by the opening of an excitation gap.

Below, we restrict our consideration to the limit of strong rung exchange $J_\perp^\pm \gg |J_\parallel|, \delta J_\perp$ and map the model onto a spin-1/2 XXZ Heisenberg chain in the presence of both longitudinal uniform and staggered magnetic fields, with the amplitude of the staggered component of the magnetic field proportional to $\sim\delta J_\perp$. We study the ground-state phase diagram of the effective spin-chain model and show that the alternation of rung exchange leads to the dynamical generation of a new energy scale in the system and to the appearance of two additional quantum phase transitions in the magnetic ground-state phase diagram. These transitions manifest themselves most clearly in the presence of a new magnetization plateau at a magnetization equal to half of its saturation value (see figure 2). The magnetic phase diagram is characterized by the following four critical fields: the field H_{c1}^-, which corresponds the the transition from a gapped rung-singlet phase to the gapless paramagnetic phase; the critical fields H_{c1}^+ and H_{c2}^- which mark end-points of the magnetization plateau, and the saturation field H_{c2}^+. The width of the plateau scales as δ^ν, where $\nu = 4/5$ in the case of a ladder with isotropic antiferromagnetic legs and $\nu = 2$ in the case of a ladder with isotropic ferromagnetic legs. Therefore this magnetic phase diagram is generic for a standard isotropic ladder with alternating rung exchange. However, in the case of a ladder with ferromagnetic legs and frustrating diagonal interleg exchange, the intermediate magnetization plateau disappears for sufficiently strong ferromagnetic diagonal coupling.

2. Derivation of the effective Hamiltonian

In this section we derive the effective spin-chain model to describe the strong rung-exchange limit $J_\perp \gg (\delta J_\perp), |J_\parallel|$ of the model (1). To obtain the spin-chain Hamiltonian, we follow

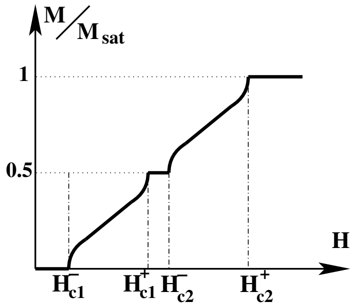

Figure 2. Schematic drawing of the magnetization (in units of saturated magnetization $M_{\rm sat}$) of a two-leg isotropic ladder with alternating rungs as a function of the external magnetic field.

the route already used to study the standard ladder models in the same limit of strong rung exchange [4, 5].

We start from the case $J_\parallel = 0$. In this limit the system decouples into a set of noninteracting rungs with couplings $J_\perp^+$ and $J_\perp^-$. In this case, an eigenstate of $\mathcal{H}$ is written as a product of rung states. At each rung, two spins $\mathbf{S}_{n,l}$ and $\mathbf{S}_{n,2}$ are either in a singlet state $|s_n^0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\ \downarrow\rangle - |\downarrow\ \uparrow\rangle)$ or in one of the triplet states $|t_n^+\rangle = |\uparrow\ \uparrow\rangle$, $|t_n^-\rangle = |\downarrow\ \downarrow\rangle$ and $|t_n^0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\ \downarrow\rangle + |\downarrow\ \uparrow\rangle)$. Their energies are, respectively, $E(s_n^0) = -3J_\perp^n/4$, $E(t_n^+) = J_\perp^n/4 - H$, $E(t_n^0) = J_\perp^n/4$ and $E(t_n^-) = J_\perp^n/4 + H$, where $J_\perp^n = J_\perp[1 + (-1)^n\delta]$.

When H is small, the ground state consists of a product of a rung singlet. As the field H increases, the energy of the state $|t_{2n-1}^+\rangle$ decreases, and at $H = H_{\rm c1} = J_\perp^-$ this state is degenerate with $|s_{2n-1}^0\rangle$. Thus, at $H = H_{\rm c1}$ the ground state of a given odd rung undergoes a transition from the singlet $|s_{2n-1}^0\rangle$ to the triplet $|t_{2n-1}^+\rangle$ and the total magnetization of the system jumps discontinuously from zero to $0.5M_{\rm sat} = N/2$. With a further increase in the magnetic field, at $H_{\rm c1} < H < H_{\rm c2}$ the magnetization remains constant. However, since for $H_{\rm c2} > J_\perp^+$ the energy of the state $|t_{2n}^+\rangle$ is lower than $|s_{2n}^0\rangle$, the magnetization once again increases discontinuously from $0.5M_{\rm sat}$ to $M_{\rm sat} = N$ for $H_{\rm c2} = J_\perp^+$.

