Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 262570, 7 pages
http://dx.doi.org/10.1155/2014/262570

Research Article

The Fixed Points of Solutions of Some q-Difference Equations

Xiu-Min Zheng,[1] Hong-Yan Xu,[2] and Jun-Feng Xu[3]

[1] *Institute of Mathematics and Information Science, Jiangxi Normal University, Nanchang, Jiangxi 330022, China*
[2] *Department of Informatics and Engineering, Jingdezhen Ceramic Institute, Jingdezhen, Jiangxi 333403, China*
[3] *Department of Mathematics, Wuyi University, Jiangmen, Guangdong 529020, China*

Correspondence should be addressed to Xiu-Min Zheng; zhengxiumin2008@sina.com

Received 22 March 2014; Accepted 12 August 2014; Published 14 October 2014

Academic Editor: Josef Diblík

The purpose of this paper is to investigate the fixed points of solutions $f(z)$ of some q-difference equations and obtain some results about the exponents of convergence of fixed points of $f(z)$ and $f(q^j z)$ $(j \in \mathbb{N}_+)$, q-differences $\Delta_q f(z) = f(qz) - f(z)$, and q-divided differences $\Delta_q f(z)/f(z)$.

1. Introduction and Main Results

Throughout this paper, we will assume that the readers are familiar with basic notations such as $m(r, f)$, $N(r, f)$, and $T(r, f)$ of Nevanlinna theory (see Hayman [1], Yang [2], and Yang and Yi [3]). We use $\rho(f)$, $\lambda(f)$, and $\lambda(1/f)$ to denote the order, the exponent of convergence of zeros, and the exponent of convergence of poles of $f(z)$, respectively, and we also use the notation $\tau(f)$ to denote the exponent of convergence of fixed points of $f(z)$, which is defined as

$$\tau(f) = \limsup_{r\to\infty} \frac{\log N(r, 1/(f(z)-z))}{\log r}, \tag{1}$$

and $S(r, f)$ to denote any quantity satisfying $S(r, f) = o(T(r, f))$ for all r on a set F of logarithmic density 1, where the logarithmic density of a set F is defined by

$$\lim_{r\to\infty} \frac{1}{\log r} \int_{[1,r]\cap F} \frac{1}{t}\, dt. \tag{2}$$

Throughout this paper, the set F of logarithmic density 1 will be not necessarily the same at each occurrence.

Recently, a number of papers (including [4–9]) focused on complex difference equations, system of complex difference equations, and difference analogues of Nevanlinna theory. Correspondingly, there are many papers focusing on the q-difference (or q-shift difference) equations, such as [10–16].

In 2013, Zhang [17] investigated the growth of meromorphic solutions of some complex q-difference equations and the exponents of convergence of fixed points and zeros of transcendental meromorphic solutions of the second order q-difference equation and obtained the following theorem.

Theorem 1 (see [17]). *Suppose that $f(z)$ is a transcendental meromorphic solution of the equation*

$$f(q^2 z) + \gamma_1 f(qz) = \frac{\alpha_0 + \alpha_1 f(z) + \alpha_2 f^2(z)}{\beta_0 + \beta_1 f(z) + \beta_2 f^2(z)}, \tag{3}$$

where $|q| < 1$, coefficients γ_1, α_0, α_1, α_2, β_0, β_1, and β_2 are constants, and at least one of α_2, β_2 is nonzero. Then, $\rho(f) = 0$ and (i) $f(z)$ has infinitely many fixed points, and (ii) $f(z)$ has infinitely many zeros, whenever $\alpha_0 \neq 0$.

Our first result of this paper is about the exponents of convergence of fixed points and zeros of transcendental meromorphic solutions of the higher order q-difference equation as follows.

Theorem 2. *Suppose that $f(z)$ is a transcendental meromorphic solution of the equation*

$$f(q^n z) + \sum_{t=1}^{n-1} \gamma_t f(q^t z) = \frac{\sum_{j=0}^{n} \alpha_j f^j(z)}{\sum_{j=0}^{n} \beta_j f^j(z)}, \tag{4}$$

where $q \in \mathbb{C}$, $|q| < 1$, *coefficients* γ_t $(t = 1, \ldots, n-1)$, α_j, β_j, $(j = 0, \ldots, n)$, *are constants, and at least one of* α_n, β_n *is nonzero. Then,* $\rho(f) = 0$ *and (i)* $f(z)$ *has infinitely many fixed points, and (ii)* $f(z)$ *has infinitely many zeros, whenever* $\alpha_0 \neq 0$.

From Theorem 2, it is a natural question to ask, What will happen if the right-hand side of (4) is a rational function in both arguments?

