Research Article

Existence and Uniqueness Results for a Coupled System of Nonlinear Fractional Differential Equations with Antiperiodic Boundary Conditions

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This paper studies the existence and uniqueness of solutions for a coupled system of nonlinear fractional differential equations of order $\alpha, \beta \in (4, 5]$ with antiperiodic boundary conditions. Our results are based on the nonlinear alternative of Leray-Schauder type and the contraction mapping principle. Two illustrative examples are also presented.

1. Introduction

In this paper, we consider the existence and uniqueness of solutions for the following coupled system of nonlinear fractional differential equations:

$$^{c}D^{\alpha}x(t) + f(t, x(t), y(t)) = 0, \quad t \in [0, T],$$

$$^{c}D^{\beta}y(t) + g(t, x(t), y(t)) = 0, \quad t \in [0, T],$$

$$x^{(i)}(0) = -x^{(i)}(T), \quad i = 0, 1, 2, 3, 4,$$

$$y^{(i)}(0) = -y^{(i)}(T), \quad i = 0, 1, 2, 3, 4,$$

(1)

where $4 < \alpha, \beta \leq 5$, and $^{c}D^{\alpha}$ denotes the Caputo fractional derivative of order α. Here our nonlinearity $f, g : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

Fractional differential equations have recently been addressed by many researchers in various fields of science and engineering, such as rheology, porous media, fluid flows, chemical physics, and many other branches of science; see [1–4]. As a matter of fact, fractional-order models become more realistic and practical than the classical integer-order models; as a consequence, there are a large number of papers

and books dealing with the existence and uniqueness of solutions to nonlinear fractional differential equations; see [5–14]. The study of a coupled system of fractional order is also very significant because this kind of system can often occur in applications; see [15–17].

Antiperiodic boundary value problems arise in the mathematical modeling of a variety of physical process; many authors have paid much attention to such problems; for examples and details of Antiperiodic boundary conditions, see [5, 18–22]. In [5], Alsaedi et al. study an Antiperiodic boundary value problem of nonlinear fractional differential equations of order $q \in (4, 5]$.

It should be noted that, in [23], Ntouyas and Obaid have researched a coupled system of fractional differential equations with nonlocal integral boundary conditions, but this paper researches a coupled system of fractional differential equations with Antiperiodic boundary conditions. On the other hand, in [5, 19], the authors have discussed some existence results of solutions for Antiperiodic boundary value problems of fractional differential equation but not the coupled system. The rest of the papers above for the coupled systems have been devoted to the case of

Riemann-Liouville fractional derivatives but not the Caputo fractional derivatives.

This paper is organized as follows. In Section 2, we recall some basic definitions and preliminary results. In Section 3, we give the existence results of (1) by means of the Leray-Schauder alternative; then we obtain the uniqueness of solutions for system (1) by the contraction mapping principle. We give two examples in Section 4 to illustrate the applicability of our results.

2. Background Materials

For the convenience of the readers, we present here some necessary definitions and lemmas which are used throughout this paper.

Definition 1 (see [2, 3]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0, \infty) \to \mathbb{R}$ is given by

$$I^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) \, ds, \tag{2}$$

provided the right hand side is pointwise defined on $(0, \infty)$.

Definition 2 (see [2, 3]). The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $y : (0, \infty) \to \mathbb{R}$ is given by

$$^cD^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{y^{(n)}(s)}{(t-s)^{\alpha-n+1}} \, ds, \tag{3}$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of number α, provided that the right side is pointwise defined on $(0, \infty)$.

Lemma 3 (see [19]). *Consider* $I^{\alpha \, c}D^\alpha x(t) = x(t) + C_0 + C_1 t + C_2 t^2 + \cdots + C_{n-1}t^{n-1}$, *for some* $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, *where* $n = [\alpha] + 1$ *and* $[\alpha]$ *denotes the integer part of number* α.