For $J_\parallel \neq 0$, these abrupt transitions are broadened into intervals $H_{\rm c1}^- < H < H_{\rm c1}^+$ and $H_{\rm c2}^- < H < H_{\rm c2}^+$, respectively. Two different scenarios are possible. Either $H_{\rm c1}^+ < H_{\rm c2}^-$ and the magnetization plateau with $M = 0.5M_{\rm sat}$ remains, or $H_{\rm c1}^+ > H_{\rm c2}^-$ and alternation of the rung exchange is irrelevant. In the latter case, the model shows a similar behaviour to the standard two-leg ladder in a magnetic field: with increasing magnetic field in the range between $H_{\rm c1}^-$ and $H_{\rm c2}^+$, the magnetization continuously evolves from a nonmagnetic phase at $H \leqslant H_{\rm c1}^-$ into the fully polarized ferromagnetic state at $H \geqslant H_{\rm c2}^+$.

The easiest way to obtain the effective model is to split the Hamiltonian (1) into three parts:

$$H = H_0^{(o)} + H_0^{(e)} + H_{\rm int}$$

$$H_0^{(o)} = J_\perp^{(-)} \sum_n \mathbf{S}_{2n-1,1} \cdot \mathbf{S}_{2n-1,2} - H_{\rm c1} \sum_{n,\alpha} S_{2n-1,\alpha}^z, \tag{2}$$

$$H_0^{(e)} = J_\perp^{(+)} \sum_{n=1}^{N/2} \mathbf{S}_{2n,1} \cdot \mathbf{S}_{2n,2} - H_{\rm c2} \sum_{n,\alpha} S_{2n,\alpha}^z \tag{3}$$

and

$$H_{\rm int} = \sum_{n,\alpha} \Big[J_\parallel \mathbf{S}_{n,\alpha}\mathbf{S}_{n+1,\alpha} - (H - H_{\rm c1})\, S_{2n-1,\alpha}^z - (H - H_{\rm c2})\, S_{2n,\alpha}^z \Big]. \tag{4}$$

The ground state of $\mathcal{H}_0$ is 2^N times degenerate, since each rung can be in the state $|s_0\rangle$ or $|t_+\rangle$ and the first excited state has an energy of the order of $J_\perp$. $\mathcal{H}_{\rm int}$ will lift the degeneracy in the ground-state manifold, leading to an effective Hamiltonian that can be derived by standard perturbation theory [4].

Let us start by introducing pseudo-spin $\tau = 1/2$ operators, $\boldsymbol{\tau}_n$, which act on these states as

$$\begin{aligned} &\tau_n^z|s_0\rangle_n = -\tfrac{1}{2}|s_0\rangle_n, \qquad \tau_n^z|t_+\rangle_n = \tfrac{1}{2}|t_+\rangle_n, \\ &\tau_n^+|s_0\rangle_n = |t_+\rangle_n, \qquad \tau_n^+|t_+\rangle_n = 0, \\ &\tau_n^-|s_0\rangle_n = 0, \qquad \tau_n^-|t_+\rangle_n = |s_0\rangle_n. \end{aligned} \tag{5}$$

The relation between the real spin operator $\mathbf{S}_n$ and the pseudo-spin operator $\boldsymbol{\tau}_n$ in this restricted subspace can easily be derived by inspection:

$$S_{n,\alpha}^{\pm} = (-1)^\alpha \frac{1}{\sqrt{2}} \tau_n^{\pm}, \qquad S_{n,\alpha}^z = \tfrac{1}{2}\left(\tfrac{1}{2} + \tau_n^z\right). \tag{6}$$

Using (6), to first order and up to a constant, we easily obtain the effective Hamiltonian

$$H_{\rm eff} = \sum_n \{J_{xy}\left(\tau_n^x\tau_{n+1}^x + \tau_n^y\tau_{n+1}^y\right) + J_z\tau_n^z\tau_{n+1}^z\} - h_{\rm eff}^0\sum_n \tau_n^z - h_{\rm eff}^1\sum_n(-1)^n\tau_n^z, \tag{7}$$

where

$$J_{xy} = J_\parallel, \qquad J_z = \tfrac{1}{2}J_\parallel, \tag{8}$$

$$h_{\rm eff}^0 = H - J_\perp - \frac{J_\parallel}{2}, \tag{9}$$

$$h_{\rm eff}^1 = \delta J_\perp. \tag{10}$$

Thus the effective Hamiltonian is nothing but the XXZ Heisenberg chain, with anisotropy $J_z/J_{xy} \equiv \Delta = 1/2$ in uniform and staggered longitudinal magnetic fields. It is worth noticing that the model (7) and closely related models have been discussed intensively in recent years [24–33].