Regarding the above question, we will investigate the exponents of convergence of fixed points of meromorphic solutions of the q-difference equation

$$f(qz) = \frac{R(z) f(z)}{Q(z) + P(z) f(z)}, \tag{5}$$

where $P(z)$, $Q(z)$, and $R(z)$ are nonzero polynomials, $q \in \mathbb{C}$, and $|q| \neq 0, 1$. Similar to [18, Page 99], we can call (5) a q-Pielou logistic equation, which is a special form of nonautonomous Schröder equations.

Theorem 3. *Let* $P(z)$, $Q(z)$, *and* $R(z)$ *be nonzero polynomials such that*

$$\deg P(z) \geq \max\{\deg R(z), \deg Q(z), 1\}. \tag{6}$$

Set $\Delta_q f(z) = f(qz) - f(z)$, *where* $q \in \mathbb{C}$ *and* $|q| \neq 0, 1$. *Then every transcendental meromorphic solution* $f(z)$ *of* (5) *satisfies the following statements:*

(i) $f(q^j z)$ *has infinitely many fixed points and* $\tau(f(q^j z)) = \rho(f)$, $(j = 0, 1, 2, \ldots)$;

(ii) *if* $R(z) - (z+1)Q(z) \not\equiv 0$, *then* $\Delta_q f(z)/f(z)$ *has infinitely many fixed points and* $\tau(\Delta_q f/f) = \rho(f)$.

We also study fixed points of transcendental meromorphic solutions of the following q-difference equations:

$$a_n(z) f(q^n z) + \cdots + a_1(z) f(qz) + a_0(z) f(z) = 0, \tag{7}$$

$$a_n(z) f(q^n z) + \cdots + a_1(z) f(qz) + a_0(z) f(z) = F(z), \tag{8}$$

where $0 < |q| < 1$, $a_j(z)$ $(j = 0, 1, \ldots, n)$, and $F(z)$ are polynomials and $a_n(z) a_0(z) \not\equiv 0$, and obtain the following results.

Theorem 4. *Let* $q \in \mathbb{C}$, $0 < |q| < 1$, *let* $a_j(z)$ $(j = 0, 1, \ldots, n)$ *be polynomials, and let* $a_n(z) a_0(z) \not\equiv 0$. *If* $a_0(z), a_1(z), \ldots, a_n(z)$ *satisfy one of the following conditions:*

(i) *there exists an integer* s $(0 \leq s \leq n)$ *such that*

$$\deg a_s(z) > \max\{\deg a_j(z), j = 0, 1, \ldots, n, j \neq s\}; \tag{9}$$

(ii)

$$q^n a_n(z) + \cdots + q a_1(z) + a_0(z) \not\equiv 0, \tag{10}$$

then every transcendental meromorphic solution $f(z)$ *of* (7) *satisfies that* $f(q^j z)$ *has infinitely many fixed points and* $\tau(f(q^j z)) = \rho(f)$ *for* $j \in \mathbb{N}$.

By using the same argument as that in Theorem 4, we can easily obtain the following theorem.

Theorem 5. *Let* $q \in \mathbb{C}$, $0 < |q| < 1$, $a_j(z)$ $(j = 0, 1, \ldots, n)$, *and* $F(z)$ *be polynomials and let* $a_n(z) a_0(z) \not\equiv 0$. *If* $a_0(z), a_1(z), \ldots, a_n(z)$, $F(z)$ *satisfy one of the following conditions:*

(i) $a_0(z), a_1(z), \ldots, a_n(z)$ *and* $F(z)$ *contain just one term of maximal total degree;*

(ii)

$$q^n a_n(z) + \cdots + q a_1(z) + a_0(z) - F(z) \not\equiv 0, \tag{11}$$

then every transcendental meromorphic solution $f(z)$ *of* (8) *satisfies that* $f(q^j z)$ *has infinitely many fixed points and* $\tau(f(q^j z)) = \rho(f)$ *for* $j \in \mathbb{N}$.

2. Some Lemmas

The following result is a difference counterpart to the standard result due to A. A. Mohon'ko and V. D. Mohon'ko [19].

Lemma 6 (see [20], Theorem 2.2). *Let* $f(z)$ *be a nonconstant zero-order meromorphic solution of* $P(z, f) = 0$, *where* $P(z, f)$ *is a q-difference polynomial in* $f(z)$. *If* $P(z, a) \not\equiv 0$ *for a slowly moving target* $a(z)$, *then*

$$m\left(r, \frac{1}{f - a}\right) = S(r, f), \tag{12}$$

on a set of logarithmic density 1.