Lemma 4 (see [5]). *For any* $y \in C[0, T]$, *the unique solution of the boundary value problem*

$$^cD^q x(t) = y(t), \quad t \in [0, T], \ 4 < q \le 5,$$
$$x^{(i)}(0) = -x^{(i)}(T), \quad i = 0, 1, 2, 3, 4 \tag{4}$$

is

$$x(t) = \int_0^T G(t, s) \, y(s) \, ds, \tag{5}$$

where $G(t, s)$ is the Green function given by

$G(t, s)$

$$= \begin{cases} \dfrac{2(t-s)^{q-1} - (T-s)^{q-1}}{2\Gamma(q)} + \dfrac{(T-2t)(T-s)^{q-2}}{4\Gamma(q-1)} \\ + \dfrac{t(T-t)(T-s)^{q-3}}{4\Gamma(q-2)} + \dfrac{(6Tt^2 - 4t^3 - T^3)(T-s)^{q-4}}{48\Gamma(q-3)} \\ + \dfrac{(2Tt^3 - tT^3 - t^4)(T-s)^{q-5}}{48\Gamma(q-4)}, \quad 0 < s < t < T, \\ -\dfrac{(T-s)^{q-1}}{2\Gamma(q)} + \dfrac{(T-2t)(T-s)^{q-2}}{4\Gamma(q-1)} \\ + \dfrac{t(T-t)(T-s)^{q-3}}{4\Gamma(q-2)} + \dfrac{(6Tt^2 - 4t^3 - T^3)(T-s)^{q-4}}{48\Gamma(q-3)} \\ + \dfrac{(2Tt^3 - tT^3 - t^4)(T-s)^{q-5}}{48\Gamma(q-4)}, \quad 0 < t < s < T. \end{cases} \tag{6}$$

Let

$G_1(t, s)$

$$= \begin{cases} \dfrac{(t-s)^{\alpha-1} - (1/2)(T-s)^{\alpha-1}}{\Gamma(\alpha)} + \dfrac{(T-2t)(T-s)^{\alpha-2}}{4\Gamma(\alpha-1)} \\ + \dfrac{t(T-t)(T-s)^{\alpha-3}}{4\Gamma(\alpha-2)} + \dfrac{(6Tt^2 - 4t^3 - T^3)(T-s)^{\alpha-4}}{48\Gamma(\alpha-3)} \\ + \dfrac{(2Tt^3 - tT^3 - t^4)(T-s)^{\alpha-5}}{48\Gamma(\alpha-4)}, \quad 0 < s < t < T, \\ -\dfrac{1}{2\Gamma(\alpha)}(T-s)^{\alpha-1} + \dfrac{(T-2t)(T-s)^{\alpha-2}}{4\Gamma(\alpha-1)} \\ + \dfrac{t(T-t)(T-s)^{\alpha-3}}{4\Gamma(\alpha-2)} + \dfrac{(6Tt^2 - 4t^3 - T^3)(T-s)^{\alpha-4}}{48\Gamma(\alpha-3)} \\ + \dfrac{(2Tt^3 - tT^3 - t^4)(T-s)^{\alpha-5}}{48\Gamma(\alpha-4)}, \quad 0 < t < s < T, \end{cases}$$

$G_2(t, s)$

$$= \begin{cases} \dfrac{(t-s)^{\beta-1} - (1/2)(T-s)^{\beta-1}}{\Gamma(\beta)} + \dfrac{(T-2t)(T-s)^{\beta-2}}{4\Gamma(\beta-1)} \\ + \dfrac{t(T-t)(T-s)^{\beta-3}}{4\Gamma(\beta-2)} + \dfrac{(6Tt^2 - 4t^3 - T^3)(T-s)^{\beta-4}}{48\Gamma(\beta-3)} \\ + \dfrac{(2Tt^3 - tT^3 - t^4)(T-s)^{\beta-5}}{48\Gamma(\beta-4)}, \quad 0 < s < t < T, \\ -\dfrac{1}{2\Gamma(\beta)}(T-s)^{\beta-1} + \dfrac{(T-2t)(T-s)^{\beta-2}}{4\Gamma(\beta-1)} \\ + \dfrac{t(T-t)(T-s)^{\beta-3}}{4\Gamma(\beta-2)} + \dfrac{(6Tt^2 - 4t^3 - T^3)(T-s)^{\beta-4}}{48\Gamma(\beta-3)} \\ + \dfrac{(2Tt^3 - tT^3 - t^4)(T-s)^{\beta-5}}{48\Gamma(\beta-4)}, \quad 0 < t < s < T. \end{cases} \tag{7}$$

We call (G_1, G_2) Green's function for problem (1).