3. Magnetic phase diagram

3.1. The first critical field H_{c1}^- and the saturation field H_{c2}^+

We first calculate of the critical field H_{c1}^-, corresponding to the transition from a gapped rung-singlet phase to a gapless paramagnetic phase, and the saturation field H_{c2}^+.

For $H < H_{c1}^-$, the ground state of the system corresponds to the gapped rung-singlet phase with zero magnetization. For $H > H_{c2}^+$, the system is in the fully polarized ferromagnetic phase. The easiest way to express H_{c1}^- and H_{c2}^+ in terms of ladder parameters $J_\parallel$, $J_\perp$ and δ is to perform the Jordan–Wigner transformation, which maps the problem onto a system of interacting spinless fermions:

$$H_{sf} = t\sum_n(a_n^+a_{n+1} + \text{h.c.}) + V\sum_n \rho_n\rho_{n+1} - \sum_n\left[\mu_0 + (-1)^n\mu_1\right]\rho_n \tag{11}$$

where

$$t = \tfrac{1}{2}J_\parallel, \qquad V = \tfrac{1}{2}J_\parallel, \qquad \mu_0 = \tfrac{1}{2}J_\parallel + h_{\rm eff}^0, \qquad \mu_1 = h_{\rm eff}^1. \tag{12}$$

The lowest critical field H_{c1}^- corresponds to that value of the chemical potential μ_{0c} for which the band of spinless fermions starts to fill up. In this limit we can neglect the interaction term in equation (11) and obtain the model of free massive particles with spectrum

$$E^\pm(k) = -\mu_0 \pm \sqrt{J_\parallel^2\cos^2(k) + \mu_1^2}. \tag{13}$$

The chemical potential corresponding to H_{c1}^- is given by $\mu_{0c} = -\sqrt{J^2 + \mu_1^2}$, i.e.

$$H_{c1}^- = J_\perp - \sqrt{J_\parallel^2 + (\delta J_\perp)^2}. \tag{14}$$

A similar argument can be used to determine H_{c2}^+. It is useful to make a particle–hole transformation and estimate H_{c2}^+ from the condition where the transformed hole band starts to fill. This gives

$$H_{c2}^+ = J_\perp + J_\parallel + \sqrt{J_\parallel^2 + (\delta J_\perp)^2}. \tag{15}$$

3.2. Magnetization plateau: H_{c1}^+ and H_{c2}^-

To determine the values of the remaining two critical fields H_{c1}^+ and H_{c2}^- we consider the model (7) for $h_{\text{eff}}^0, h_{\text{eff}}^1 \ll J_\parallel$.

For $h_{\text{eff}}^0 = h_{\text{eff}}^1 = 0$, the Hamiltonian (7) with anisotropy parameter $|\Delta| < 1$ is known to be critical. The long-wavelength excitations are described by the standard Gaussian theory with Hamiltonian [34]

$$\mathcal{H}_{\text{leg}} = \int \mathrm{d}x\, \frac{v_s}{2}[(\partial_x\phi)^2 + (\partial_x\theta)^2]. \tag{16}$$

Here $\phi(x)$ and $\theta(x)$ are dual bosonic fields, $\partial_t\phi = v_s\partial_x\theta$, and satisfy the following commutational relation

$$[\phi(x), \theta(y)] = \mathrm{i}\Theta(y - x), \qquad [\phi(x), \theta(x)] = \mathrm{i}/2. \tag{17}$$

The velocity of spin excitation v_s is fixed from the Bethe ansatz solution as

$$v_s = J_\parallel \frac{K}{2K - 1} \sin(\pi/2K), \tag{18}$$

where the spin-stiffness parameter K is given by

$$K = \frac{1}{2\left(1 - \frac{1}{\pi}\arccos\Delta\right)}. \tag{19}$$

Thus the parameter K increases monotonically along the XXZ critical line $-1 < \Delta < 1$ from its minimal value $K = 1/2$ at $\Delta = 1$ (isotropic antiferromagnetic chain) to unity at $\Delta = 0$ (the XY chain) and diverges at the ferromagnetic instability point of a single chain $\Delta = -1$.

