

Research Article

Generalized Numerical Index and Denseness of Numerical Peak Holomorphic Functions on a Banach Space

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The generalized numerical index of a Banach space is introduced, and its properties on certain Banach spaces are studied. Ed-dari's theorem on the numerical index is extended to the generalized index and polynomial numerical index of a Banach space. The denseness of numerical strong peak holomorphic functions is also studied.

1. Introduction and Preliminaries

Let X and Y be Banach spaces over a scalar field \mathbb{F} , where \mathbb{F} is the real field \mathbb{R} or the complex field \mathbb{C} . We denote by B_X and S_X its closed unit ball and unit sphere, respectively. Let X^* be the dual space of X . An N -homogeneous polynomial P from X to Y is a mapping such that there is an N -linear (bounded) mapping L from X to Y such that $P(x) = L(x, \dots, x)$ for every x in X . $\mathcal{P}^N(X : Y)$ denotes the Banach space of all N -homogeneous polynomials from X to Y , endowed with the norm $\|P\| = \sup_{x \in B_X} \|P(x)\|$. A mapping $Q : X \rightarrow Y$ is a *polynomial* if there exist a nonnegative integer m and $P_k \in \mathcal{P}^k(X : Y)$, $k = 0, 1, \dots, m$ such that $Q = P_0 + P_1 + \dots + P_m$. If $P_m \neq 0$, then we say that Q is a *polynomial of degree m* . We denote by $\mathcal{P}(X : Y)$ the normed space of all polynomials from X to Y , endowed with the norm $\|Q\| = \sup_{x \in B_X} \|Q(x)\|$. We refer to [1] for background on polynomials on a Banach space.

For two Banach spaces X, Y over a field \mathbb{F} and a Hausdorff topological space K , let

$$C_b(K : Y) := \{f : K \rightarrow Y : f \text{ be a bounded continuous function on } K\}. \quad (1)$$

Then $C_b(K : Y)$ is a Banach space under the sup norm $\|f\| := \sup\{\|f(t)\|_Y : t \in K\}$ and $\mathcal{P}^N(X : Y)$ is a closed subspace of

$C_b(B_X : Y)$ for each $N \geq 1$. We just write $C_b(K)$ and $\mathcal{P}^N(X)$ instead of $C_b(K : \mathbb{F})$ and $\mathcal{P}^N(X : \mathbb{F})$, respectively.

For complex Banach spaces X and Y , we denote that

$$A_b(B_X : Y) := \{f \in C_b(B_X : Y) : f \text{ is holomorphic on } B_X^\circ\}$$
$$A_u(B_X : Y) := \{f \in A_b(B_X : Y) : f \text{ is uniformly continuous}\}, \quad (2)$$

where B_X° is the interior of B_X . Then $A_b(B_X : Y)$ and $A_u(B_X : Y)$ are closed subspaces of $C_b(B_X : Y)$. In case that Y is the complex scalar field \mathbb{C} , we write $A_b(B_X)$ and $A_u(B_X)$ instead of $A_b(B_X : Y)$ and $A_u(B_X : Y)$, respectively. The closed subspace of $A_u(B_X : Y)$ consisting of all weakly uniformly continuous functions is denoted by $A_{wu}(B_X : Y)$. We denote by $A(B_X : X)$ one of $A_b(B_X : X)$, $A_u(B_X : X)$, and $A_{wu}(B_X : X)$. Notice that if X is finite dimensional, $A_b(B_X : X) = A_u(B_X : X) = A_{wu}(B_X : X)$.

Given a real or complex Banach space X , we denote by τ the product topology of the set $S_X \times S_{X^*}$, where the topologies on S_X and S_{X^*} are the norm topology of X and the weak- $*$ topology of X^* , respectively. The set $\Pi(X) := \{(x, x^*) \in S_X \times S_{X^*} : \|x\| = \|x^*\| = 1 = x^*(x)\}$ is a τ -closed subset of $S_X \times S_{X^*}$. The *spatial numerical range* of f in $C_b(B_X : X)$ is defined [2] by $W(f) = \{x^*(f(x)) : (x, x^*) \in \Pi(X)\}$, and the *numerical*

radius of f is defined by $v(f) = \sup\{|\lambda| : \lambda \in W(f)\}$. Let f be an element of $C_b(K : X)$. We say that f attains its norm if there is some $t \in K$ such that $\|f\| = \|f(t)\|_X$. f is said to be a (norm) peak function at t if there exists a unique $t \in K$ such that $\|f\| = \|f(t)\|_X$. It is clear that every (norm) peak function in $C_b(K : X)$ is norm attaining. A peak function f at t is said to be a (norm) strong peak function if whenever there is a sequence $\{t_k\}_{k=1}^\infty$ in K with $\lim_k \|f(t_k)\|_X = \|f\|$, $\{t_k\}_{k=1}^\infty$ converges to t in K . It is easy to see that if K is compact, then every peak function is a strong peak function. Given a subspace H of $C_b(K)$, we denote by ρH the set of all points $t \in K$ such that there is a strong peak function f in H with $\|f\| = |f(t)|$.