We define the space $X = \{x(t) \mid x(t) \in C[0, T]\}$ endowed with $\|x\|_X = \max_{t \in [0, T]} |x(t)|$; for $(x, y) \in X \times X$,

let $\|(x, y)\|_{X \times X} = \|x\|_X + \|y\|_X$. Obviously, $(X, \|\cdot\|_X)$ is a Banach space, and the product space $(X \times X, \|(\cdot, \cdot)\|_{X \times X})$ is also a Banach space.

Consider the following coupled system of the integral equations:

$$x(t) = \int_0^T G_1(t, s) f(s, x(s), y(s)) \, ds,$$

$$y(t) = \int_0^T G_2(t, s) g(s, x(s), y(s)) \, ds. \tag{8}$$

As a result, differential problem (1) turns into integral problem (8), and here is a conclusion about the relationship between their solutions.

Lemma 5. *Assume that* $f, g : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *are continuous functions. Then* $(x, y) \in (X, X)$ *is a solution of* (1) *if and only if* $(x, y) \in (X, X)$ *is a solution of system* (8).

Proof. The proof is immediate from the discussion above, and we omit the details here. $\square$

Let $F : X \times X \to X \times X$ be an operator defined as $F(x, y)(t) = (F_1(x, y)(t), F_2(x, y)(t))$, where

$$F_1(x, y)(t) = \int_0^T G_1(t, s) f(s, x(s), y(s)) \, ds,$$

$$F_2(x, y)(t) = \int_0^T G_2(t, s) g(s, x(s), y(s)) \, ds. \tag{9}$$

It is obvious that a fixed point of the operator F is a solution of problem (1).

3. Main Results

In this section, we will discuss the existence and uniqueness of solutions for problem (1).

Lemma 6. *One can conclude that the Green functions* $G_1(t, s)$, $G_2(t, s)$ *satisfy the following estimates:*

$$\int_0^T |G_1(t, s)| \, ds$$

$$\leq \frac{T^\alpha}{\Gamma(\alpha + 1)} \left(\frac{3}{2} + \frac{5\alpha^4 - 14\alpha^3 + 55\alpha^2 + 146\alpha}{768} \right) = U_1,$$

$$t \in [0, T], \tag{10}$$

$$\int_0^T |G_2(t, s)| \, ds$$

$$\leq \frac{T^\beta}{\Gamma(\beta + 1)} \left(\frac{3}{2} + \frac{5\beta^4 - 14\beta^3 + 55\beta^2 + 146\beta}{768} \right) = U_2,$$

$$t \in [0, T], \tag{11}$$

$$\int_0^T \left| \frac{\partial G_1(t, s)}{\partial t} \right| \, ds$$

$$\leq \frac{T^{\alpha-1}}{\Gamma(\alpha)} \left(\frac{3}{2} + \frac{\alpha^3 - 3\alpha^2 + 14\alpha - 12}{48} \right) = U_3, \tag{12}$$

$$t \in [0, T],$$

$$\int_0^T \left| \frac{\partial G_2(t, s)}{\partial t} \right| \, ds$$

$$\leq \frac{T^{\beta-1}}{\Gamma(\beta)} \left(\frac{3}{2} + \frac{\beta^3 - 3\beta^2 + 14\beta - 12}{48} \right) = U_4, \tag{13}$$

$$t \in [0, T].$$

Proof. For any $t \in [0, T]$,

$$\int_0^T |G_1(t, s)| \, ds$$

$$\leq \left| \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} \, ds \right| + \frac{1}{2} \left| \int_0^T \frac{(T - s)^{\alpha-1}}{\Gamma(\alpha)} \, ds \right|$$

