An Efficient Matched Filtering Algorithm for the Detection of Continuous Gravitational Wave Signals

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INTRODUCTION

Neutron stars are perhaps the most promising class of gravitational wave (GW) sources, and searches for such GW signals is particularly suited to the characteristics of the GEO600 detector (see Schutz "Getting Ready for GEO600 Data" gr–qc9910033). However, the instantaneous GW frequency of such a source will evolve due to both intrinsic spindown effects and Doppler modulations induced by the motion of the Earth. Thus because of the large parameter space of likely signals, directly implemented optimal matched filtering is not computationally feasible.

In response to this problem, Schutz and Papa have developed an alternative strategy: the Hough–Hierarchical search algorithm (see Schutz and Papa "End–to–End Algorithm for Hierarchical Area Searches for Long–Duration GW Sources for GEO600" gr–qc9905018). In order to carry out a blind search over a range of intrinsic GW frequencies, the following three stages must be calculated for each point in the parameter space of sky positions and intrinsic spindown parameters:

Stage I: Calculate demodulated Fourier transforms (DeFTs) on an intermediate time baseline (of order 1 day) by combining FFTs of short durations (approximately 30 minutes) of the time series data. In this context demodulated means that if there is a source at the sky position in question, and with the intrinsic spindown parameters in question, then all spindown and modulatory effects will have been correctly removed from the DeFTs: all signal power will be confined to one and the same frequency bin in each DeFT. This frequency is the intrinsic frequency of the source measured at the start of the observing time. It is expected that the total observing time will be of order 4 months, and thus roughly 120 of these DeFTs will be calculated for each point in parameter space.

Stage II: In general source parameters will not coincide exactly with those searched for, and residual frequency evolution and modulation will remain in the DeFTs. Thus, the peak in power associated with a given source may

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change frequency bins from DeFT to DeFT. Because of the relatively small time baseline of these DeFTs and the resultant poor signal–to–noise of any expected continuous GW signal, this evolution will be not directly apparent in the DeFTs, but can be recovered statistically using the Hough Transform algorithm.

Stage III: Calculate DeFTs for candidate sources with the full frequency resolution of the total observation time, by combining the intermediate baseline DeFTs produced in stage I.

Thus, during stage II, regions of the parameter space in which it is statistically unlikely that there are GW sources are eliminated from the search. Thereby, in stage III, the most computationally expensive part of the algorithm, the long time baseline DeFTs are calculated over only a very small fraction of parameter space and over a very small range of frequencies.

In this paper we outline the methods used in the first and third stages of this algorithm in constructing a longer time baseline DeFT from a number of shorter time baseline FFTs or DeFTs.

THE METHOD

Consider a time series x_a of total duration T, which has been divided into M short time series, each having N data points. Then the DeFT for a signal with a time independent amplitude and phase $2\pi\Phi_{ab}(\vec{\lambda})$ is

$$\hat{x}_b(\vec{\lambda}) = \sum_{a=0}^{NM-1} x_a e^{-2\pi i \Phi_{ab}(\vec{\lambda})} = \sum_{\alpha=0}^{M-1}\sum_{j=0}^{N-1} x_{\alpha j} e^{-2\pi i \Phi_{\alpha j b}(\vec{\lambda})}, \tag{1}$$

where the time indices are related by $N\alpha + j = a$, and b is a long time baseline frequency index. In the following discussion Latin indices j, k, l always sum over N, while Greek indices sum over M. Note that $\Phi_{ab}(\vec{\lambda})$ is dependent on a vector $\vec{\lambda}$ of parameters which characterize the signal one is searching for. In searching for GW signals from neutron stars these will include intrinsic spindown parameters, and the position of the source in the sky. If $\tilde{x}_{\alpha k}$ is the matrix formed by carrying out Fourier transforms along the short time index j in $x_{\alpha j}$, then equation 1 can be written as

$$\hat{x}_b(\vec{\lambda}) = \sum_{\alpha=0}^{M-1}\sum_{k=0}^{N-1} \tilde{x}_{\alpha k} \left[\frac{1}{N}\sum_{j=0}^{N-1} e^{-2\pi i\left(\Phi_{\alpha j b}(\vec{\lambda}) - \frac{jk}{N}\right)}\right] = \sum_{\alpha=0}^{M-1} Q_\alpha(b,\vec{\lambda}) \sum_{k=0}^{N-1} \tilde{x}_{\alpha k} P_{\alpha k}(b,\vec{\lambda}), \tag{2}$$

where the product $Q_\alpha(b,\vec{\lambda})P_{\alpha k}(b,\vec{\lambda})$ is defined by the terms in square brackets, and $Q_\alpha(b,\vec{\lambda})$ contains all parts of the square brackets independent of the short time index j and short frequency index k.

In equation 2 we have effectively re–written equation 1, a long time baseline DeFT in the time domain, as a sum (α index) of short time baseline DeFTs in the frequency domain (k index), where $Q_\alpha(b,\vec{\lambda})P_{\alpha k}(b,\vec{\lambda})$ are these frequency domain filters. In the presence of stationary noise with a flat spectrum, equation 2 is the optimal detector. However, through applying various approximations, the detector can be made "acceptably sub–optimal", in the sense that only a small fraction of power from a signal is lost in comparison to the optimal case, while achieving vast savings in computational cost.

To illustrate these mathematical approximations it is instructive to discuss equation 2 for a specific case of $\Phi_{\alpha jb}(\vec{\lambda})$: a linearly varying frequency model, i.e. in the continuum limit $\Phi(t,f_0,\dot{f}_0) = f_0 t + \dot{f}_0 t^2$, where f_0 and $\dot{f}_0$ are the intrinsic frequency and spindown of the source respectively, and t is time. In the case of an actual search for GW signals from pulsars, $\Phi_{\alpha jb}(\vec{\lambda})$ will not be so simple. However, this model is sufficiently complex to effectively demonstrate all of the approximations to be discussed here.