Similarly we introduce the notion of numerical peak functions. Let f be an element of $C_b(B_X : X)$. If there is some $(x, x^*) \in \Pi(X)$ such that $v(f) = |x^*(f(x))|$, we say [3] that f attains its numerical radius. f is said ([4, 5]) to be a numerical peak function at (x, x^*) if there exist a unique $(x, x^*) \in \Pi(X)$ such that $v(f) = |x^*(f(x))|$. In this case, (x, x^*) is said to be the numerical peak point of f . It is clear that every numerical peak function in $C_b(B_X : X)$ is numerical radius attaining. The numerical peak function f at (x, x^*) is called a numerical strong peak function if whenever there is a sequence $\{(x_k, x_k^*)\}_{k=1}^\infty$ in $\Pi(X)$ such that $\lim_k |x_k^*(f(x_k))| = v(f)$, then $\{(x_k, x_k^*)\}_{k=1}^\infty$ converges to (x, x^*) in τ -topology. In this case, (x, x^*) is said to be the numerical strong peak point of f . We say that a numerical strong peak function f at (x, x^*) is said to be a very strong numerical peak function if whenever there is a sequence $\{(x_k, x_k^*)\}_{k=1}^\infty$ in $\Pi(X)$ satisfying $\lim_n |x_k^*(f(x_k))| = v(f)$, we get $\lim_k x_k = x$ and $\lim_k x_k^* = x^*$ in the norm topology. If X is finite dimensional, then every numerical peak function is a very strong numerical peak function.

In 1996, Choi and Kim [6] initiated the study of denseness of norm or numerical radius attaining nonlinear functions, especially homogeneous polynomials on a Banach space. Using the perturbed optimization theorem of Bourgain [7] and Stegall [8], they proved that if a real or complex Banach space X has the Radon-Nikodým property, then the set of all norm attaining functions in $\mathcal{P}^k(X)$ is norm-dense. For the definition and properties of the Radon-Nikodým property, see [9]. Concerning the numerical radius, it was also shown that if X has the Radon-Nikodým property, then the set of all numerical radii attaining functions in $\mathcal{P}^k(X : X)$ is norm-dense. Acosta et al. [10] proved that if a complex Banach space X has the Radon-Nikodým property, then the set of all norm attaining functions in $A_b(B_X)$ is norm-dense. Recently, it was shown in [11] that if X has the Radon-Nikodým property, the set of all (norm) strong peak functions in $A_b(B_X)$ is dense. Concerning the numerical radius, Acosta and Kim [3] showed that the set of all numerical radii attaining functions in $A_b(B_X : X)$ is dense if X has the Radon-Nikodým property. When X is a smooth (complex) Banach space with the Radon-Nikodým property, it is shown in [5] that the set of all numerical strong peak functions is dense in $A(B_X : X)$. As a corollary, if $1 < p < \infty$ and $X = L_p(\mu)$ for a measure space μ , then the set of all norm and numerical strong peak functions in $A(B_X : X)$ is a dense G_δ -subset of $A(B_X : X)$.

In this case, every numerical strong peak function is a very strong numerical peak function. It is also shown in [5] that the set of all norm and numerical strong peak functions in $A(B_{l_1} : l_1)$ is a dense G_δ -subset of $A(B_{l_1} : l_1)$.

Let us briefly sketch the content of this paper. In Section 2, to extend the results of a finite dimensional space to an infinite dimensional space by approximation, we introduce the following notions. A Banach space X has the (FPA)-property with $\{\pi_i, F_i\}_{i \in I}$ if

- (1) each π_i is a norm-one projection with the finite dimensional range F_i ,
- (2) given $\epsilon > 0$, for every finite-rank operator T from X into a Banach space F and for every finite dimensional subspace G of X , there is π_i such that

$$\|T - T\pi_i\| \leq \epsilon, \quad \|I_G - \pi_i|_G\| \leq \epsilon. \tag{3}$$

As examples, we show that X has the (FPA)-property if at least one of the following conditions is satisfied.

- (a) It has a shrinking and monotone finite-dimensional decomposition.
- (b) $X = L_p(\mu)$, where μ is a finite measure and $1 \leq p < \infty$.

We show that if X has the (FPA)-property, then the set of all polynomials $Q \in \mathcal{P}(X : X)$ such that there exist a finite dimensional subspace F and norm-one projection $\pi : X \rightarrow F$ such that $\pi \circ Q \circ \pi = Q$ and $Q|_F$ is a norm, and numerical peak function as a mapping from B_F into F is dense in $A_{wu}(B_X : X)$.

A subset Γ of $\Pi(X)$ is called a numerical boundary for a subspace H of $C_b(B_X : X)$ if $v(f) = \sup\{|x^*(f(x))| : (x, x^*) \in \Gamma\}$ for every $f \in H$ (see [4, 12]). The projections $\{\pi_i, F_i\}_{i \in I}$ are said to be parallel to a numerical boundary Γ of H if each π_i has the image F_i and

$$\left| \langle x^*|_{F_i}, \pi_i(x) \rangle \right| = \|x^*|_{F_i}\| \cdot \|\pi_i(x)\|, \quad \forall (x, x^*) \in \Gamma, \forall i \in I. \tag{4}$$

A projection $\pi : X \rightarrow X$ is said to be strong if whenever $\{\pi(x_k)\}_{k=1}^\infty$ is norm-convergent to $y \in S_X$ for a sequence $\{x_k\}_{k=1}^\infty$ in B_X , $\{x_k\}_{k=1}^\infty$ is norm-convergent to y .

Recall that a Banach space X is said to be locally uniformly convex if $x \in S_X$, and there is a sequence $\{x_n\}$ in B_X satisfying $\lim_n \|x_n + x\| = 2$, then $\lim_n \|x_n - x\| = 0$. Notice that if X is locally uniformly convex, then every norm-one projection is strong. We prove that if a smooth Banach space X has the (FPA)-property and the corresponding projections are strong and parallel to $\Pi(X)$, then the set of all norm and numerical strong peak functions in $A_{wu}(B_X : X)$ is dense. We also prove that if a Banach space X has the (FPA)-property with $\{(\pi_i, F_i)\}_{i \in I}$, the corresponding projections are strong, parallel to $\Pi(X)$, and if each $\pi_i^* : X^* \rightarrow X^*$ is strong, then the set of all very strong numerical and norm strong peak functions is dense in $A_{wu}(B_X : X)$.