Lemma 7 (see [21, 22]). *Let* $a_j(z)$, $j = 0, 1, \ldots, n$, *and* $Q(z)$ *be rational functions, and let* $a_0(z) \not\equiv 0$, $a_n(z) \equiv 1$, *and* q $(0 < |q| < 1)$. *Then*

(i) *all meromorphic solutions of the equation*

$$\sum_{j=0}^{n} a_j(z) f(q^j z) = Q(z) \tag{13}$$

satisfy $T(r, f) = O((\log r)^2)$;

(ii) *all transcendental meromorphic solutions of* (13) *satisfy* $(\log r)^2 = O(T(r, f))$.

Lemma 8 (see [17], Theorem 2). *Suppose that* $f(z)$ *is a nonconstant meromorphic solution of the equation*

$$\sum_{j=1}^{n} \gamma_j(z) f(q^j z) = R(z, f(z)) = \frac{\sum_{i=0}^{s} \alpha_i(z) f^i(z)}{\sum_{i=0}^{t} \beta_i(z) f^i(z)}, \tag{14}$$

where q $(0 < |q| < 1)$ *is a complex number,* $\alpha_j(z)$ $(j = 0, 1, \ldots, s)$, $\alpha_s(z) \not\equiv 0$, $\beta_j(z)$ $(j = 0, 1, \ldots, t)$, $\beta_t(z) \not\equiv 0$, $\gamma_n(z) \equiv 1$, *and* $\gamma_j(z)$ $(j = 0, 1, \ldots, n)$ *are small functions of* $f(z)$, *and* $R(z, f)$ *is irreducible in* $f(z)$. *Then,* $d = \max\{s, t\} \leq n$ *and* $\rho(f) \leq (\log n - \log d)/-\log|q|$.

Lemma 9 (see [21, page 249] or [23, Theorem 1.1]). *Let $f(z)$ be a transcendental meromorphic function of zero-order and let q be a nonzero complex constant. Then*

$$T(r, f(qz)) = T(r, f) + S(r, f) \tag{15}$$

on a set of logarithmic density 1.

3. Proof of Theorem 2

Suppose that $f(z)$ is a transcendental meromorphic solution of (4). From the assumptions of Theorem 2, it follows from Lemma 8 that $\rho(f) \leq 0 = (\log n - \log n)/-\log|q|$. Thus, $\rho(f) = 0$. Clearly, we have $\lambda(f) = \tau(f) = \rho(f) = 0$.

(i) Firstly, we prove that $f(z)$ has infinitely many fixed points. Set $g(z) = f(z) - z$. Then $g(z)$ is transcendental, $T(r, g) = T(r, f) + O(\log r)$, and $S(r, f) = S(r, g)$. So, $g(z)$ is of zero-order. Then substituting $f(z) = g(z) + z$ into (4), we get that

$$\begin{aligned} g(q^n z) + \sum_{t=1}^{n-1} \gamma_t g(q^t z) + q^n z + \sum_{t=1}^{n-1} \gamma_t q^t z \\ = \frac{\sum_{j=0}^{n} \alpha_j (g(z) + z)^j}{\sum_{j=0}^{n} \beta_j (g(z) + z)^j}. \end{aligned} \tag{16}$$

Set $A(z) = g(q^n z) + \sum_{t=1}^{n-1} \gamma_t g(q^t z) + q^n z + \sum_{t=1}^{n-1} \gamma_t q^t z$ and

$$P_1(z, g(z)) := A(z) \sum_{j=0}^{n} \beta_j (g(z) + z)^j - \sum_{j=0}^{n} \alpha_j (g(z) + z)^j. \tag{17}$$

It follows from (17) that

$$\begin{aligned} & P_1(z, 0) \\ & = \left(q^n + \sum_{t=1}^{n-1} \gamma_t q^t\right) \sum_{j=0}^{n} \beta_j z^{j+1} - \sum_{j=0}^{n} \alpha_j z^j \\ & = \left(q^n + \sum_{t=1}^{n-1} \gamma_t q^t\right) \beta_n z^{n+1} \\ & \quad + \sum_{j=0}^{n-1} \left[\left(q^n + \sum_{t=1}^{n-1} \gamma_t q^t\right) \beta_j - \alpha_{j+1}\right] z^{j+1} - \alpha_0. \end{aligned} \tag{18}$$

Suppose that $P_1(z, 0) \equiv 0$. If $q^n + \sum_{t=1}^{n-1} \gamma_t q^t = 0$, then it follows from (18) that $\alpha_0 = \alpha_1 = \cdots = \alpha_n = 0$. Thus, the right-hand side of (4) is 0, which is in contradiction with the assumption of Theorem 2. If $q^n + \sum_{t=1}^{n-1} \gamma_t q^t \neq 0$, it follows from (18) that $\beta_n = \alpha_0 = 0$ and

$$\frac{\alpha_{j+1}}{\beta_j} = q^n + \sum_{t=1}^{n-1} \gamma_t q^t, \quad j = 0, 1, \ldots, n-1. \tag{19}$$