To obtain the continuum version of the Hamiltonian (7), we use the standard bosonization expression of the spin operator [35]

$$\tau_n^z = \sqrt{\frac{K}{\pi}}\partial_x\phi + (-1)^n\frac{A}{\pi}\sin(\sqrt{4\pi K}\phi), \tag{20}$$

where A is a non-universal real constant of the order of unity [36], and get the continuum Hamiltonian

$$H_{\text{Bos}} = \int \mathrm{d}x \left\{ \frac{v_s}{2}[(\partial_x\phi)^2 + (\partial_x\theta)^2] + \frac{h_{\text{eff}}^1}{\pi a_0}\sin(\sqrt{4\pi K}\phi) - h_{\text{eff}}^0\sqrt{\frac{K}{\pi}}\partial_x\phi \right\}. \tag{21}$$

The Hamiltonian (21) is the standard Hamiltonian for the commensurate–incommensurate transition, which has been studied intensively in the past using bosonization [37] and the Bethe ansatz [38]. Below, we use these results to describe the magnetization plateau and the transitions from a gapped (plateau) to gapless paramagnetic phases.

Let us first consider $h_{\text{eff}}^0 = 0$. In this case, the continuum theory of the initial ladder model in the magnetic field $H = J_\perp + J_\parallel/2$ is given by the quantum sine-Gordon (SG) model with a massive term $\sim h^1_{\rm eff}\sin(\sqrt{4\pi K}\phi)$. From the exact solution of the SG model [39] it is known that the excitation spectrum is gapless for $K \geqslant 2$ and has a gap in the interval $0 < K < 2$. At $K = 1$ the sine-Gordon model is equivalent to the theory of free massive fermions with $m = h^1_{\rm eff}$. At $1 < K < 2$ the excitation spectrum of the model consists of solitons and antisolitons with mass M, while for $0 < K < 1$ the spectrum also contains soliton–antisoliton bound states ('breathers'). The exact relation between the soliton mass M and the bare mass $h^1_{\rm eff} = \delta J_\perp$ is given by [40]

$$M = J_\parallel \mathcal{C}(K)\left(\delta J_\perp/J_\parallel\right)^{1/(2-K)}, \tag{22}$$

where

$$\mathcal{C}(K) = \frac{2\Gamma(\frac{1}{2\nu})}{\sqrt{\pi}\Gamma(\frac{1}{2}+\frac{1}{2\nu})}\cdot\left[\frac{\Gamma(1-K/2)}{2\Gamma(K/2)}\right]^{1/(2-K)}. \tag{23}$$

It is straightforward to get, from (19), that at $\Delta = 1/2$ the spin-stiffness parameter $K = 3/4$. Therefore, for $h^0_{\rm eff} = 0$ the sine-Gordon Hamiltonian (21) is in the strong coupling (massive) regime. In this case, the low-energy behaviour of the system is determined by the strongly relevant staggered magnetic field (i.e. alternating part of the rung exchange), represented by the term $h^1_{\rm eff}\sin(\sqrt{3\pi}\phi)$. In the ground state the field ϕ is pinned in one of the minima of the staggered field potential

$$\langle 0|\sqrt{3\pi}\phi|0\rangle = -\pi/2 + 2\pi n. \tag{24}$$

In view of (20), we conclude that this state corresponds to a long-range-ordered antiferromagnetic phase of the effective Heisenberg chain (7), i.e. to a phase of the initial ladder system, where odd rungs have a dominant triplet character and even rungs are predominantly singlets.

At $h^0_{\rm eff} \neq 0$ (i.e. $H \neq J_\perp + J_\parallel/2$) the very presence of the gradient term in the Hamiltonian (21) makes it necessary to consider the ground state of the sine-Gordon model in sectors with nonzero topological charge. The effective chemical potential $\sim h^0_{\rm eff}\sqrt{\frac{K}{\pi}}\partial_x\phi$ tends to change the number of particles in the ground state, i.e. to create finite and uniform density solitons. It is clear that the gradient term in (21) can be eliminated by a gauge transformation $\phi \to \phi_s + h^0_{\rm eff}\sqrt{\frac{K}{\pi}}x$, however this immediately implies that the vacuum distribution of the fieled ϕ will be shifted with respect to the minima (24). This competition between contributions of the smooth and staggered components of magnetic field is resolved as a continuous phase transition from a gapped state at $|h^0_{\rm eff}| < M$ to a gapless (paramagnetic) phase at $|h^0_{\rm eff}| > M$, where M is the soliton mass [37].

For our effective Hamiltonian (7) with $\Delta = 1/2$, the spin-stiffness parameter K is 3/4 model (equation (19)) and the commensurate–incommensurate transition in the effective sine-Gordon theory at $\pm h^0_{\rm eff} = M$ gives two additional critical values of the magnetic field,

$$H^+_{c1} = J_\perp + J_\parallel/2 - J_\parallel\mathcal{C}_0\left(\delta J_\perp/J_\parallel\right)^{4/5} \tag{25}$$

and

$$H^-_{c2} = J_\perp + J_\parallel/2 + J_\parallel\mathcal{C}_0\left(\delta J_\perp/J_\parallel\right)^{4/5}, \tag{26}$$

where $\mathcal{C}_0 = \mathcal{C}(3/4) = 1.114\,28$.