In Section 3, we extend the recent result of Ed-dari [13]. Let X be a complex Banach space and H a subspace of

$A_b(B_X : X)$. We introduce the H -numerical index by $N(H) := \inf\{\nu(f) : f \in H, \|f\| = 1\}$. When $H = \mathcal{P}^k(X : X)$ for some $k \geq 1$, the polynomial numerical index $N(H)$ is usually denoted by $n^{(k)}(X)$, which was first introduced and studied by Choi et al. [14]. We refer to [15–20] for some recent results about polynomial numerical index. For a norm-one projection π with range F and for any subspace H of $A_b(B_X : X)$, define $H_F = \{\pi \circ f \circ \pi|_F : B_F \rightarrow F : f \in H\}$. We prove that if X has the (FPA)-property with $\{(\pi_i, F_i)\}_{i \in I}$ and the corresponding projections are parallel to a numerical boundary of a subspace H , then $N(H) = \inf_{i \in I} N(H_{F_i})$. In fact, $N(H)$ is a decreasing limit of the right-hand side with respect to the inclusion partial order. If X is a real Banach space, we get a similar result (see Theorem 14). As a corollary we also extended Ed-dari's result to the polynomial numerical indices of l_p . In fact, Kim [17] extended Ed-dari's result [13, Theorem 2.1] to the polynomial numerical indices of (real or complex) l_p of order k as follows: Let $1 < p < \infty$ and $k \in \mathbb{N}$ be fixed. Then $n^{(k)}(l_p) = \inf\{n^{(k)}(l_p^m) : m \in \mathbb{N}\}$ and the sequence $\{n^{(k)}(l_p^m)\}_{m \in \mathbb{N}}$ is decreasing.

2. Banach Spaces with the (FPA)-Property and Denseness of Numerical Peak Holomorphic Functions

Following [21, Definition 1.g.1], a Banach space X has a *finite-dimensional Schauder decomposition* (FDD for short) if there is a sequence $\{X_n\}$ of finite-dimensional spaces such that every $x \in X$ has a unique representation of the form $x = \sum_{n=1}^{\infty} x_n$, where $x_n \in X_n$ for every n . In such a case, the projections given by $P_n(x) = \sum_{i=1}^n x_i$ are linear and bounded operators. If, moreover, for every $x^* \in X^*$, it is satisfied that $\|P_n^* x^* - x^*\| \rightarrow 0$, the FDD is called *shrinking*. The FDD is said to be *monotone* if $\|P_n\| = 1$ for every n .

The following proposition is easy to prove and its proof is omitted.

Proposition 1. *The following two conditions on a Banach space are equivalent.*

- (1) *A Banach space X has the (FPA)-property.*
- (2) *Given $\epsilon > 0$, $\{x_1, \dots, x_m\} \subset X$ and $\{x_1^*, \dots, x_m^*\} \subset X^*$, there is a norm-one projection $P : X \rightarrow X$ such that P has a finite rank, and for each $i = 1, \dots, m$ and for each $j = 1, \dots, n$, there exist $y_i \in X$ and $y_j^* \in X^*$ such that $\|Py_i - x_i\| \leq \epsilon$ and $\|P^* y_j^* - x_j^*\| \leq \epsilon$.*

Example 2. Assume that X is a complex Banach space satisfying at least one of the following conditions.

- (1) It has a shrinking and monotone finite-dimensional decomposition.
- (2) $X = L_p(\mu)$, where μ is a finite measure and $1 \leq p < \infty$.

Then X has the (FPA)-property.

Proof. Let $T : X \rightarrow F$ be a linear operator from X to a finite dimensional space F and G a finite dimensional subspace G

of X . Given $\epsilon > 0$, there is an $\epsilon/3$ -net $\{g_1, \dots, g_n\}$ in B_G and T can be written as $\sum_{i=1}^m x_i^* \otimes y_i$ for some $x_1^*, \dots, x_m^* \in X^*$ and $y_1, \dots, y_m \in F$.

(1) Suppose that X has a shrinking monotone finite-dimensional decomposition. Then there is $N \in \mathbb{N}$ such that

$$\max_{1 \leq i \leq m} \|y_i\| \cdot \sum_{i=1}^m \|P_N^* x_i^* - x_i^*\| \leq \epsilon, \quad \max_{1 \leq j \leq n} \|P_N g_j - g_j\| \leq \frac{\epsilon}{3}. \tag{5}$$

Then for any $x \in B_X$,

$$\begin{aligned} \|TP_N x - Tx\| &= \left\| \sum_{i=1}^m (P_N^* x_i^*)(x) y_i - \sum_{i=1}^m x_i^*(x) y_i \right\| \\ &\leq \max_{1 \leq j \leq n} \|y_j\| \cdot \sum_{i=1}^m \|P_N^* x_i^* - x_i^*\| \leq \epsilon, \end{aligned} \tag{6}$$

hence $\|TP_N - T\| \leq \epsilon$. For any $x \in B_G$, there is g_j such that $\|x - g_j\| \leq \epsilon/3$, then because the decomposition is monotone,

$$\begin{aligned} \|P_N x - x\| &\leq \|P_N(x - g_j)\| + \|P_N g_j - g_j\| + \|x - g_j\| \\ &\leq 2\|x - g_j\| + \|P_N g_j - g_j\| \leq \epsilon. \end{aligned} \tag{7}$$

So taking $P = P_N$, we obtained the desired result.