Thus, we have from (4) and (19) that

$$f(q^n z) + \sum_{t=1}^{n-1} \gamma_t f(q^t z) = \left(q^n + \sum_{t=1}^{n-1} \gamma_t q^t\right) f(z), \tag{20}$$

which is in contradiction with the assumption of Theorem 2. Hence, we have $P_1(z, 0) \not\equiv 0$. By Lemma 6, we get that

$$m\left(r, \frac{1}{g}\right) = S(r, g) = S(r, f) \tag{21}$$

on a set of logarithmic density 1. Thus, it follows from (21) that

$$N\left(r, \frac{1}{f - z}\right) = N\left(r, \frac{1}{g}\right) = T(r, f) + S(r, f) \tag{22}$$

on a set of logarithmic density 1. Since $f(z)$ is a transcendental meromorphic solution of (4), then it follows from (22) that $f(z)$ has infinitely many fixed points.

(ii) From (4), we have

$$\begin{aligned} P_2(z, f(z)) := \left[f(q^n z) + \sum_{t=1}^{n-1} \gamma_t f(q^t z)\right] \sum_{j=0}^{n} \beta_j f^j(z) \\ - \sum_{j=0}^{n} \alpha_j f^j(z). \end{aligned} \tag{23}$$

Since $\alpha_0 \neq 0$ and from (23), we derive that

$$P_2(z, 0) = \alpha_0 \not\equiv 0. \tag{24}$$

Thus, it follows from Lemma 6 that

$$m\left(r, \frac{1}{f}\right) = S(r, f) \tag{25}$$

on a set of logarithmic density 1; that is,

$$N\left(r, \frac{1}{f}\right) = T(r, f) + S(r, f) \tag{26}$$

on a set of logarithmic density 1. Since $f(z)$ is a transcendental solution of (4), then it follows from (26) that $f(z)$ has infinitely many zeros.

Thus, this completes the proof of Theorem 2.

4. Proof of Theorem 3

Suppose that $f(z)$ is a transcendental meromorphic solution of (5). Since $q \in \mathbb{C}$, $|q| \neq 0, 1$, and $P(z)$, $Q(z)$, and $R(z)$ are polynomials, it follows from Lemma 8 and [11] that $f(z)$ is of zero-order.

(i) We first prove that $f(z)$ has infinitely many fixed points and $\tau(f) = \rho(f)$. Set $g(z) = f(z) - z$. Then $g(z)$ is transcendental, $T(r, g) = T(r, f) + O(\log r)$, and $S(r, g) = S(r, f)$. Then it follows that $g(z)$ is of zero-order. Set

$$\begin{aligned} P_3(z, f(z)) := P(z) f(z) f(qz) \\ + f(qz) Q(z) - R(z) f(z) \equiv 0. \end{aligned} \tag{27}$$

Then substituting $f(z) = g(z) + z$ into (27), we have

$$\begin{aligned} P_4(z, g(z)) = P(z)(g(z) + z)(g(qz) + qz) \\ + Q(z)(g(qz) + qz) - R(z)(g(z) + z) = 0. \end{aligned} \tag{28}$$

It follows from (28) that

$$P_4(z,0) = qz^2 P(z) + qzQ(z) - zR(z). \tag{29}$$

Thus, we derive by (6) and (29) that $P_4(z,0) \not\equiv 0$. Thus, by Lemma 6 and $P_4(z,0) \not\equiv 0$, we have

$$m\left(r, \frac{1}{g}\right) = S(r,g) = S(r,f) \tag{30}$$

on a set of logarithmic density 1; that is,

$$N\left(r, \frac{1}{f-z}\right) = N\left(r, \frac{1}{g}\right) = T(r,f) + S(r,f) \tag{31}$$

on a set of logarithmic density 1.

Since $f(z)$ is a transcendental meromorphic solution of (5), then it follows from (31) that $f(z)$ has infinitely many fixed points.