As usual in the case of quantum commensurate–incommensurate transition transitions, the magnetic susceptibility of the system shows a square-root divergence at the transition points:

$$\chi(H) = \begin{cases} \left(H^+_{c1} - H\right)^{-1/2} & \text{for } H < H^+_{c1} \\ 0 & \text{for } H^+_{c1} < H < H^-_{c2} \\ \left(H - H^-_{c2}\right)^{-1/2} & \text{for } H > H^-_{c2}. \end{cases} \tag{27}$$

3.3. Magnetic phase diagram

Summarizing the results of the previous subsections, we obtain the following magnetic phase diagram for a ladder with alternating rung exchange (see figure 2). For $H < H_{c1}^-$, the system is in a rung-singlet phase with zero magnetization and vanishing magnetic susceptibility. For $H > H_{c1}^-$, some of the singlet rungs melt and the magnetization increase as $\left(H - H_{c1}^-\right)^{1/2}$. With a further increase in the magnetic field, the system gradually crosses to a regime with linearly increasing magnetization. However, in the vicinity of the magnetization plateau, for $H \leqslant H_{c1}^+$ this linear dependence changes and the magnetization once again shows a square-root behaviour $M - \frac{1}{2}M_{\rm sat} \sim -(H_{c1}^+ - H)^{1/2}$. For fields in the interval between H_{c1}^+ and H_{c2}^-, the magnetization is constant: $M = 0.5M_{\rm sat}$. At $H > H_{c2}^-$ the magnetization increases as $M \sim 0.5M_{\rm sat} + \left(H - H_{c2}^-\right)^{1/2}$, then passes again through a linear regime until, in the vicinity of the saturation field H_{c2}^+, it becomes $M \sim M_{\rm sat} - \left(H_{c2}^+ - H\right)^{1/2}$.

The width of the magnetization plateau at $M = 0.5M_{\rm sat}$ is given by

$$H_{c2}^- - H_{c1}^+ \simeq 2J_\parallel \left(\delta J_\perp / J_\parallel\right)^{4/5}. \qquad (28)$$

3.4. Ladder with ferromagnetic legs

The existence of a magnetization plateau at $M = 0.5M_{\rm sat}$ is not limited to the ladder with antiferromagnetic exchange, but is also found for a ladder with *isotropic ferromagnetic* legs ($J_\parallel = -|J_\parallel| < 0$) coupled by an antiferromagnetic rung exchange ($J_\perp > 0$). Despite the fact that, up to now, no ladder materials with ferromagnetic legs have been synthesized, from the theoretical point of view these systems are extremely interesting, since they open up a new large class of complicated quantum behaviour, unexpected in more conventional spin systems [17, 41–45].

In the case of a ladder with ferromagnetic legs, the effective spin-chain model is also given by the Hamiltonian (7), but with different parameters:

$$J_{xy} = |J_\parallel|, \qquad J_z = -\tfrac{1}{2}|J_\parallel|, \qquad (29)$$

$$h_{\rm eff}^0 = H - J_\perp + \tfrac{1}{2}|J_\parallel|, \qquad (30)$$

$$h_{\rm eff}^1 = \delta J_\perp. \qquad (31)$$

Thus, in this case the anisotropy parameter of the effective XXZ chain is $\Delta = -1/2$ and consequently the spin-stiffness parameter K is given by $K = 3/2$. The equivalent sine-Gordon theory (21) with a massive term $\sim h_{\rm eff}^1 \sin(\sqrt{6\pi}\phi)$ remains in the gapped strong-coupling regime. Using equation (22), we find an excitation gap $M \sim |J_\parallel| \left(\delta J_\perp / |J_\parallel|\right)^2$. Correspondingly, the width of the magnetization plateau in this case equals

$$H_{c2}^- - H_{c1}^+ \simeq 2J_\parallel \left(\delta J_\perp / J_\parallel\right)^2. \qquad (32)$$

4. Generalized ladder

In this section we consider a generalized ladder model with frustrating (diagonal) interleg interactions. The Hamiltonian is given by

$$\mathcal{H} = J_\parallel \sum_{n,\alpha} \mathbf{S}_{n,\alpha} \cdot \mathbf{S}_{n+1,\alpha} + J_\perp \sum_n \left[1 + (-1)^n \delta\right] \mathbf{S}_{n,1} \cdot \mathbf{S}_{n,2}$$
$$+ J_\perp' \sum_n \left(\mathbf{S}_{n,1} \cdot \mathbf{S}_{n+1,2} + \mathbf{S}_{n,2} \cdot \mathbf{S}_{n+1,1}\right) + H \sum_{n,\alpha} S_{n,\alpha}^z. \qquad (33)$$

