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# Two-dimensional iterated morphisms and discrete planes

Pierre Arnoux[a,*], Valérie Berthé[b], Anne Siegel[c]

[a] *IML UPR-CNRS 9016, Campus de Luminy case 907, 13288 Marseille Cedex 9, France*
[b] *LIRMM UMR-CNRS 5506, 161 rue Ada, 34392 Montpellier Cedex 5, France*
[c] *IRISA UMR-CNRS 6074,Campus de Beaulieu, 35042 Rennes Cedex, France*

**Abstract**

Iterated morphisms of the free monoid are very simple combinatorial objects which produce infinite sequences by replacing iteratively letters by words. The aim of this paper is to introduce a formalism for a notion of two-dimensional morphisms; we show that they can be iterated by using local rules, and that they generate two-dimensional patterns related to discrete approximations of irrational planes with algebraic parameters. We associate such a two-dimensional morphism with any usual Pisot unimodular one-dimensional iterated morphism over a three-letter alphabet.


*Keywords:* Discrete plane; Multidimensional combinatorics on words; Iterated morphism; $\mathbb{Z}^2$-action; Tiling

## 1. Introduction

Iterated morphisms (also called *substitutions* or *inflation rules*) are very simple combinatorial objects which produce infinite sequences by iteration: roughly speaking, a morphism replaces a letter by a word. They can be seen as one of the mathematical translations of a macro in computer science (replacement of the name of the macro by its definition). These morphisms are widely studied and have a rich structure, shown by their natural interactions with combinatorics on words, ergodic theory, linear algebra, spectral theory, geometry of tilings, theoretical computer science, diophantine approximation, transcendence, graph theory, and so on (see [17] and the references in [1,16]).

* Corresponding author.
*E-mail addresses:* arnoux@iml.univ-mrs.fr (P. Arnoux), berthe@iml.univ-mrs.fr (V. Berthé), anne.siegel@irisa.fr (A. Siegel).

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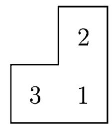



Fig. 1. A nonpointed pattern (with no location information).

This paper aims at introducing a formalism and some examples for a notion of two-dimensional morphism that can be iterated, either by means of global placing rules, or by local rules. One of the purposes of the introduction of such a device is to make possible the iteration of the two-dimensional morphism to get in specific cases an explicit construction of a discrete approximation of a plane.

Since we will in the rest of the paper try to extend the theory to higher dimension, let us point here a "trivial" fact in dimension 1 which becomes much more complicated in higher dimension: letters in finite words are naturally ordered by their rank of apparition. As a consequence, the set of finite words has a structure of monoid, that is, two finite words $W_1, W_2$ can be naturally "combined" to give the word $W_1W_2$ by putting the two words side by side. This allows a simple definition of iterated morphisms using the rule $\sigma(W_1W_2)=\sigma(W_1)\sigma(W_2)$. This definition is obviously consistent, and can be extended in a natural way to finite and infinite one-dimensional sequences.

### 1.1. Two-dimensional patterns

It is a mathematical reflex to try to extend a one-dimensional theory to several dimensions. But the theory of words seems so strongly one-dimensional that the tentative might seem artificial in this case, although it is quite fun to work on. However, a number of recent advances in mathematics and physics (tilings, quasi-crystals, $\mathbb{Z}^d$-actions and higher-dimensional symbolic dynamics, see for instance [5,14,19,20]) point to the need of a good theory of higher-dimensional words. The basic setting is not yet completely clear. In particular, many results in tilings seem to depend on Delone sets or similar sets, which have weaker structures than lattices. The theory seems difficult in this case, and we will restrict in this paper to the first nontrivial case: two-dimensional infinite sequences, seen as sequences $(U_{i,j})$ indexed by $\mathbb{Z}^2$.

Following the definition of one-dimensional words, it is natural to define two-dimensional words as geometrical patterns that contain no information on the location of the pattern inside $\mathbb{Z}^2$. More precisely, a two-dimensional *nonpointed pattern* is a map from a finite subset of $\mathbb{Z}^2$ to the alphabet up to a translation. A three-letter example is given by a 1, put on the left-hand side of a 3 and below a 2 (see Fig. 1).