(2) Suppose that $X = L^p(\mu)$. We may assume that μ is a probability measure. For each $1 \leq i \leq m$, there is $s_i \in L_q(\mu)$ such that $1/p + 1/q = 1$ and $x_i^*(f) = \int f s_i d\mu$ ($f \in L_p(\mu)$). Then there is a sub- σ -algebra \mathcal{F} generated by finite disjoint subsets such that

$$\begin{aligned} \max_{1 \leq j \leq n} \|y_j\| \cdot \sum_{i=1}^m \|E(s_i | \mathcal{F}) - s_i\|_q &\leq \frac{\epsilon}{2}, \\ \max_{1 \leq i \leq n} \|E(g_i | \mathcal{F}) - g_i\|_p &\leq \frac{\epsilon}{3}. \end{aligned} \tag{8}$$

Define a projection $P : X \rightarrow X$ as $Pf = E(f | \mathcal{F})$. It is clear that P is a norm-one projection. For any $f \in B_X$,

$$\begin{aligned} \|TPf - Tf\| &= \left\| \sum_{i=1}^m (x_i^*)(Pf) y_i - \sum_{i=1}^m x_i^*(f) y_i \right\| \\ &\leq \max_{1 \leq j \leq n} \|y_j\| \cdot \sum_{i=1}^m |x_i^*(Pf) - x_i^*(f)| \\ &\leq \max_{1 \leq j \leq n} \|y_j\| \\ &\quad \cdot \sum_{i=1}^m \left| \int_K (E(f | \mathcal{F}) - f) E(s_i | \mathcal{F}) d\mu \right| \\ &\quad + \max_{1 \leq j \leq n} \|y_j\| \\ &\quad \cdot \sum_{i=1}^m \left| \int_K (E(f | \mathcal{F}) - f) (E(s_i | \mathcal{F}) - s_i) d\mu \right| \end{aligned}$$

$$\begin{aligned}
 &= 0 + \max_{1 \leq j \leq n} \|y_j\| \\
 &\quad \cdot \sum_{i=1}^m \left| \int_K (E(f | \mathcal{F}) - f) \right. \\
 &\quad \quad \left. \times (E(s_i | \mathcal{F}) - s_i) d\mu \right| \\
 &\leq \max_{1 \leq j \leq n} \|y_j\| \cdot 2 \sum_{i=1}^m \|f\|_p \|E(s_i | \mathcal{F}) - s_i\|_q \leq \epsilon.
 \end{aligned} \tag{9}$$

On the other hand, for any $f \in B_G$, there is g_j such that $\|f - g_j\| \leq \epsilon/3$. So

$$\begin{aligned}
 \|Pf - f\| &\leq \|P(f - g_j)\| + \|Pg_j - g_j\| + \|x - g_j\| \\
 &\leq 2\|f - g_j\| + \|Pg_j - g_j\| \leq \epsilon.
 \end{aligned} \tag{10}$$

We obtained the desired result. The proof is complete. \square

We will say that a k -linear mapping $L : X \times \dots \times X \rightarrow Y$ is of *finite-type* if it can be written as

$$L(x_1, \dots, x_k) = \sum_{i=1}^m x_{1,i}^*(x_1) \cdots x_{k,i}^*(x_k) y_i, \quad \forall x_1, \dots, x_k \in X \tag{11}$$

for some $m \in \mathbb{N}$, $x_{1,1}^*, \dots, x_{k,m}^*$ in X^* and y_1, \dots, y_m in Y . We will denote by $L_f^{(k)}(X : Y)$ the space of all k -linear mappings from X to Y of finite type. If a polynomial P is associated with such a k -linear mapping, we will say that it is a *finite-type polynomial*.

Proposition 3. *Suppose that a Banach space X has the (FPA)-property with $\{(\pi_i, F_i)\}_i$. Then the set of all polynomials $Q \in \mathcal{P}(X : X)$ such that there exists a projection $\pi_i : X \rightarrow F_i$ such that $\pi_i \circ Q \circ \pi_i = Q$ and $Q|_{F_i}$ is a norm and numerical peak function as a mapping from B_{F_i} to F_i is dense in $A_{wu}(B_X : X)$.*

Proof. We follow the ideas in [10]. The subset of continuous polynomials is always dense in $A_u(B_X : X)$. Given $f \in A_u(B_X : X)$ and $n \in \mathbb{N}$, it is the limit in $A_u(B_X : X)$ of sequence of functions $\{f_n\}_n$ defined by $f_n(x) := f((n/(n+1))x)$. Then f_n belongs to $A_b(((n+1)/n)B_X : X)$. Thus the Taylor series expansion of f_n at 0 converges uniformly on B_X for all n .

We will also use the fact that if $\sum_{k=0}^\infty P_k$ is the Taylor series expansion of $f \in A_{wu}(B_X : X)$ at 0, then P_k is weakly uniformly continuous on B_X for all k .

Since X has the (FPA)-property, X^* has the approximation property (see [22, Lemma 3.1]). Then the subspace of k -homogeneous polynomials of finite-type restricted on B_X is dense in the subspace of all k -homogeneous polynomials which are weakly uniformly continuous on B_X (see [1, Proposition 2.8]). Thus the subspace of the polynomials of finite-type restricted to the closed unit ball of X is dense in $A_{wu}(B_X : X)$.