$$+ \frac{|T - 2t|}{4} \int_0^T \frac{(T - s)^{\alpha-2}}{\Gamma(\alpha - 1)} \, ds$$

$$+ \frac{|t(T - t)|}{4} \int_0^T \frac{(T - s)^{\alpha-3}}{\Gamma(\alpha - 2)} \, ds$$

$$+ \frac{|6Tt^2 - 4t^3 - T^3|}{48} \int_0^T \frac{(T - s)^{\alpha-4}}{\Gamma(\alpha - 3)} \, ds \tag{14}$$

$$+ \frac{|2Tt^3 - tT^3 - t^4|}{48} \int_0^T \frac{(T - s)^{\alpha-5}}{\Gamma(\alpha - 4)} \, ds$$

$$\leq \frac{T^\alpha}{\Gamma(\alpha + 1)} + \frac{T^\alpha}{2\Gamma(\alpha + 1)} + \frac{T^\alpha}{4\Gamma(\alpha)}$$

$$+ \frac{T^\alpha}{16\Gamma(\alpha - 1)} + \frac{T^\alpha}{48\Gamma(\alpha - 2)} + \frac{5T^\alpha}{768\Gamma(\alpha - 3)}$$

$$= \frac{T^\alpha}{\Gamma(\alpha + 1)} \left(\frac{3}{2} + \frac{5\alpha^4 - 14\alpha^3 + 55\alpha^2 + 146\alpha}{768} \right).$$

On the other hand,

$$\int_0^T \left| \frac{\partial G_1(t, s)}{\partial t} \right| \, ds$$

$$= \int_0^t \frac{(t - s)^{\alpha-2}}{\Gamma(\alpha - 1)} \, ds$$

$$+ \int_0^T \left| -\frac{(T - s)^{\alpha-2}}{2\Gamma(\alpha - 1)} + \frac{(T - 2t)(T - s)^{\alpha-3}}{4\Gamma(\alpha - 2)} \right.$$

$$+ \frac{(Tt - t^2)(T - s)^{\alpha-4}}{4\Gamma(\alpha - 3)}$$

$$\left. + \frac{(6Tt^2 - T^3 - 4t^3)(T - s)^{\alpha-5}}{48\Gamma(\alpha - 4)} \right| \, ds$$

$$\leq \frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^{\alpha-1}}{2\Gamma(\alpha)} + \frac{|T-2t|\,T^{\alpha-2}}{4\Gamma(\alpha-1)}$$

$$+ \frac{\left|Tt-t^2\right|T^{\alpha-3}}{4\Gamma(\alpha-2)} + \frac{\left|6Tt^2-T^3-4t^3\right|T^{\alpha-4}}{48\Gamma(\alpha-3)}$$

$$\leq \frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^{\alpha-1}}{2\Gamma(\alpha)} + \frac{T^{\alpha-1}}{4\Gamma(\alpha-1)}$$

$$+ \frac{T^{\alpha-1}}{16\Gamma(\alpha-2)} + \frac{T^{\alpha-1}}{48\Gamma(\alpha-3)}$$

$$\leq \frac{T^{\alpha-1}}{\Gamma(\alpha)}\left(\frac{3}{2} + \frac{\alpha^3-3\alpha^2+14\alpha-12}{48}\right). \tag{15}$$

Inequalities (11) and (13) can be proved in the same way. $\square$

The first result is based on Leray-Schauder alternative.

Lemma 7 (see [24]). *Let $F : E \to E$ be a completely continuous operator (i.e., a map that is restricted to any bounded set in E is compact). Let*

$$\varepsilon(F) = \{x \in E : x = \lambda F(x) \text{ for some } 0 < \lambda < 1\}. \tag{16}$$

Then either the set $\varepsilon(F)$ is unbounded or F has at least one fixed point.