The main problem is that unlike the one-dimensional case, there is here no natural monoid structure: there is no privileged way to put two finite nonpointed patterns side by side. It is also very unclear whether a given collection of nonpointed patterns can tile the plane, while in the case of one-dimensional words, this is obviously always possible. Remark however that, in the particular case of rectangular patterns of the same size, it is possible to define two canonical operations (putting one rectangle on the side, or on the top, of the other) [10].

### 1.2. Two-dimensional substitution rules

We want to define in this setting two-dimensional morphisms. Following the one-dimensional definition, we call a *two-dimensional substitution rule* a map that associates with each letter a finite (two-dimensional) nonpointed pattern. To be called a *morphism*, we need to be able to apply this substitution rule, not only to letters, but also to patterns and sequences.

Associating with each letter a nonpointed pattern is not enough to realize this: we first need more information to know where to place the image of the letter at the origin (there is an obvious solution in the one-dimensional case: it is natural to place at the origin the first letter of the word image of the initial letter); then, we need to know the relative locations of the patterns substituted to adjacent letters, and a consistency problem arises. Indeed, it is not clear that there is a good way to apply the morphism to a finite pattern or to a two-dimensional sequence: there might be overlaps. Furthermore, if this is possible, do we obtain in this way a fixed point? Can we obtain all of a two-dimensional sequence?

Note that there is one case where the existence of the two-dimensional iterated morphism is not problematic, and the theory is quite easy: if we associate to each letter a rectangular pattern of fixed size, it is clear that the image of any pattern or sequence is well defined; one can consider that such a morphism naturally splits into one-dimensional iterated morphisms [1,11].

Our motivation in this paper is to show, on a nontrivial example (that is, not rectangular), that the obstructions above can be overcome. The answer to the iteration problem itself is not easy to prove, and requires some additional geometric constructions. Indeed, in the rest of the paper, we study a very simple example of a two-dimensional substitution rule on the three-letter alphabet $\{1,2,3\}$:

$$\Sigma_0:\quad 1 \mapsto \begin{matrix}2\\1\end{matrix} \qquad 2 \mapsto 3 \qquad 3 \mapsto 1.$$

It is clear on the first letter that this formula is not a usual one-dimensional morphism. However, the way to extend the definition from letters to finite nonpointed patterns and sequences is unclear: for example, should the image of 12 be $\begin{matrix}2 & 3\\1 & \end{matrix}$ or $\begin{matrix} & 2\\3 & 1\end{matrix}$, or some other possibility?

We will give two solutions to this problem: we endow a two-dimensional morphism $\Sigma_0$ first with global rules $\mathbf{v}$, second with a set of local rules $\mathscr{S}$ and an initial rule $\mathscr{I}$. Local rules are more convenient to iterate, but it is easier to prove consistency for global rules; we will prove consistency for $(\Sigma_0, \mathbf{v})$ on a particular sequence $U$, and then prove that $(\Sigma_0, \mathbf{v})$ and $(\Sigma_0, \mathscr{I}, \mathscr{S})$ act in the same way on $U$; this will prove consistency of the local rules on this particular sequence which is a fixed point of both morphisms.

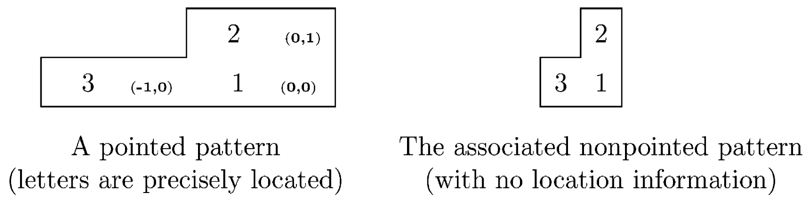



Fig. 2. The pointed pattern $((\mathbf{0},\mathbf{0}),1)((\mathbf{0},\mathbf{1}),2)((-\mathbf{1},\mathbf{0}),3)$ and its nonpointed pattern.

### 1.3. Global placing rules

A first step to solve the problem of iterating such a two-dimensional substitution rule is to extend its definition to pointed patterns: let us define a two-dimensional *pointed pattern* as a map from a finite subset of $\mathbb{Z}^2$ to the alphabet (so that two pointed patterns represent the same nonpointed pattern if and only if they are translate of each other; the difference between nonpointed and pointed patterns is the same as the difference between a finite word, and its occurrences in a sequence). The set of pointed patterns is denoted by $\mathscr{L}^*$, that is, the set of finite words on the alphabet $\mathscr{L}=\mathbb{Z}^2\times\{1,2,3\}$ with some combinatorial restrictions (the pointed letters have different locations, see Section 2).

