Internat. J. Math. & Math. Sci.
Vol. 1 (1978) 93-96

A NOTE ON RIESZ ELEMENTS IN C*-ALGEBRAS

DAVID LEGG

Department of Mathematics
Indiana University-Purdue University
Fort Wayne, Indiana

(Received December 5, 1977)

ABSTRACT. It is known that every Riesz operator R on a Hilbert space can be written R = Q + C, where C is compact and both Q and CQ - QC are quasinilpotent. This result is extended to a general C*-algebra setting.

1. INTRODUCTION.

In [3], Smyth develops a Riesz theory for elements in a Banach algebra with respect to an ideal of algebraic elements. In [1], Chui, Smith and Ward show that every Riesz operator on a Hilbert space is decomposible into R = Q + C, where C is compact and both Q and CQ - QC are quasinilpotent. In this paper we use Smyth's work to show that the analogous result holds in an arbitrary C*-algebra.

2. DEFINITIONS AND NOTATION.

Let A be a C*-algebra, and let F be a two-sided ideal of algebraic elements of A. An element $T \varepsilon A$ is a Riesz element if its coset $T + \overline{F}$ in $A/\overline{F}$ has spectral radius 0. A point $\lambda \varepsilon \sigma(T)$ is a finite pole of T if it is isolated in $\sigma(T)$ and the corresponding spectral projection lies in F. Let $E\sigma(T) = \{\lambda \varepsilon \sigma(T): \lambda$ is not a finite pole of T$\}$. Smyth has shown that T is a Riesz element if and only if $E\sigma(T) \subseteq \{0\}$, [3, Thm. 5.3]. Smyth also showed that if T is a Riesz element, then $T = Q + U$, where Q is quasinilpotent and $U \varepsilon \overline{F}$. [3, Thm. 6.9]. This is a generalization of West's result [4, Thm. 7.5]. We now extend the result of Chui, Smith and Ward [1, Thm. 1] by showing that $UQ - QU$ is quasinilpotent, where $T = Q + U$ is the Smyth decomposition.

3. OUTLINE OF SMYTH'S CONSTRUCTION.

Let T be a Riesz element, and label the elements of $\sigma(T) \setminus E\sigma(T)$ by λ_n, $n = 1, 2, \ldots$, in such a way that $|\lambda_n| \geq |\lambda_{n+1}|$, $\lambda_n \to 0$ as $n \to \infty$. Each λ_n is a finite pole, so each spectral projection P_n is in F. Let $S_n = P_1 + \ldots + P_n$, then find a self-adjoint projection Q_n satisfying $S_n Q_n = Q_n$ and $Q_n S_n = S_n$. Let $V_n = Q_n - Q_{n-1}$, and define $U = \sum \lambda_k V_k$. U is clearly in $\overline{F}$ and $Q = T - U$ is shown to be quasinilpotent.

4. THEOREM 1 $UQ - QU$ is quasinilpotent.

PROOF. For any $S \varepsilon A$, let $\tilde{S}$ denote the left regular representation of S. Then by Lemma 6.6 in Smyth [3], we have that $Q_n A$ is an invariant subspace of $\tilde{Q}$. Since $Q_n = Q_n Q_n$, we have $Q_n \varepsilon Q_n A$. Hence $\tilde{Q}(Q_n) \varepsilon Q_n A$, say $\tilde{Q}(Q_n) = Q_n S$ for some $S \varepsilon A$. That is, $QQ_n = Q_n S$. Now let $v \varepsilon$ range Q_n, say $v = Q_n x$. Then $Qv = QQ_n x = Q_n Sx$ belongs to range Q_n. Hence we see that range Q_n is an invariant subspace of Q. It follows that Q has an operator matrix representation of the form

$$Q = \begin{array}{cccc|c} A_{11} & A_{12} & A_{13} & --- & \\ 0 & A_{22} & A_{23} & --- & * \\ 0 & 0 & A_{33} & --- & \\ \vdots & \vdots & \vdots & \ddots & \\ \hline & 0 & & & * \end{array}$$

where $A_{ij} = V_i Q V_j$. With respect to this blocking, we have

$$U = \begin{array}{cccc|c} \lambda_1 I_1 & 0 & 0 & --- & \\ 0 & \lambda_2 I_2 & 0 & --- & \\ 0 & 0 & \lambda_3 I_3 & --- & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ \hline & 0 & & & 0 \end{array}$$

Hence

$$UQ - QU = \begin{array}{cccc|c} 0 & (\lambda_1-\lambda_2)A_{12} & --- & (\lambda_1-\lambda_n)A_{1n} \quad --- & \\ 0 & 0 & & & * \\ \vdots & \vdots & \ddots & (\lambda_{n-1}-\lambda_n)A_{n-1,n} \quad --- & \\ \vdots & \vdots & & 0 & \\ \hline & & 0 & & 0 \end{array}$$

Now let P be the orthogonal projection onto $\bigcup_n$ range Q_n, and let $A_n = (P - Q_n)(UQ - QU)(P - Q_n)$. It is easy to see that $||A_n|| \leq |\lambda_n| \, ||Q - \text{diag. } Q|| \to 0$ as $n \to \infty$. Hence $UQ - QU - A_n$ converges in the uniform norm to $UQ - QU$ as $n \to \infty$. But $UQ - QU - A_n$ has the form

$$\begin{array}{c|c|c} N & * & * \\ \hline 0 & 0 & * \\ \hline 0 & 0 & 0 \end{array}$$

where N is nilpotent. It follows that $UQ - QU - A_n$ has no non-zero eigenvalues. Thm. 3.1, p. 14 of [2] can now be easily modified to show that $UQ - QU$ has no non-zero eigenvalues. Since $UQ - QU$ belongs to $\overline{F}$, this means $\sigma(UQ - QU) \subseteq \{0\}$, i.e., $UQ - QU$ is quasinilpotent.

REFERENCES

1. Chui, C. K., Smith, P. W., and Ward, J. D., A note on Riesz operators, Proc. Amer. Math. Soc., 60, (1976), 92-94.

2. Gohberg, I. C., and Krein, M. G., Introduction to the theory of linear non-selfadjoint operators, "Nauka," Moscow, 1965; English transl., Transl. Math Monographs, vol. 18, Amer. Math. Soc., Providence, R. I. 1969.

3. Smyth, M. R. F., Riesz theory in Banach algebras, Math Z., 145, (1975), 145-155.

4. West, T. T., The decomposition of Riesz operators, Proc. London Math. Soc., III, Ser. 16, (1966), 737-752.

AMS (MOS) Subject Classification numbers 47 B 05, 47 C 10

KEY WORDS AND PHRASES. *C^* algebra, quasinilpotent operators, Riesz elements.*


Advances in
Operations Research
Advances in
Decision Sciences
Journal of
Applied Mathematics
Algebra
Journal of
Probability and Statistics
The Scientific
World Journal
International Journal of
Differential Equations
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of
Combinatorics
Advances in
Mathematical Physics
Journal of
Complex Analysis
Journal of
Mathematics
Mathematical Problems
in Engineering
Abstract and
Applied Analysis
Discrete Dynamics in
Nature and Society
International
Journal of
Mathematics and
Mathematical
Sciences
Journal of
Discrete Mathematics
Journal of
Function Spaces
International Journal of
Stochastic Analysis
Journal of
Optimization