Let us first consider the case of antiferromagnetic legs ($J_\parallel > 0$). Assuming $J_\perp \gg J_\parallel, |J'_\perp|, \delta J_\perp$, we easily obtain the parameters of $H_{\rm eff}$:

$$\Delta = \frac{1}{2}\frac{J_\parallel + J'_\perp}{J_\parallel - J'_\perp}, \tag{34}$$

$$h^0_{\rm eff} = H - J_\perp - \frac{J_\parallel}{2} - \frac{J'_\perp}{2}, \tag{35}$$

$$h^1_{\rm eff} = \delta J_\perp. \tag{36}$$

Below, in this section we will separately discuss several interesting limiting cases corresponding to the: (i) antiferromagnetic ladder ($J_\parallel, J'_\perp > 0$); (ii) ladder with ferromagnetic legs and ferromagnetic diagonal interleg exchange ($J_\parallel, J'_\perp < 0$); and (iii) ladder with competing intraleg and diagonal interleg exchange ($J_\parallel \cdot J'_\perp < 0$).

4.1. Antiferromagnetic ladder

In the case of an antiferromagnet ladder, an increase in the antiferromagnetic diagonal exchange leads to a reduction in the spin-stiffness parameter K from its value $K = 3/4$ at $J'_\perp = 0$ until $K = 1/2$ at $J'_\perp = 1/3J_\parallel$. In this parameter range, the effective spin chain anisotropy parameter $\Delta \leqslant 1$, and therefore at $h^1_{\rm eff} = 0$ (i.e. $\delta = 0$) the system is in the gapless Luttinger-liquid phase. In complete analogy with the cases discussed above, at $\delta \neq 0$ the system exhibits a magnetization plateau $\sim(\delta J_\perp/J_\parallel)^\nu$ and the plateau width scaling index ν changes from 4/5 at $J'_\perp = 0$ to 2/3 at $J'_\perp = 1/3J_\parallel$, where the width of the magnetization plateau, determined only by the rung-exchange alternation, reaches its maximum given by $\Delta H \equiv H^-_{\rm c2} - H^+_{\rm c1} \simeq 2J_\parallel \left(\delta J_\perp/J_\parallel\right)^{2/3}$.

For $J'_\perp > 1/3J_\parallel$, the effective spin-chain anisotropy parameter $\Delta > 1$, and therefore the system, is gapped at $h^0_{\rm eff} = 0$ already at $h^1_{\rm eff} = 0$ (i.e. $\delta = 0$). In marked contrast to the cases considered above, for $h^0_{\rm eff} = h^1_{\rm eff} = 0$ the system is no more in the XY universality class but in the Ising universality class and shares common features with the class of extended ladder models showing a magnetization plateau at $M = 0.5M_{\rm sat}$ and studied previously by several authors [3–5, 16]. Since in the present model for $J'_\perp > 1/3J_\parallel$ and $\delta \neq 0$ two sources of gap formation (Ising-type antiferromagnetic exchange and alternating magnetic field) are complementary to each other, the widths of the magnetization plateaux will further increase with increasing diagonal antiferromagnetic exchange. For $J'_\perp \leqslant J_\parallel$ (i.e. $\Delta \gg 1$), one reaches the ultimate Ising limit, where the width of the magnetization plateau $\Delta H \simeq 2\left(\Delta J_\parallel + \delta J_\perp\right)$ is given by the excitation gap of the Ising model in the presence of a staggered magnetic field.

4.2. Ferromagnetic ladder with ferromagnetic diagonal exchange

In the case of a ladder with ferromagnetic legs ($J_\parallel < 0$) and ferromagnetic diagonal exchange ($J'_\perp < 0$), the anisotropy parameter of the effective ferromagnetic spin-chain model is given by

$$\Delta = -\frac{1}{2}\frac{|J_\parallel| - J'_\perp}{|J_\parallel| + J'_\perp}. \tag{37}$$

Therefore, with increasing ferromagnetic diagonal exchange the spin-stiffness parameter K increases from its value $K = 3/2$ ($\Delta = -1/2$) at $J'_\perp = 0$ to $K = 2$ ($\Delta = -1/\sqrt{2}$) at $J'^c_\perp = -\frac{\sqrt{2}-1}{\sqrt{2}+1}|J_\parallel|$. With increasing K, the excitation gap (and therefore the widths of the magnetization plateaux) reduces and finally disappears at $K = 2$ ($J = J'^c_\perp$). For $-1 < J < J'^c_\perp$, the effective model shows properties of a Luttinger liquid with strong ferromagnetic correlations, which makes perturbations caused by a staggered magnetic field irrelevant [24]. Therefore, in this sector of the model parameters, the phase stays massless after the introduction of a small $h^1_{\rm eff} = 0$ and the magnetization plateau is absent.