Theorem 8. *Let f and g satisfy the following growth condition:*

$$|f(t,x,y)| \leq m_0 + m_1\,|x| + m_2\,|y|,$$
$$m_i \geq 0 \quad (i=1,2), \quad m_0 > 0,$$
$$|g(t,x,y)| \leq n_0 + n_1\,|x| + n_2\,|y|, \tag{17}$$
$$n_i \geq 0 \quad (i=1,2), \quad n_0 > 0.$$

In addition, it is assumed that

$$U_1 m_1 + U_2 n_1 < 1, \qquad U_1 m_2 + U_2 n_2 < 1. \tag{18}$$

Then problem (1) has at least one solution.

For sake of convenience, we set $U_0 = \min\{1 - (U_1 m_1 + U_2 n_1), 1 - (U_1 m_2 + U_2 n_2)\}$.

Proof. First, we show that operator $F : X \times X \to X \times X$ is completely continuous.

Step 1. In view of the continuity of f, g, $x(t)$, $y(t)$ and $G_1(t,s)$, $G_2(t,s)$, it is obvious that the operator F is continuous.

Step 2. Let $\Omega \subset X \times X$ be bounded; then there exist positive constants K_1 and K_2 such that

$$|f(t,x(t),y(t))| \leq K_1, \qquad |g(t,x(t),y(t))| \leq K_2,$$
$$\forall (x,y) \in \Omega, \ t \in [0,T]. \tag{19}$$

Then for any $(x,y) \in \Omega$, according to the inequalities (10) and (11), we have

$$|F_1(x,y)(t)| = \left|\int_0^T G_1(t,s)\,f(s,x(s),y(s))\,ds\right| \leq K_1 U_1. \tag{20}$$

Similarly, we get

$$|F_2(x,y)(t)| = \left|\int_0^T G_2(t,s)\,g(s,x(s),y(s))\,ds\right| \leq K_2 U_2. \tag{21}$$

Thus, it follows from the above inequalities that the operator F is uniformly bounded.

Step 3. We show that F is equicontinuous:

$$\left|(F_1(x,y))'_t(t)\right| = \left|\int_0^T \frac{\partial G_1(t,s)}{\partial t} f(s,x(s),y(s))\,ds\right|$$
$$\leq K_1 \int_0^T \left|\frac{\partial G_1(t,s)}{\partial t}\right| ds \leq K_1 U_3. \tag{22}$$

Hence, for $0 \leq t_1 \leq t_2 \leq T$, by the inequalities (12) and (13), we have

$$|F_1(x,y)(t_2) - F_1(x,y)(t_1)|$$
$$\leq \int_{t_1}^{t_2} \left|(F_1(x,y))'_s(s)\right| ds \leq K_1 U_3\,|t_2-t_1|. \tag{23}$$

Analogously, we can obtain

$$|F_2(x,y)(t_2) - F_2(x,y)(t_1)| \leq \int_{t_1}^{t_2} \left|(F_2(x,y))'_s(s)\right| ds$$
$$\leq K_2 U_4\,|t_2-t_1|. \tag{24}$$

Therefore, the operator $F(x,y)$ is equicontinuous, and thus the operator $F(x,y)$ is completely continuous.

Finally, it will be verified that the set $\varepsilon = \{(x,y) \in X \times X : (x,y) = \lambda F(x,y),\ 0 \leq \lambda \leq 1\}$ is bounded.

Let $(x,y) \in \varepsilon$, then $(x,y) = \lambda F(x,y)$. For any $t \in [0,T]$, we have $x(t) = \lambda F_1(x,y)(t)$, $y(t) = \lambda F_2(x,y)(t)$; then,

$$|x(t)| \leq |F_1(x,y)(t)| \leq (m_0 + m_1\,|x(t)| + m_2\,|y(t)|)\,U_1,$$
$$|y(t)| \leq (n_0 + n_1\,|x(t)| + n_2\,|y(t)|)\,U_2. \tag{25}$$