An example of a pointed pattern is given by a 1 located at the index $(0,0)$, a 2 at the index $(0,1)$ and a 3 at the index $(-1,0)$. This pointed pattern can be written as the word $((\mathbf{0},\mathbf{0}),1)((\mathbf{0},\mathbf{1}),2)((-\mathbf{1},\mathbf{0}),3)$. The underlying nonpointed pattern is shown in Fig. 2. The interest of pointed patterns is that a natural structure lies on $\mathscr{L}^*$, that is, the concatenation. A two-dimensional substitution rule $\Sigma$ associates with each letter a nonpointed pattern. To extend its definition to pointed patterns, we add a *global placing rule* $\mathbf{v}$: for each nonpointed pattern $\Sigma(i)$, choose a special pointed pattern that represents it; to a letter $i$ in position $(\mathbf{m},\mathbf{n})$, the global rule associates a pointed pattern $\Sigma(i)+\mathbf{v}((\mathbf{m},\mathbf{n}),i)$ which is a translate of the special representative of $\Sigma(i)$ by a translation vector $\mathbf{v}((\mathbf{m},\mathbf{n}),i)$. We will define such global placing rules precisely in Section 2. For example, global placing rules for the substitution rule $\Sigma_0$ are given by $\mathbf{v}((\mathbf{m},\mathbf{n}),i)=(\mathbf{1}-\mathbf{n},\mathbf{m}-\mathbf{n}-\mathbf{r}(\mathbf{m},\mathbf{n}))$, where $\mathbf{r}$ is an explicit function.

$$
\begin{aligned}
(\Sigma_0,\mathbf{v}):\ &((\mathbf{m},\mathbf{n}),1)\mapsto((\mathbf{1}-\mathbf{n},-\mathbf{1}+\mathbf{m}-\mathbf{n}-\mathbf{r}),1)((\mathbf{1}-\mathbf{n},\mathbf{m}-\mathbf{n}-\mathbf{r}),2)\\
&((\mathbf{m},\mathbf{n}),2)\mapsto((\mathbf{1}-\mathbf{n},\mathbf{m}-\mathbf{n}-\mathbf{r}),3)\\
&((\mathbf{m},\mathbf{n}),3)\mapsto((\mathbf{1}-\mathbf{n},\mathbf{m}-\mathbf{n}-\mathbf{r}),1).
\end{aligned}
$$

We prove in Section 6 that these global rules can be iterated, since, at least on some particular sequence obtained by iterating an original letter, two disjoint pointed patterns map to disjoint patterns. Hence, a *two-dimensional substitution rule endowed with a global rule* appears to be a first appropriate definition for a *two-dimensional morphism*. To differentiate this from what will follow, we can call this a *two-dimensional morphism defined by a global rule*.

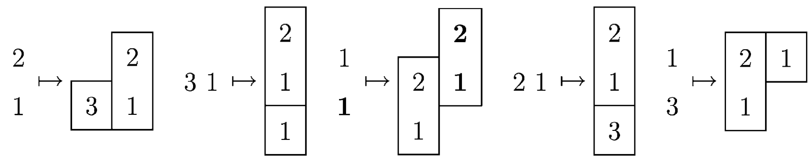


Fig. 3. A set of local rules for the substitution rule $\Sigma_0$.

Such a definition is however inconvenient for explicit computation, since one needs at each step global information. In particular, it is difficult to iterate it in order to generate an infinite two-dimensional sequence. Moreover, this can work explicitly only for very particular sequences, as can be seen from the one-dimensional case. Indeed, one-dimensional infinite sequences have a natural reference point: their initial letter. Giving such a placing rule in dimension one means that we know that a letter $i$ in position $\mathbf{n}$ maps to a word starting in a position $\mathbf{v}(\mathbf{n}, i)$ which depends on $\mathbf{n}$. But, by construction, this position $\mathbf{v}(\mathbf{n}, i)$ depends on the whole prefix of length $n$, so that for iterated morphisms of nonconstant length, there cannot be a rule that is valid for all one-dimensional infinite sequences. This does not prevent, of course, of giving a rule that is valid, for example, only for one of the fixed points of the iterated morphism: this is what we do in Section 6.