Next, we prove that $f(qz)$ has infinitely many fixed points and $\tau(f(qz)) = \rho(f)$. From (5), we have

$$\begin{aligned} f(qz) - z &= \frac{(R(z) - zP(z)) f(z) - zQ(z)}{Q(z) + P(z) f(z)} \\ &= \frac{(R(z) - zP(z)) \left[f(z) - zQ(z)/(R(z) - zP(z))\right]}{Q(z) + P(z) f(z)}. \end{aligned} \tag{32}$$

By (6), we have $R(z) - zP(z) \not\equiv 0$. Since $f(z)$ is transcendental and $P(z)$, $Q(z)$, and $R(z)$ are polynomials, we have by (32) the fact that $f(z) - zQ(z)/(R(z) - zP(z))$ and $Q(z) + P(z)f(z)$ have the same poles, except possibly finitely many poles. Moreover, we can get that $(R(z) - zP(z))f(z) - zQ(z)$ and $Q(z) + P(z)f(z)$ have at most finitely many common zeros. In fact, suppose that z_0 is a common zero of $(R(z) - zP(z))f(z) - zQ(z)$ and $Q(z) + P(z)f(z)$. Then $(R(z_0) - z_0P(z_0))f(z_0) - z_0Q(z_0) = 0$; that is, $f(z_0) = z_0Q(z_0)/(R(z_0) - z_0P(z_0))$. Substituting it into $Q(z_0) + P(z_0)f(z_0)$, we have

$$\frac{z_0 Q(z_0)}{R(z_0) - z_0 P(z_0)} P(z_0) + Q(z_0) = \frac{R(z_0) Q(z_0)}{R(z_0) - z_0 P(z_0)} = 0. \tag{33}$$

Thus, this shows that z_0 must be the zeros of $R(z)Q(z)/(R(z) - zP(z))$. Since $P(z)$, $Q(z)$, and $R(z)$ are polynomials, then $R(z)Q(z)/(R(z) - zP(z))$ has only finitely many zeros. So, $f(z) - zQ(z)/(R(z) - zP(z))$ and $Q(z) + P(z)f(z)$ have at most finitely many common zeros. Then it follows from (32) that

$$\tau(f(qz)) = \lambda(f(qz) - z) = \lambda\left(f(z) - \frac{zQ(z)}{R(z) - zP(z)}\right). \tag{34}$$

From (27), we have

$$\begin{aligned} &P_3\left(z, \frac{zQ(z)}{R(z) - zP(z)}\right) \\ &= P(z) \frac{zQ(z)}{R(z) - zP(z)} \frac{qzQ(qz)}{R(qz) - qzP(qz)} \\ &\quad + \frac{qzQ(qz)}{R(qz) - qzP(qz)} Q(z) - R(z) \frac{zQ(z)}{R(z) - zP(z)} \\ &= \left(qz^2 P(qz) Q(z) R(z) + qzQ(qz) Q(z) R(z) \right. \\ &\quad \left. - zQ(z) R(z) R(qz)\right) \\ &\quad \times \left((R(z) - zP(z)) (R(qz) - qzP(qz))\right)^{-1}. \end{aligned} \tag{35}$$

Since $\deg P(z) \geq \max\{\deg R(z), \deg Q(z)\}$ and $\deg P(qz) = \deg P(z)$, then we have $\deg\{qz^2P(qz)Q(z)R(z) + qzQ(qz)Q(z)R(z) - zQ(z)R(z)R(qz)\} \geq 1$. Thus, it follows from (35) that $P_3(z, zQ(z)/(R(z) - zP(z))) \not\equiv 0$. Since $f(z)$ is transcendental function of zero-order and $zQ(z)/(R(z) - zP(z))$ is a rational function, then we have by Lemma 6 the fact that

$$m\left(r, \frac{1}{f(z) - zQ(z)/(R(z) - zP(z))}\right) = S(r,f) \tag{36}$$

on a set of logarithmic density 1; that is,

$$\begin{aligned} &N\left(r, \frac{1}{f(z) - zQ(z)/(R(z) - zP(z))}\right) \\ &\quad = T(r,f) + S(r,f) \end{aligned} \tag{37}$$

on a set of logarithmic density 1. Since $f(z)$ is transcendental, we can derive from (34) and (37) that $f(qz)$ has infinitely many fixed points and $\tau(f(qz)) = \rho(f)$.

Now, we prove that $f(q^2z)$ has infinitely many fixed points and $\tau(f(q^2z)) = \rho(f)$. From (5), we have

$$f_1(qz) = \frac{R(qz) f_1(z)}{Q(qz) + P(qz) f_1(z)}, \tag{38}$$

where $f_1(z) = f(qz)$. By Lemma 9, we have $\rho(f_1) = \rho(f) = 0$. Obviously, $\deg P(qz) = \deg P(z) \geq 1$, $\deg R(qz) = \deg R(z)$, and $\deg Q(qz) = \deg Q(z)$. Thus, by using the same argument as in the proof of $\tau(f(qz)) = \rho(f)$, we can prove that $f_1(qz) = f(q^2z)$ has infinitely many fixed points and $\tau(f(q^2z)) = \tau(f_1(qz)) = \rho(f_1) = \rho(f)$.