Assume that P is a finite-type polynomial that can be written as a finite sum $P = \sum_{k=0}^n P_k$, where each P_k

is an homogeneous finite-type polynomial with degree k . Consider the symmetric k -linear form A_k associated with the corresponding polynomial P_k . Since P_k is a finite-type polynomial, then $T_k : X \rightarrow L_f^{(k-1)}(X : X)$ given by

$$T_k(x)(x_1, \dots, x_{k-1}) := A_k(x, x_1, \dots, x_{k-1}), \quad \forall x \in X \tag{12}$$

is a linear finite-rank operator for any $1 \leq k \leq n$.

The direct sum of these operators, that is, the operator

$$T : X \rightarrow \bigoplus_{k=1}^n L_f^{(k-1)}(X : X) \tag{13}$$

given by $T(x) := (T_1(x), \dots, T_n(x))$, for all $x \in X$, is also of finite rank.

By the assumption on X , given any $\epsilon > 0$, there is a norm-one projection $\pi := \pi_i : X \rightarrow X$ with a finite-dimensional range such that $\|T - T\pi\| \leq \epsilon$ and $\|\pi|_G - I_G\| \leq \epsilon$, where G is the span of $\bigcup_{k=1}^n P_k(X)$.

Let B_k be the symmetric k -linear mapping given by $B_k := A_k \circ (\pi, \dots, \pi)$, and let Q_k be the associated polynomial. It happens that $Q_k = P_k \circ \pi$. Now for $\|x\| \leq 1$, we have

$$\begin{aligned}
 &\|P_k \circ \pi(x) - P_k(x)\| \\
 &= \left\| \sum_{j=0}^{k-1} \binom{k}{j} A_k((x - \pi(x))^{k-j}, \pi(x)^j) \right\| \\
 &= \left\| \sum_{j=0}^{k-1} \binom{k}{j} (T_k - T_k \circ \pi)(x) ((x - \pi(x))^{k-j-1}, \pi(x)^j) \right\| \\
 &\leq \sum_{j=0}^{k-1} \binom{k}{j} \|T_k - T_k \circ \pi\| \|x\| \|x - \pi(x)\|^{k-j-1} \|\pi(x)\|^j \\
 &\leq \epsilon \sum_{j=0}^{k-1} \binom{k}{j} 2^{k-j-1} \leq 4^k \epsilon.
 \end{aligned} \tag{14}$$

Then $\|P_k \circ \pi - P_k\| \leq 4^k \epsilon$ and

$$\begin{aligned}
 &\|\pi \circ P_k \circ \pi - P_k\| \\
 &\leq \|\pi \circ P_k \circ \pi - \pi \circ P_k\| + \|\pi \circ P_k - P_k\| \leq 2 \cdot 4^k \epsilon.
 \end{aligned} \tag{15}$$

Let $R_k = \pi \circ P_k \circ \pi$ and $R = P_0 + \sum_{k=1}^n R_k$. Then $\|R - P\| \leq 2n4^n \epsilon$. By [5, Theorem 2.9], there is a numerical and norm peak polynomial $Q' : \pi(X) \rightarrow \pi(X)$ of degree $\leq n$ such that $\|R|_{\pi(X)} - Q'\| \leq \epsilon$. Setting $Q := Q' \circ \pi$, $\|P - Q\| \leq (2n4^n + 2)\epsilon$. The proof is done. \square

Remark 4. If X is a Banach space satisfying the (FPA)-property, then the set of polynomials in $B_{A_{wu}(B_X : X)}$ which has a nontrivial invariant subspace and has a fixed point is dense in $B_{A_{wu}(B_X : X)}$.

Notice that if X is locally uniformly convex, then every norm-one projection is strong. Indeed, suppose that if $\pi :$

$X \rightarrow F$ is a norm-one projection and if $\{\pi(x_k)\}_{k=1}^\infty$ in B_X converges to $y \in S_F$, then

$$\begin{aligned} 1 &= \lim_k \left\| \frac{\pi(x_k) + y}{2} \right\| \\ &= \lim_k \left\| \frac{\pi(x_k + y)}{2} \right\| \leq \lim_k \left\| \frac{x_k + y}{2} \right\| \leq 1 \end{aligned} \tag{16}$$

shows that $\lim_k \|x_k + y\| = 2$ and $\lim_k \|x_k - y\| = 0$ since X is locally uniformly convex.

The following lemma is proved in [5].

Lemma 5 (see [5]). *Let X be a complex Banach space and $f \in A_b(B_X : X)$. Suppose that there are $y \in B_X$ and $y^* \in B_{X^*}$ such that $|y^*(y)| = \|y^*\| \cdot \|y\|$. Then $|y^*(f(y))| \leq v(f)$. In particular, $\|f(0)\| \leq v(f)$.*

Theorem 6. *Suppose that a smooth Banach space X has the (FPA)-property with $\{\pi_i, F_i\}_{i \in I}$ and the corresponding projections are strong and parallel to $\Pi(X)$. Then the set of all numerical and norm strong peak functions in $A_{wu}(B_X : X)$ is dense.*

Proof. By Proposition 3, the set of all polynomials Q such that there exists norm-one projection $\pi := \pi_i : X \rightarrow F$ such that $\pi \circ Q \circ \pi = Q$ and $Q|_F$ is a norm and numerical peak function as a mapping from B_F to F is dense in $A_{wu}(B_X : X)$.

Fix corresponding Q and π and assume that $v_F(Q) = |y_0^*(Q(y_0))|$ and $\|Q(y_1)\| = \|Q\|$ for some $(y_0^*, y_0) \in \Pi(F)$ and $y_1 \in B_F$, where $v_F(Q)$ is the numerical radius of the map $Q|_F : B_F \rightarrow F$.