### 1.4. Local rules

It is much more convenient to be able to use a local information: a two-dimensional substitution rule defines the images of letters as nonpointed patterns; in addition, the *initial rule* defines the image of a particular pointed letter, thus giving a starting point for the iteration, and *local rules* will define the images of a finite number of well-chosen nonpointed patterns, so that the morphism can be iterated, using paths made of nonpointed patterns. This is exactly what is done when computing one-dimensional iterated morphisms: one does not compute the exact position of a given letter, but one only uses the fact that letters follow each other. Roughly speaking, a local rule says how to place the image of a letter with respect to the images of the neighbouring letters. If we know where to locate the image of the initial letter, we can compute the values of adjacent letters by using a finite number of patterns, and in this way, compute the image of the complete sequence.

For example, we will see that the local rules shown in Fig. 3, in addition to a natural initial rule for the pointed letter $((\mathbf{1},\mathbf{0}),1)$, can be used to define a two-dimensional morphism $(\Sigma_0, \mathscr{I}, \mathscr{S})$ with local rules in a consistent way; it turns out that $(\Sigma_0, \mathscr{I}, \mathscr{S})$ and $(\Sigma_0, \mathbf{v})$ have the same fixed point.

There are however significant problems in this approach also; we can raise four questions:

- One must choose a good set of patterns, sufficient to allow iteration, and minimal if possible, for simplicity. One checks in our case that one can restrict to some two-letter patterns.

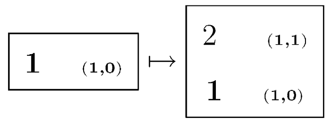



Fig. 4. An initial rule for the substitution rule $\Sigma_0$.

- There is then a problem of consistency: if two points, of coordinates, say, $(\mathbf{0},\mathbf{0})$ and $(\mathbf{i},\mathbf{j})$ of a finite pointed pattern can be joined by two different paths of nonpointed patterns corresponding to local rules, then this gives two independent ways to place the pattern corresponding to letter in position $(\mathbf{i},\mathbf{j})$ with respect to the pattern image of $(\mathbf{0},\mathbf{0})$. For consistency, these two placement rules must be the same.
- Furthermore, the images of different letters must not overlap.
- The image of an infinite sequence must not have "holes": all positions in the image must be included in one (and exactly one) pattern image of a letter.

A consequence of these problems is that, in general, the two-dimensional morphism will only be defined on a subset of all possible finite patterns and infinite sequences.

The main result of this paper is that the problems above can be solved for the substitution rule $\Sigma_0$:

**Theorem.** *The two-dimensional substitution rule $\Sigma_0$ endowed with the set $\mathscr{S}$ of five local rules given in Fig.* 3 *and the initial rule* $\mathscr{I}:((\mathbf{1},\mathbf{0}),1)\mapsto((\mathbf{1},\mathbf{0}),1)((\mathbf{1},\mathbf{1}),2)$ (*see Fig.* 4) *defines a two-dimensional morphism with local rules* $(\Sigma_0,\mathscr{I},\mathscr{S})$. *The rules are consistent and can be iterated on the patterns* $(\Sigma_0,\mathscr{I},\mathscr{S})^n((\mathbf{1},\mathbf{0}),1)$. *The successive images of* $((\mathbf{1},\mathbf{0}),1)$ *appear to be subpatterns of each others*; *they generate an infinite two-dimensional sequence denoted* $(U(\mathbf{m},\mathbf{n}))_{\mathbb{Z}^2}$ *and called the* fixed point *of* $(\Sigma_0,\mathscr{I},\mathscr{S})$.

Unfortunately, although this theorem appears to be a purely combinatorial result, we do not know any combinatorial proof of it, and we would be very interested in such a proof. To prove this combinatorial result, we need to use a quite devious path, giving to the two-dimensional morphism $(\Sigma_0,\mathscr{I},\mathscr{S})$ a geometric interpretation in terms of discrete approximation of a plane, as we explain below. Indeed, the class of two-dimensional morphisms with local rules introduced in this paper can be seen as a symbolic translation of a geometric formalism inspired by Rauzy's construction of its well-known fractal [18] and studied in [3,4,12]. A first example of a family of such two-dimensional morphisms with local rules has been introduced in [2] associated with the Jacobi–Perron continued fraction algorithm. For more details, see also Chapter 8 in [16].