Thus, by using the same method as above, we can obtain that $f(q^jz)$ has infinitely many fixed points and $\tau(f(q^jz)) = \rho(f)$ for $j = 0, 1, \ldots$.

(ii) Now, we prove that $\Delta_q f(z)/f(z)$ has infinitely many fixed points and

$$\tau\left(\frac{\Delta_q f}{f}\right) = \rho(f). \tag{39}$$

By (5) and from $R(z)-(z+1)Q(z)\not\equiv 0$, we have

$$\begin{aligned}\frac{\Delta_q f(z)}{f(z)}-z &= \frac{f(qz)-f(z)}{f(z)}-z\\ &=\frac{R(z)-(z+1)Q(z)-(z+1)P(z)f(z)}{Q(z)+P(z)f(z)}\\ &=-(z+1)P(z)\\ &\quad\times\left(f(z)-\frac{R(z)-(z+1)Q(z)}{(z+1)P(z)}\right)\\ &\quad\times\left(Q(z)+P(z)f(z)\right)^{-1}.\end{aligned}\tag{40}$$

Since $R(z)-(z+1)Q(z)\not\equiv 0$, $f(z)$ is transcendental, and $P(z)$, $Q(z)$, and $R(z)$ are polynomials, we have by (40) the fact that $f(z)-(R(z)-(z+1)Q(z))/(z+1)P(z)$ and $Q(z)+P(z)f(z)$ have the same poles, except possibly finitely many poles. Moreover, by using the same argument as in (i), we can get that $R(z)-(z+1)Q(z)-(z+1)P(z)f(z)$ and $Q(z)+P(z)f(z)$ have at most finitely many common zeros. Then it follows from (40) that

$$\begin{aligned}\tau\left(\frac{\Delta_q f}{f}\right)&=\lambda\left(\frac{\Delta_q f}{f}-z\right)\\ &=\lambda\left(f(z)-\frac{R(z)-(z+1)Q(z)}{(z+1)P(z)}\right).\end{aligned}\tag{41}$$

From (27), we have

$$\begin{aligned}P_3&\left(z,\frac{R(z)-(z+1)Q(z)}{(z+1)P(z)}\right)\\ &=P(z)\frac{R(z)-(z+1)Q(z)}{(z+1)P(z)}\frac{R(qz)-(qz+1)Q(qz)}{(qz+1)P(qz)}\\ &\quad+\frac{R(qz)-(qz+1)Q(qz)}{(qz+1)P(qz)}Q(z)\\ &\quad-R(z)\frac{R(z)-(z+1)Q(z)}{(z+1)P(z)}\\ &:=\frac{B(z)}{(qz+1)(z+1)P(qz)P(z)},\end{aligned}\tag{42}$$

where

$$\begin{aligned}B(z)&=(qz+1)(z+1)P(qz)Q(z)R(z)\\ &\quad-(qz+1)P(z)Q(qz)R(z)\\ &\quad-(qz+1)P(qz)R^2(z)+P(z)R(z)R(qz).\end{aligned}\tag{43}$$

Since $P(z)$, $Q(z)$, and $R(z)$ are polynomials satisfying (6), then it follows from (43) that $B(z)$ is a polynomial of degree $t\geq 1$. Thus, from (42) we have $P_3(z,(R(z)-(z+1)Q(z))/(z+1)P(z))\not\equiv 0$. Since $f(z)$ is transcendental function of zero-order and $(R(z)-(z+1)Q(z))/(z+1)P(z)$ is a rational function, then we have by Lemma 6 the fact that

$$\begin{aligned}m&\left(r,\frac{1}{f(z)-(R(z)-(z+1)Q(z))/(z+1)P(z)}\right)\\ &=S(r,f)\end{aligned}\tag{44}$$

on a set of logarithmic density 1; that is,

$$\begin{aligned}N&\left(r,\frac{1}{f(z)-(R(z)-(z+1)Q(z))/(z+1)P(z)}\right)\\ &=T(r,f)+S(r,f)\end{aligned}\tag{45}$$

on a set of logarithmic density 1. Since $f(z)$ is transcendental, we can derive from (41) and (45) that $\Delta_q f(z)/f(z)$ has infinitely many fixed points and $\tau(\Delta_q f/f)=\rho(f)$.

Thus, this completes the proof of Theorem 3.