4.3. Ladder with competing interleg and diagonal intraleg exchange

An intermediate situation, between the cases discussed above, is realized in the case of a ladder with competing interleg and diagonal intraleg interactions. In particular, for $J'_\perp = -J_\parallel$, $K = 1$ ($\Delta = 0$) and the effective continuum theory becomes the theory of free massive fermions. At $K = 1$ and $h^0_{\rm eff} = 0$, the excitation gap $M = \delta J_\perp$ and, respectively, the width of the magnetization plateaux also scale linearly in δ.

5. Conclusion

We have studied the ground-state phase diagram of a spin $S = 1/2$ two-leg ladder with alternating rung exchange $J^n_\perp = J_\perp[1 + (-1)^n\delta]$ in a magnetic field. We have shown that, in a wide parameter range, the magnetization curve exhibits a plateau at one half of its saturation value. The width of the plateau is proportional to the excitation gap in the system at $M = 0.5M_{\rm sat}$ and scales as δ^ν. The critical exponent has a value $\nu = 4/5$ in the case of a ladder with isotropic antiferromagnetic legs and $\nu = 2$ in the case of a ladder with isotropic ferromagnetic legs. We have also shown that, in a ladder with frustrating diagonal interleg exchange, the plateau effect is stronger and, for realistic values of diagonal exchange, the critical exponent reaches the value $\nu = 2/3$.

We have also shown that, in the case of a ladder with ferromagnetic legs and with strong diagonal ferromagnetic intraleg interaction, the magnetization plateau $M = 0.5M_{\rm sat}$ is absent.

We also predict the possibility of realizing the unconventional spin-Peierls transition in a ladder in the case of an applied magnetic field. This spin–lattice instability takes place at a magnetization equal to half of its saturation value and is characterized by a spontaneous doubling of the lattice unit via the spontaneous staggered modulation of the inter-rung distance. Assuming that, in the harmonic approximation, the lattice deformation energy per rung is given by $E_{\rm def} \sim \delta^2$ and, estimating the magnetic condensation energy associated with gap opening as $E_{\rm mag}(\delta) - E_{\rm mag}(0) \sim -\delta^{2\nu}$, we conclude that the spin-phonon instability is possible for an antiferromagnetic ladder at magnetization $M = 0.5M_{\rm sat}$. We also note that this spin-Peierls transition could be more manifestly pronounced in the case of an applied uniaxial (along the ladder) static pressure.

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References

[1] For a review see Dagotto E 1999 *Rep. Prog. Phys.* **62** 1525
Dagotto E and Rice T M 1996 *Science* **271** 618
[2] Chitra R and Giamarchi T 1997 *Phys. Rev.* B **55** 5816
[3] Cabra D C, Honecker A and Pujol P 1997 *Phys. Rev. Lett.* **79** 5126
Cabra D C, Honecker A and Pujol P 1998 *Phys. Rev.* B **58** 6241