Hence, we have

$$\|x\| \leq (m_0 + m_1\,\|x\| + m_2\,\|y\|)\,U_1,$$
$$\|y\| \leq (n_0 + n_1\,\|x\| + n_2\,\|y\|)\,U_2, \tag{26}$$

which imply that

$$\|x\| + \|y\| \leq (U_1 m_0 + U_2 n_0) + (U_1 m_1 + U_2 n_1)\,\|x\|$$
$$+ (U_1 m_2 + U_2 n_2)\,\|y\|. \tag{27}$$

As a result,

$$\|(x, y)\| \leq \frac{U_1 m_0 + U_2 n_0}{U_0}, \tag{28}$$

for any $t \in [0, T]$, which proves that ε is bounded; thus by Lemma 7, the operator F has at least one fixed point. Hence the boundary value problem (1) has at least one solution; this completes the proof. □

In the second result, we prove uniqueness of solutions of the boundary value problem (1) via contraction mapping principle.

Theorem 9. *Let f and g satisfy the following growth conditions:*

(H_1) there exist two constants $L_i > 0$ and $\lambda_i > 0$, $i = 1, 2$ such that

$$\left| f(t, x_1, y_1) - f(t, x_2, y_2) \right| \leq L_1 \left| x_1 - x_2 \right| + L_2 \left| y_1 - y_2 \right|,$$

$$\left| g(t, x_1, y_1) - g(t, x_2, y_2) \right| \leq \lambda_1 \left| x_1 - x_2 \right| + \lambda_2 \left| y_1 - y_2 \right|,$$

$$t \in [0, T], \ x_1, x_2, y_1, y_2 \in \mathbb{R}; \tag{29}$$

(H_2) Consider

$$\max \left\{ L_1 U_1 + \lambda_1 U_2, L_2 U_1 + \lambda_2 U_2 \right\} = L < 1. \tag{30}$$

Then problem (1) has a unique solution.

Proof. Let $(x_1, y_1), (x_2, y_2) \in X \times X$; then

$$\left| F_1(x_2, y_2)(t) - F_1(x_1, y_1)(t) \right|$$

$$= \left| \int_0^T G_1(t, s) f(s, x_2(s), y_2(s)) \, ds \right.$$

$$\left. - \int_0^T G_1(t, s) f(s, x_1(s), y_1(s)) \, ds \right| \tag{31}$$

$$\leq \int_0^T \left| G_1(t, s) \right| \left| f(s, x_2(s), y_2(s)) \right.$$

$$\left. - f(s, x_1(s), y_1(s)) \right| \, ds$$

$$\leq U_1 \left(L_1 \left\| x_2 - x_1 \right\| + L_2 \left\| y_2 - y_1 \right\| \right).$$

Analogously,

$$\left| F_2(x_2, y_2)(t) - F_2(x_1, y_1)(t) \right|$$

$$\leq U_2 \left(\lambda_1 \left\| x_2 - x_1 \right\| + \lambda_2 \left\| y_2 - y_1 \right\| \right). \tag{32}$$

Thus,

$$\left\| F(x_2, y_2) - F(x_1, y_1) \right\|$$

$$= \left\| F_1(x_2, y_2) - F_1(x_1, y_1) \right\|$$

$$+ \left\| F_2(x_2, y_2) - F_2(x_1, y_1) \right\|$$

$$\leq \left(L_1 U_1 + \lambda_1 U_2 \right) \left\| x_2 - x_1 \right\|$$

$$+ \left(L_2 U_1 + \lambda_2 U_2 \right) \left\| y_2 - y_1 \right\|$$

$$\leq L \left(\left\| x_2 - x_1 \right\| + \left\| y_2 - y_1 \right\| \right)$$

$$= L \left\| (x_2, y_2) - (x_1, y_1) \right\|. \tag{33}$$

Hence, we conclude that problem (1) has a unique solution by (H_2) and the contraction mapping principle; this ends the proof. □

4. Examples

In this section, two examples are given in order to verify the validity of Theorems 8 and 9.