5. Proof of Theorem 4

Suppose that $f(z)$ is a transcendental meromorphic solution of (7). Since $q\in\mathbb{C}$, $0<|q|<1$, and $a_j(z)$, $j=0,1,\ldots,n$, are polynomials, by Lemma 7, we see that $f(z)$ is of zero-order. Set

$$\begin{aligned}P_5(z,f(z)):=a_n(z)f(q^n z)+\cdots+a_1(z)f(qz)\\ +a_0(z)f(z)=0.\end{aligned}\tag{46}$$

Thus, it follows from (46) that

$$\begin{aligned}P_5(z,z)&=a_n(z)q^n z+\cdots+a_1(z)qz+a_0(z)z\\ &=z\left[q^n a_n(z)+\cdots+qa_1(z)+a_0(z)\right].\end{aligned}\tag{47}$$

(i) Suppose that $a_0(z),\ldots,a_n(z)$ satisfy condition (9). Then it follows that $P_5(z,z)\not\equiv 0$. Since $f(z)$ is a transcendental solution of zero-order, then it follows from Lemma 6 that

$$m\left(r,\frac{1}{f-z}\right)=S(r,f)\tag{48}$$

on a set of logarithmic density 1. So,

$$N\left(r,\frac{1}{f-z}\right)=T(r,f)+S(r,f)\tag{49}$$

on a set of logarithmic density 1. Thus, it follows that $f(z)$ has infinitely many fixed points and $\tau(f)=\rho(f)$.

Now, we prove that $f(qz)$ has infinitely many fixed points and $\tau(f(qz))=\rho(f)$. By (7), we derive

$$a_n(qz)f_1(q^n z)+\cdots+a_1(qz)f_1(qz)+a_0(qz)f_1(z)=0,\tag{50}$$

where $f_1(z)=f(qz)$. Since $f(z)$ is a transcendental meromorphic function of zero-order, then we have by Lemma 9 the fact that $f_1(z)$ is a transcendental and $\rho(f_1) = \rho(f)$. By $\deg a_j(qz) = \deg a_j(z)$, $j = 0, 1, \ldots, n$, and (9), we have

$$\deg a_s(qz) = \deg a_s(z) > \max\left\{a_j(qz), j = 0, 1, \ldots, n, j \neq s\right\}. \quad (51)$$

Thus, by the above proof of $\tau(f) = \rho(f)$, we see that $f_1(z) = f(qz)$ has infinitely many fixed points and $\tau(f_1) = \tau(f(qz)) = \rho(f_1) = \rho(f)$. Continuing to use the same method as the above, we can prove that $f(q^j z)$ has infinitely many fixed points and $\tau(f(q^j z)) = \rho(f)$ for $j = 0, 1, \ldots$.

(ii) Suppose that $a_0(z), \ldots, a_n(z)$ satisfy the condition (10).

By using the same argument as the one above, we can prove that $f(z)$ has infinitely many fixed points and $\tau(f) = \rho(f)$ easily.

Now, we prove that $f(qz)$ has infinitely many fixed points and $\tau(f(qz)) = \rho(f)$. Set

$$P_6(z, f_1(z)) := a_n(qz) f_1(q^n z) + \cdots + a_1(qz) f_1(qz) + a_0(qz) f_1(z) = 0. \quad (52)$$

Thus, it follows from (10) that

$$P_6(z, z) = z\left[q^n a_n(qz) + \cdots + q a_1(qz) + a_0(qz)\right] \not\equiv 0. \quad (53)$$

In fact, if $P_6(z, z) \equiv 0$, replacing z by z/q into (53), we have

$$P_6\left(\frac{z}{q}, \frac{z}{q}\right) = \frac{z}{q}\left[q^n a_n(z) + \cdots + q a_1(z) + a_0(z)\right] \equiv 0, \quad (54)$$

which is in contradiction with the condition (10). Since $f_1(z) = f(qz)$ and $f(z)$ is transcendental meromorphic of zero-order, then it follows from (53) and Lemma 6 that $f_1(z) = f(qz)$ has infinitely many fixed points and $\tau(f_1) = \tau(f(qz)) = \rho(f_1) = \rho(f)$. Continuing to use the same method as the one above, we can prove that $f(q^j z)$ has infinitely many fixed points and $\tau(f(q^j z)) = \rho(f)$ for $j = 0, 1, \ldots$.

Thus, this completes the proof of Theorem 4.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors thank the referees for reading the paper very carefully and making a number of valuable and kind comments which improve the presentation. This work was supported by the NSF of China (11301233 and 61202313), the Natural Science Foundation of Jiangxi Province in China (20132BAB211001, GJJ14644, and GJJ14271), Sponsored Program for Cultivating Youths of Outstanding Ability in Jiangxi Normal University of China, Foundation for Distinguished Young Talents in Higher Education of Guangdong China (2013LYM0093), and Training Plan for the Outstanding Young Teachers in Higher Education of Guangdong (Yq 2013159).