Suppose that there is a sequence $\{(x_k, x_k^*)\}_{k=1}^\infty$ in $\Pi(X)$ such that $\lim_k |x_k^*(Q(x_k))| = v(Q)$. Then

$$|\langle x_k^*, Q(x_k) \rangle| = |\langle x_k^*|_F, Q(\pi(x_k)) \rangle| \rightarrow v(Q). \tag{17}$$

We may assume that the sequence $\{(\pi(x_k), x_k^*|_F)\}_{k=1}^\infty$ converges to $(y, y^*) \in B_F \times B_{F^*}$ in the norm topology. So $v(Q) = |y^*(Q(y))| \geq v_F(Q)$. Since π is parallel to $\Pi(X)$, $|\langle y^*, y \rangle| = \|y^*\| \cdot \|y\|$. By Lemma 5,

$$v(Q) = |y^*(Q(y))| \leq v_F(Q). \tag{18}$$

So $v(Q) = |y^*(Q(y))| = v_F(Q)$. Since $Q|_F$ is a numerical peak function, $\|y\| = 1 = \|y^*\|$ and $y = y_0$ and $y^* = y_0^*$.

Since π is strong, $\lim_n x_n = y_0$. Let x^* be the weak- $*$ limit point of the sequence $\{x_n^*\}$. Then $x^*(y) = 1$ and $\|x^*\| = 1 = \|x^*|_F\|$, and

$$v(Q) = |x^*(Q(y))| = |y^*(Q(y))| = v_F(Q) \tag{19}$$

implies that $x^*|_F = y^*$ since $Q|_F$ is a numerical strong peak function. Hence x^* is unique because X is smooth. Therefore $\{x_n^*\}_{n=1}^\infty$ converges weak- $*$ to x^* . The proof is complete. \square

Theorem 7. *Suppose that a Banach space X space has the (FPA)-property with $\{\pi_i, F_i\}_{i \in I}$ and the corresponding projections are strong and parallel to $\Pi(X)$. One also assumes that each $\pi_i^* : X^* \rightarrow X^*$ is strong. Then the set of all very strong numerical and norm strong peak functions is dense in $A_{wu}(B_X : X)$.*

Proof. By Proposition 3, the set of all polynomials Q such that there exists norm-one projection $\pi := \pi_i : X \rightarrow F$ such that $\pi \circ Q \circ \pi = Q$ and $Q|_F$ is a norm and numerical peak function as a mapping from B_F to F is dense in $A_{wu}(B_X : X)$.

Fix corresponding Q and π and assume that $v_F(Q) = |y_0^*(Q(y_0))|$ and $\|Q(y_1)\| = \|Q\|$ for some $(y_0^*, y_0) \in \Pi(F)$ and $y_1 \in B_F$, where $v_F(Q)$ is the numerical radius of the map $Q|_F : B_F \rightarrow F$.

Suppose that there is a sequence $\{(x_k, x_k^*)\}_{k=1}^\infty$ in $\Pi(X)$ such that $\lim_k |x_k^*(Q(x_k))| = v(Q)$. Then

$$|\langle x_k^*, Q(x_k) \rangle| = |\langle x_k^*|_F, Q(\pi(x_k)) \rangle| \rightarrow v(Q). \tag{20}$$

We may assume that the sequence $\{(\pi(x_k), x_k^*|_F)\}_{k=1}^\infty$ converges to $(y, y^*) \in B_F \times B_{F^*}$ in the norm topology. So $v(Q) = |y^*(Q(y))| \geq v_F(Q)$. Since π is parallel to $\Pi(X)$, $|\langle y^*, y \rangle| = \|y^*\| \cdot \|y\|$. By Lemma 5,

$$v(Q) = |y^*(Q(y))| \leq v_F(Q). \tag{21}$$

So $v(Q) = |y^*(Q(y))| = v_F(Q)$. Since $Q|_F$ is a numerical peak function, $\|y\| = 1 = \|y^*\|$ and $y = y_0$ and $y^* = y_0^*$.

Since π is strong, $\lim_n x_n = y_0$. Fix $z^* \in S_{X^*}$ to be a Hahn-Banach extension of y^* . Let x^* be the weak- $*$ limit point of the sequence $\{x_n^*\}_{n=1}^\infty$. Then $x^*(y) = 1$ and $\|x^*\| = 1 = \|\pi^*(x^*)\|$ and

$$v(Q) = |x^*(Q(y))| = |y^*(Q(y))| = v_F(Q) \tag{22}$$

implies that $\pi^*(x^*)|_F = y^*$ since $Q|_F$ is a numerical strong peak function so $\pi^*(x^*) = \pi^*(z^*)$.

Hence $\lim_n \pi^*(x_n^*) = \pi^*(z^*)$ and $\|\pi^*(z^*)\| = 1$. Now we get $\|x_n^* - \pi^*(z^*)\| \rightarrow 0$ by the assumption. This shows that $\lim_n \|x_n^* - \pi^*(z^*)\| = 0$. Therefore $x^* = \pi^*(z^*)$ and Q is a very strong numerical peak function at $(y, \pi^*(z^*))$. This completes the proof. \square

Corollary 8. *Suppose that $X = \ell_p$ with $1 < p < \infty$. Then the set of all very strong numerical and norm strong peak functions is dense in $A_{wu}(B_X : X)$.*

Proof. Let $\{\pi_i, F_i\}_{i=1}^\infty$ be a projection consisting of i th natural projections. Then these projections satisfy the conditions in Theorem 7. The proof is done. \square