[4] Mila F 1998 *Eur. Phys. J.* B **6** 201
[5] Totsuka K 1998 *Phys. Rev.* B **57** 3454
[6] Usami M and Suga S I 1998 *Phys. Rev.* B **58** 14401
[7] Giamarchi T and Tsvelik A M 1999 *Phys. Rev.* B **59** 11398
[8] Hagiwara M, Katori H A, Schollwöck U and Mikeska H-J 2000 *Phys. Rev.* B **62** 1051
[9] Wang X and Yu L 2000 *Phys. Rev. Lett.* **84** 5399
[10] Langari A, Abolfath M and Martin-Delgado M A 2000 *Phys. Rev.* B **61** 343
[11] Langari A and Martin-Delgado M A 2000 *Phys. Rev.* B **62** 11725
[12] Hikihara T and Furusaki A 2001 *Phys. Rev.* B **63** 134438
[13] Wessel S, Olshanii M and Haas S 2001 *Phys. Rev. Lett.* **87** 206407
[14] Wang Y-J, Essler F H L, Fabrizio M and Nersesyan A A 2002 *Phys. Rev.* B **66** 024412
[15] Wang Y-J 2003 *Phys. Rev.* B **68** 214428
[16] Hida K, Shino M and Chen W 2004 *J. Phys. Soc. Japan* **73** 1587
[17] Vekua T, Japaridze G I and Mikeska H-J 2004 *Phys. Rev.* B **70** 014425
[18] Chaboussant G, Crowell P A, Lèvy L P, Piovesana O, Madouri A and Mailly D 1997 *Phys. Rev.* B **55** 3046
[19] Chaboussant G, Julien M-H, Fagot-Revurat Y, Hanson M, Lèvy L P, Berthier C, Horvatic M and Piovesana O 1998 *Eur. Phys. J.* B **6** 167
[20] Chaboussant G, Fagot-Revurat Y, Julien M-H, Hanson M E, Berthier C, Horvatic M, Lèvy L P and Piovesana O 1998 *Phys. Rev. Lett.* **80** 2713
[21] Arcon D, Lappas A, Margadonna S, Prassides K, Ribera E, Veciana J, Rovira C, Henriques R T and Almeida M 1999 *Phys. Rev.* B **60** 4191
[22] Mayaffre H, Horvatic H, Berthier C, Julien M-H, Söransan P, Lèvy L P and Piovesana O 2000 *Phys. Rev. Lett.* **85** 4795
[23] Watson B C, Kotov V N, Meisel N W, Hall D W, Granroth G E, Montfrooij W T, Nagler S E, Jensen D A, Backov R, Petruska M A, Fanucci G E and Talham D R 2001 *Phys. Rev. Lett.* **86** 5168
[24] den Nijs M P M 1981 *Phys. Rev.* B **23** 6111
[25] Alcaraz F C and Malvezzi A L 1995 *J. Phys. A: Math. Gen.* **28** 1521
[26] Oshikawa M and Affleck I 1997 *Phys. Rev. Lett.* **79** 2883
Oshikawa M and Affleck I 1999 *Phys. Rev.* B **60** 1038
[27] Essler F H L and Tsvelik A M 1998 *Phys. Rev.* B **57** 10592
[28] Huang H and Affleck I 2004 *Phys. Rev.* B **69** 184414
[29] Fouet J-B, Tchernyshyov O and Mila F 2004 *Phys. Rev.* B **70** 174427
[30] Zvyagin S A, Kolezhuk A K, Krzystek J and Feyerherm R 2004 *Phys. Rev. Lett.* **95** 027207
[31] Dmitriev D V and Krivnov V Ya 2004 *JETP Lett.* **80** 303
[32] Lou J, Chen C, Zhao J, Wang X, Xiang T, Su Z and Yu L 2005 *Phys. Rev. Lett.* **94** 217207
[33] Zhao J, Wang X, Xiang T, Su Z, Yu L, Lou J and Chen C 2006 *Phys. Rev.* B **73** 012411
[34] Luther A and Peschel I 1975 *Phys. Rev.* B **12** 3908
[35] Gogolin A O, Nersesyan A A and Tsvelik A M 1998 *Bosonization and Strongly Correlated Systems* (Cambridge: Cambridge University Press)
[36] Hikihara T and Furusaki A 1998 *Phys. Rev.* B **58** R583
[37] Japaridze G I and Nersesyan A A 1978 *JETF Pis.* **27** 356
Japaridze G I and Nersesyan A A 1978 *JETP Lett.* **27** 334 (Engl. Transl.)
Japaridze G I and Nersesyan A A 1979 *J. Low Temp. Phys.* **37** 95
Pokrovsky V L and Talapov A L 1979 *Phys. Rev. Lett.* **42** 65
Schulz H J 1980 *Phys. Rev.* B **22** 5274
[38] Japaridze G I, Nersesyan A A and Wiegmann P B 1984 *Nucl. Phys.* B **230** 511
[39] Dashen R F, Hasslacher B and Neveu A 1975 *Phys. Rev.* D **10** 3424
Takhtadjan A and Faddeev L D 1975 *Sov. Theor. Math. Phys.* **25** 147
[40] Zamolodchikov Al B 1995 *Int. J. Mod. Phys.* A **10** 1125
[41] Schulz H J 1989 *Phys. Rev.* B **34** 6372
[42] Kolezhuk A K and Mikeska H-J 1998 *Int. J. Mod. Phys.* **12** 2325
[43] Wiessner R M, Fledderjohann A, Mütter K-H and Karbach M 1999 *Phys. Rev.* B **60** 6545
[44] Vekua T, Japaridze G I and Mikeska H-J 2003 *Phys. Rev.* B **67** 064419
[45] Bibikov P N 2006 *Phys. Rev.* B **73** 132402