Example 1. Consider the system

$$^c D^{17/4} x(t) + \frac{1}{2} \left(t + x(t) + y(t) \right) = 0, \quad 0 < t < 1,$$

$$^c D^{9/2} y(t) + \frac{3}{2} \left(\sqrt{t} + x(t) + y(t) \right) = 0, \quad 0 < t < 1,$$

$$x^{(i)}(0) = -x^{(i)}(1), \quad i = 0, 1, 2, 3, 4,$$

$$y^{(i)}(0) = -y^{(i)}(1), \quad i = 0, 1, 2, 3, 4, \tag{34}$$

where $T = 1, |f(t, x, y)| \leq 1 + |x| + |y|, |g(t, x, y)| \leq 2 + 2|x| + 2|y|, \alpha = 17/4, \beta = 9/2, m_0 = m_1 = m_2 = 1, n_0 = n_1 = n_2 = 2$. Consider

$$U_1 = \frac{T^\alpha}{\Gamma(\alpha+1)} \left(\frac{3}{2} + \frac{5\alpha^4 - 14\alpha^3 + 55\alpha^2 + 146\alpha}{768} \right) \approx 0.1229,$$

$$U_2 = \frac{T^\beta}{\Gamma(\beta+1)} \left(\frac{3}{2} + \frac{5\beta^4 - 14\beta^3 + 55\beta^2 + 146\beta}{768} \right) \approx 0.0920. \tag{35}$$

Then $U_1 m_1 + U_2 n_1 < 1$ and $U_1 m_2 + U_2 n_2 < 1$; by Theorem 8, the existence of the solution for the system (34) is obvious.

Example 2. Consider the system

$$^c D^{17/4} x(t) + t + \sin x(t) + 2 \frac{|y(t)|}{1 + |y(t)|} = 0, \quad 0 < t < 1,$$

$$^c D^{9/2} y(t) + t^2 + 2 \frac{|x(t)|}{1 + |x(t)|} + \arctan y(t) = 0, \quad 0 < t < 1,$$

$$x^{(i)}(0) = -x^{(i)}(1), \quad i = 0, 1, 2, 3, 4,$$

$$y^{(i)}(0) = -y^{(i)}(1), \quad i = 0, 1, 2, 3, 4, \tag{36}$$

where $T = 1$, $f(t, x, y) = t + \sin x(t) + 2(|y(t)|/(1 + |y(t)|))$, $g(t, x, y) = t^2 + 2(|x(t)|/(1 + |x(t)|)) + \arctan y(t)$. $\alpha = 17/4$, $\beta = 9/2$.

Noting that

$$\left|(\sin x)'\right| = |\cos x| \le 1, \qquad \left|(\arctan y)'\right| = \frac{1}{1+y^2} \le 1, \tag{37}$$

we have

$$\left|f\left(t, x_2, y_2\right) - f\left(t, x_1, y_1\right)\right|$$

$$= \left|\sin x_2(t) - \sin x_1(t)\right| + 2\left|\frac{\left|y_2(t)\right|}{1 + \left|y_2(t)\right|} - \frac{\left|y_1(t)\right|}{1 + \left|y_1(t)\right|}\right|$$

$$\le \left\|x_2 - x_1\right\| + 2\left\|y_2 - y_1\right\|,$$

$$\left|g\left(t, x_2, y_2\right) - g\left(t, x_1, y_1\right)\right| \le 2\left\|x_2 - x_1\right\| + \left\|y_2 - y_1\right\|,$$

$$U_1 = \frac{T^\alpha}{\Gamma(\alpha+1)}\left(\frac{3}{2} + \frac{5\alpha^4 - 14\alpha^3 + 55\alpha^2 + 146\alpha}{768}\right) \approx 0.1229,$$

$$U_2 = \frac{T^\beta}{\Gamma(\beta+1)}\left(\frac{3}{2} + \frac{5\beta^4 - 14\beta^3 + 55\beta^2 + 146\beta}{768}\right) \approx 0.0920. \tag{38}$$

Obviously,

$$\max\left\{L_1 U_1 + \lambda_1 U_2, L_2 U_1 + \lambda_2 U_2\right\} < 1; \tag{39}$$

then we can conclude from Theorem 9 that system (36) has a unique solution.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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