3. Generalized Numerical Index

Proposition 9. *Let X be a (real or complex) Banach spaces and let H be a closed subspace of $C_b(B_X : X)$. If X has the (FPA)-property with $\{\pi_i, F_i\}_{i \in I}$, then $N(H) \geq \inf_{i \in I} N(H_{F_i})$. In particular, $n^{(k)}(X) \geq \inf_{i \in I} n^{(k)}(F_i)$ for each $k \geq 1$.*

Proof. Let $f \in S_H$. Given $\epsilon > 0$, there is a norm one projection π with a finite dimensional range F such that $\|\pi \circ f \circ \pi\| \geq 1 - \epsilon$. Let $g = \pi \circ f \circ \pi|_F$ as a map in H_F and

$$v_F(g) \geq N(H_F) \|g\| \geq N(H_F) (1 - \epsilon). \tag{23}$$

Then there is $(y, y^*) \in \Pi(H_F)$ such that $v_{H_F}(g) = |y^*(g(y))|$ since F is finite dimensional. Notice that $(y, \pi^*(y^*)) \in \Pi(X)$ and so

$$v_F(g) = |\pi^* y^*(f(\pi(y)))| = |\pi^* x^*(f(y))| \leq v_H(f). \tag{24}$$

Hence $v_H(f) \geq (1-\epsilon)N(H_F) \geq (1-\epsilon)\inf_{i \in I} N(H_{F_i})$. Therefore $N(H) \geq \inf_{i \in I} N(H_{F_i})$. \square

Proposition 10. *Let X be a complex Banach space and let H be a subspace of $A_b(B_X : X)$ with a numerical boundary Γ . Suppose that a norm-one finite dimensional projection (π, F) is parallel to Γ . Then for any $f \in H_F$,*

$$v_F(f) = v_X(f \circ \pi), \tag{25}$$

where $v_X(f \circ \pi)$ is a numerical radius as a function $f \circ \pi : B_X \rightarrow X$.

Proof. It is clear that $v_F(f) \leq v_X(f \circ \pi)$. For the converse, choose a sequence $\{(x_n, x_n^*)\}_{n=1}^\infty$ in Γ such that

$$v_X(f \circ \pi) = \lim_n |x_n^*(f(\pi(x_n)))| = \lim_n \langle x_n^*|_F, f(\pi(x_n)) \rangle. \tag{26}$$

Since $\{\pi(x_n)\}_{n=1}^\infty$ is in the finite dimensional space F , we may assume that $\{\pi(x_n)\}_{n=1}^\infty$ converges to $y \in B_F$ and $\{x_n^*|_F\}_{n=1}^\infty$ converges to $y^* \in B_{F^*}$. Then $|\langle y^*, y \rangle| = \|y^*\| \cdot \|y\|$. Thus by Lemma 5,

$$v_X(f \circ \pi) = |y^*(f(y))| \leq v_F(f). \tag{27}$$

The proof is complete. \square

For the real Banach spaces, we get the following lemma for a homogeneous polynomial.

Lemma 11. *Let X be a real or complex Banach space, and let f be a k -homogeneous polynomial. If there are $y \in B_X$ and $y^* \in B_{X^*}$ such that $|y^*(y)| = \|y^*\| \cdot \|y\|$, then $|y^*(f(y))| \leq v(f)$.*

Proof. If $y^* = 0$, then it is clear. So we may assume that $y^* \neq 0$. We may assume that $y \neq 0$. The

$$\begin{aligned} |y^*(f(y))| &\leq \left| \frac{y^*}{\|y^*\| \|y\|^k} (f(y)) \right| \\ &= \left| \frac{y^*}{\|y^*\|} \left(f \left(\frac{y}{\|y\|} \right) \right) \right| \leq v(f). \end{aligned} \tag{28}$$

This completes the proof. \square

If we use Lemma 11 instead of Lemma 5 in the proof of Proposition 10, we get the following.

Proposition 12. *Let X be a real or complex Banach space, and let Γ be a numerical boundary of $\mathcal{P}^k(X : X)$, where k is a natural number. Suppose that a norm-one finite dimensional projection (π, F) is parallel to Γ . Then for any $f \in \mathcal{P}^k(F : F)$,*

$$v_F(f) = v_X(f \circ \pi), \tag{29}$$

where $v_X(f \circ \pi)$ is a numerical radius as a function $f \circ \pi : B_X \rightarrow X$.

Now we get the extensions of the results of Ed-dari [13] and Kim [17] in the complex case.

Theorem 13. *Let X be a complex Banach space, and let H be a subspace of $A_b(B_X : X)$ with a numerical boundary Γ . Suppose that the Banach space X has the (FPA)-property with $\{\pi_i, F_i\}_{i \in I}$ and that the corresponding projections are parallel to Γ . Then*

$$N(H) = \inf_{i \in I} N(H_{F_i}). \tag{30}$$

In fact, $N(H)$ is a decreasing limit of the right-hand side with respect to the inclusion partial order.

Proof. For any $f \in H_F$, $v_{F_i}(f) = v_X(f \circ \pi_i)$ by Proposition 10. $v_{F_i}(f) = v_X(f \circ \pi) \geq \|f \circ \pi\| N(H) = \|f\| N(H)$. Hence $N(H_{F_i}) \geq N(H)$ and it is easy to see that if $F_i \subset F_j$, then $N(H) \leq N(H_{F_j}) \leq N(H_{F_i})$. Hence $N(H) \leq \inf_{i \in I} N(H_{F_i})$. The converse is clear by Proposition 9. \square

For the general case we get a similar result about the polynomial numerical index if we use Proposition 12 in the proof of Theorem 13.