In discrete form, $\Phi(t,f_0,\dot{f}_0)$ can be written as $\Phi_{\alpha j\beta l}(\gamma) = (\beta+Ml)(N\alpha+j)/NM + \gamma(N\alpha+j)^2/N^2M^2$, where the long time baseline frequency index $b = \beta + Ml$. The chosen discretization of the spindown parameter $\dot{f}_0 \equiv \gamma/T^2$ is not practically appropriate. However, in an actual search, a grid of points in spindown parameter space will be chosen to ensure an acceptable loss of power from unresolved signals. Thus, in the following discussion, only searches for resolved $\dot{f}_0$ parameters will be considered.

Approximation 1: By Taylor expanding the model phase function $\Phi(t)$ about the middle of each short duration time series (i.e. about $j = N/2$) and discarding terms of order $(j/N)^2 \equiv t^2$ and higher, in the limit $N \to \infty$ the function $P_{\alpha k}(\beta,l,\gamma)$ is Re $P_{\alpha k}(\beta,l,\gamma) = \text{sinc}\, x$ and Im $P_{\alpha k}(\beta,l,\gamma) = (1-\cos x)/x$. In the phase model considered here $x = -2\pi\,(\beta/M + l + (2\alpha+1)\gamma/M^2 - k)$ and $Q_\alpha(\beta,l,\gamma) = \exp\{-2\pi i(\alpha\beta/M + \alpha\gamma^2/M^2)\}$.

Approximation 2: Consider the case where the short time baseline is chosen such that the instantaneous model frequency $f(t) = \dot{\Phi}(t,f_0,\dot{f}_0,\ddot{f}_0,\ldots)$ does not move by more than one short time baseline frequency bin over the duration of a short time baseline data set, i.e. in the model discussed here $|\dot{f}_0|T/M < M/T$. Then for a given α, the function $P_{\alpha k}(b,\vec{\lambda})$ will be peaked in power about the model frequency averaged over the duration of time associated with the αth short data set, i.e. about $x = 0$ (the first three terms in the above definition of x are the index of this average model frequency). Thus only a few terms around this model frequency will contribute significantly to the summation over k in equation 2.

Approximation 3: The semi–periodic nature of $P_{\alpha k}(b,\vec{\lambda})$ means that this function can be efficiently evaluated from a look–up table of values containing the periodic parts, and three further operations: to calculate one instance of $P_{\alpha k}(b,\vec{\lambda})$ will require only 8 floating point operations.

Approximation 4: If one approximates the model frequency parameter β in the calculation of $P_{\alpha k}(\beta, l, \gamma)$ as a fixed value, for example with $\beta = \beta_0$, equation 2 can be calculated as an FFT, i.e.

$$\hat{x}_{\beta l}(\gamma) = \sum_{\alpha=0}^{M-1} \left[Q'_{\alpha}(\gamma) \sum_{k}^{n_{term}} \tilde{x}_{\alpha k} P_{\alpha k}(\beta_0, l, \gamma) \right] e^{-2\pi i \frac{\alpha\beta}{M}}, \tag{3}$$

where n_{term} relates to approximation 2, $P_{\alpha k}(\beta, l, \gamma)$ is defined above, and for the phase model discussed here $Q'_{\alpha}(\gamma) = \exp\{-2\pi i (\alpha^2\gamma/M^2 - \gamma/4M^2)\}$. Thus for values of β sufficiently near to β_0, the loss in power due to this approximation will be small. To obtain $\hat{x}_{\beta l}(\gamma)$ for other values of β, the calculation must be repeated using another β_0.

RESULTS AND DISCUSSION

Numerical tests have shown that if one chooses 10% as an acceptable loss in power in comparison to the optimal case, then $N_{FFT} = 8$ and $n_{term} = 16$ are the preferred parameter combination, if the short time baseline T/M is chosen such that in the phase model discussed here $|\dot{f}_0|T/M < M/T$. If one decides that only a 5% loss in optimal power is acceptable, then this can be achieved with the same parameters, but choosing T/M such that $|\dot{f}_0|T/M < M/2T$.

The computational cost of calculating one DeFT in stage I in floating point operations is

$$C_{DeFT} \simeq 5.3 \times 10^{10} \left(\frac{B}{300\ \text{Hz}}\right) \left(\frac{T}{1\ \text{day}}\right) \left(\frac{N_{FFT}}{8}\right) \left(\frac{n_{term.}}{16}\right), \tag{4}$$

where B is the bandwidth of the search. This is comparable to the computational cost of the corresponding steps in the Hierarchical Stack / Slide algorithm of Brady and Creighton ("Searching for Periodic Sources with LIGO: Hierarchical Searches" gr–qc9812014). The Hough–Hierarchical search algorithm also has a number of computational advantages. To calculate a given bandwidth of a DeFT requires only the FFT data from this bandwidth and an additional small overlap. Thus the algorithm can be easily parallelized by distributing data and work by bandwidth; and no communication between processors is required. Also, the complete three stage algorithm can be arranged in such a way that once a bandwidth of FFT data is read from disk by a processor, all computation required on this data can be carried out while this data is held in memory, thus time spent reading data from disk is a negligible fraction of the total computational time: each processor will need to read roughly 40 Mb from disk once every two weeks. Furthermore, little additional memory is required as workspace for stages I and III: less than 100 kb.

The GEO600 data analysis team are currently working on coding this algorithm in a computationally optimal manner, as well as integrating this with the Hough Transform part of the procedure.