References

[1] W. K. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford, UK, 1964.

[2] L. Yang, *Value Distribution Theory*, Springer, Berlin, Germany, 1993.

[3] C. C. Yang and H. X. Yi, *Uniqueness Theory of Meromorphic Functions*, Kluwer Academic, Dordrecht, The Netherlands; Chinese Original: Science Press, Beijing, China, 2003.

[4] W. Bergweiler and J. K. Langley, "Zeros of differences of meromorphic functions," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 142, no. 1, pp. 133–147, 2007.

[5] Z. Chen, Z. Huang, and X. Zheng, "On properties of difference polynomials," *Acta Mathematica Scientia B*, vol. 31, no. 2, pp. 627–633, 2011.

[6] Z. Chen and K. H. Shon, "On zeros and fixed points of differences of meromorphic functions," *Journal of Mathematical Analysis and Applications*, vol. 344, no. 1, pp. 373–383, 2008.

[7] Z. X. Chen and K. H. Shon, "Properties of differences of meromorphic functions," *Czechoslovak Mathematical Journal*, vol. 61, no. 136, pp. 213–224, 2011.

[8] Y. Chiang and S. Feng, "On the Nevanlinna characteristic of $f(z + \eta)$ and difference equations in the complex plane," *Ramanujan Journal*, vol. 16, no. 1, pp. 105–129, 2008.

[9] I. Laine and C. Yang, "Value distribution of difference polynomials," *Proceedings of Japan Academy A*, vol. 83, no. 8, pp. 148–151, 2007.

[10] A. Fletcher, J. K. Langley, and J. Meyer, "Slowly growing meromorphic functions and the zeros of differences," *Mathematical Proceedings of the Royal Irish Academy*, vol. 109, no. 2, pp. 147–154, 2009.

[11] G. G. Gundersen, J. Heittokangas, I. Laine, and D. Yang, "Meromorphic solutions of generalized Schröder equations," *Aequationes Mathematicae*, vol. 63, no. 1-2, pp. 110–135, 2002.

[12] H. Wang, H.-Y. Xu, and B.-X. Liu, "The poles and growth of solutions of systems of complex difference equations," *Advances in Difference Equations*, vol. 2013, article 75, 11 pages, 2013.

[13] H. Y. Xu, "On the value distribution and uniqueness of difference polynomials of meromorphic functions," *Advances in Difference Equations*, vol. 2013, article 90, 2013.

[14] H. Xu, B. Liu, and K. Tang, "Some properties of meromorphic solutions of systems of complex q-shift difference equations," *Abstract and Applied Analysis*, vol. 2013, Article ID 680956, 6 pages, 2013.

[15] H. Y. Xu, J. Tu, and X. M. Zheng, "Some properties of solutions of complex q-shift difference equations," *Annales Polonici Mathematici*, vol. 108, no. 3, pp. 289–304, 2013.

[16] J. Xu and X. Zhang, "The zeros of q-shift difference polynomials of meromorphic functions," *Advances in Difference Equations*, vol. 2012, article 200, 2012.

[17] G. Zhang, "Growth of meromorphic solutions of some q-difference equations," *Abstract and Applied Analysis*, vol. 2013, Article ID 943209, 6 pages, 2013.

[18] S. Elaydi, *An Introduction to Difference Equations*, Undergraduate Texts in Mathematics, Springer, New York, NY, USA, 3rd edition, 2005.

[19] A. A. Mohon'ko and V. D. Mohon'ko, "Estimates of the Nevanlinna characteristics of certain classes of meromorphic functions, and their applications to differential equations," *Sibirskii Matematicheskii Zhurnal*, vol. 15, pp. 1305–1322, 1974 (Russian).

[20] D. C. Barnett, R. G. Halburd, W. Morgan, and R. J. Korhonen, "Nevanlinna theory for the q-difference operator and meromorphic solutions of q-difference equations," *Proceedings of the Royal Society of Edinburgh A; Mathematics*, vol. 137, no. 3, pp. 457–474, 2007.

[21] W. Bergweiler, K. Ishizaki, and N. Yanagihara, "Meromorphic solutions of some functional equations," *Methods and Applications of Analysis*, vol. 5, no. 3, pp. 248–258, 1998.

[22] W. Bergweiler, K. Ishizaki, and N. Yanagihara, "Meromorphic solutions of some functional equations (correction)," *Methods and Applications of Analysis*, vol. 6, pp. 617–618, 1999.

[23] J. Zhang and R. Korhonen, "On the Nevanlinna characteristic of $f(qz)$ and its applications," *Journal of Mathematical Analysis and Applications*, vol. 369, no. 2, pp. 537–544, 2010.