Theorem 14. *Let X be a real or complex Banach space, and let Γ be a numerical boundary of $\mathcal{P}^k(X : X)$, where k is a natural number. Suppose that X has the (FPA)-property with $\{\pi_i, F_i\}_{i \in I}$ and that the corresponding projections are parallel to Γ . Then*

$$n^{(k)}(X) = \inf_{i \in I} n^{(k)}(F_i). \tag{31}$$

In fact, $n^{(k)}(X)$ is a decreasing limit of the right-hand side with respect to the inclusion partial order.

Proposition 15. *Let X be a real Banach space, and let Γ be a numerical boundary of $\mathcal{P}^k(X : X)$, where k is a natural number. Suppose that X has the (FPA)-property with $\{\pi_i, F_i\}_{i \in I}$ and that the corresponding projections are parallel to Γ . If $n^{(k)}(X) = 1$ and $k \geq 2$, then X is one-dimensional.*

Proof. We will use the fact [20] that if X is a real finite-dimensional Banach space with $n^{(k)}(X) = 1$ and $k \geq 2$, then X is one-dimensional. By Theorem 14, we get $1 = n^{(k)}(X) = \inf_{i \in I} n^{(k)}(F_i)$ and $n^{(k)}(F_i) = 1$ for all $i \in I$ and F_i 's are one-dimensional. Suppose on the contrary that X is not one-dimensional. Then we can choose two dimensional subspace G , and there is $i \in I$ with $\|id_G - \pi_i|_G\| \leq 1/2$. Then there are $x^* \in S_{X^*}$ and $a \in X$ such that $\pi_i(x) = x^*(x)a$ for all $x \in X$. Because G is two-dimensional, there exists $w \in G \cap S_X$ with $x^*(w) = 0$. So $\|w - \pi_i(w)\| = \|w\| \leq 1/2$, which is a contradiction to $\|w\| = 1$. Therefore, X is one-dimensional, and the proof is done. \square

Example 16. Let $(F_i)_{i=1}^\infty$ be a sequence of finite-dimensional Banach spaces, and consider the following spaces. For each $1 < p < \infty$,

$$X_p := \{(x_i)_{i=1}^\infty : x_i \in F_i, (\|x_i\|)_{i=1}^\infty \in \ell_p\} \tag{32}$$

with the norm

$$\|(x_i)_{i=1}^\infty\| = \left(\sum_{i=1}^\infty \|x_i\|^p \right)^{1/p} \quad (33)$$

is a Banach space with the shrinking and monotone finite-dimensional decomposition with the projections

$$P_n((x_i)_{i=1}^\infty) = (x_1, \dots, x_n, 0, 0, \dots), \quad (n \in \mathbb{N}). \quad (34)$$

The space $X_0 := \{(x_i)_{i=1}^\infty : x_i \in F_i, (\|x_i\|)_{i=1}^\infty \in c_0\}$ with the norm $\|(x_i)_{i=1}^\infty\| = \sup_{i \in \mathbb{N}} \|x_i\|$ is also a Banach space with the shrinking and monotone finite-dimensional decomposition with the same projections P_n . Then it is easy to check that $\Pi(X_p)$ is parallel to the projections $(P_n, P_n(X_p))$ for each $1 < p < \infty$ and $p = 0$. So we get the following result. For each $k \in \mathbb{N}$ and each $1 < p < \infty$ (or $p = 0$),

$$n^{(k)}(X_p) = \lim_{n \rightarrow \infty} n^{(k)}(X_p^n), \quad (35)$$

where $X_p^n = F_1 \oplus_p F_2 \oplus_p \dots \oplus_p F_n$ and for complex Banach spaces; we get

$$N(A(B_{X_p} : X_p)) = \lim_{n \rightarrow \infty} N(A(B_{X_p^n} : X_p^n)). \quad (36)$$

Corollary 17. *Let $k \geq 1$ be a natural number and $1 < p < \infty$. Then for a real or complex case,*

$$\lim_{m \rightarrow \infty} n^{(k)}(\ell_p^m) = n^{(k)}(\ell_p) \leq n^{(k)}(L_p(0, 1)). \quad (37)$$

For the complex case we get

$$\begin{aligned} & \lim_{m \rightarrow \infty} N(A_b(B_{\ell_p^m} : \ell_p^m)) \\ &= N(A_b(B_{\ell_p} : \ell_p)) \leq N(A_b(B_{L_p(0,1)} : L_p(0, 1))). \end{aligned} \quad (38)$$

Proof. We give only the first part, since the proof of the next is similar. Let $H = \mathcal{P}({}^k \ell_p)$. Then ℓ_p has the (FPA)-property with projections $\{\pi_i, \ell_p^i\}_{i=1}^\infty$, where each π_i is the i th natural projection. Notice that given projections are parallel to $\Pi(X)$. Hence $N(H) = \inf_{i \in I} N(H_{F_i})$ by Theorem 13. Notice that H_{F_i} is isometrically isomorphic to $\mathcal{P}({}^k \ell_p^i)$.

On the other hand, if we let $H = \mathcal{P}({}^k L_p(0, 1))$. Then $L_p(0, 1)$ has (FPA)-property with projections $\{\pi_i, F_i\}$, where each π_i is the conditional expectation with respect to the sub- σ -algebra generated by finitely many disjoint subsets. Hence $N(H) \geq \inf_{i \in I} N(H_{F_i})$. Notice also that F_i is isometrically isomorphic to ℓ_p^m for some m . So H_{F_i} is isometrically isomorphic to $\mathcal{P}({}^k \ell_p^m)$. The proof is complete. \square

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