## *1.5. Discrete planes*

Our approach is the following: we lift $\mathbb{Z}^2$ into $\mathbb{R}^3$ by introducing the transpose ${}^t\mathbf{M}$ of the matrix $\mathbf{M}$ of incidence of the substitution rule $\Sigma_0$. The action of this matrix is strictly contracting on a plane determined by the eigenvalues of modulus strictly less than 1. We introduce the *discrete plane* approximation $\mathfrak{P}$ of the contracting plane following [3,7,12,21] as the upper boundary of the union of all unit cubes with integral vertices that intersect the contracting plane. This construction is inspired by the cut-and-project formalism in quasicrystals [20].

We then introduce *generalized substitutions* from [3]; these are rules $\Sigma_{\mathfrak{P}}$ that act on faces of the discrete plane and map them onto finite unions of faces.

There exists a bijection $\pi$ between the points of the discrete plane and a lattice in the diagonal plane $x+y+z=0$, given by the projection on the diagonal plane along the direction $(1,1,1)$. We use the bijection $\pi$ to express the formalism of generalized substitution $\Sigma_{\mathfrak{P}}$ as a two-dimensional morphism with global rules $(\Sigma,\mathbf{v})$ acting on two-dimensional patterns. This morphism $(\Sigma,\mathbf{v})$ appears to be our example $(\Sigma_0,\mathbf{v})$.

There is no problem to generalize such a construction. Indeed, a generalized substitution $\Sigma_{\mathfrak{P}}$ is attached to a one-dimensional iterated morphism that satisfies the so-called *Pisot unimodular property*. For example, the generalized substitution which produces $\Sigma_0$ is attached to $\sigma_0: 1\mapsto 13,\ 2\mapsto 1,\ 3\mapsto 2$. Observe that the incidence matrix of $\sigma_0$ is the transpose of that of $\Sigma_0$; this duality property is the core of the definition of the generalized substitutions [3,4].

The deep relationship between $\Sigma_0$ and the underlying geometry is expressed in the following result.

**Theorem.** *The fixed point* $(U(\mathbf{m},\mathbf{n}))_{\mathbb{Z}^2}$ *of the two-dimensional morphism with global rules* $(\Sigma_0,\mathbf{v})$ *turns out to be a bijective coding for the discrete plane associated with the one-dimensional iterated morphism* $\sigma_0: 1\mapsto 13,\ 2\mapsto 1,\ 3\mapsto 2$.

### 1.6. Sketch of the paper

We first introduce in Section 2 the example $\Sigma_0$ and discuss its combinatorial properties. Global rules and local rules are defined for this substitution rule. The remaining of this paper is then devoted to the proof of consistency of these rules.

More precisely, we introduce in Section 3 the notion of a discrete plane associated with a plane in $\mathbb{R}^3$. We prove that the vertices in the discrete plane project onto a regular lattice $\Gamma$ in the main diagonal plane $x+y+z=0$. This allows to code the discrete plane by an infinite two-dimensional sequence $U$, that contains all the information necessary to rebuild the discrete plane. Let us emphasize that it is quite unexpected that a discrete plane can be recoded by using a regular lattice. We study this two-dimensional sequence in Section 4.

We then recall in Section 5 the notion of generalized substitution from [3], that is, of a morphism which acts on faces of the discrete plane. We extend this definition to $\mathbb{Z}^2$ in Section 6 via the projection of the discrete plane onto the regular lattice $\Gamma$. Hence we obtain a two-dimensional morphism endowed with global rules and we prove that it can be iterated. We deduce local rules and prove that they are consistent in Section 6.2.

For the sake of clarity, some technical results are proved in Appendix A.

Observe that everything done in the present paper works in the $n$-dimensional case. We restrict ourselves to the two-dimensional case in order to be able to give pictures of our objects.

## 2. An example of a two-dimensional morphism

Before introducing the notion of two-dimensional morphisms, we need a precise formalism to describe the objects on which the two-dimensional morphism will act, namely, patterns.

### 2.1. Patterns

Roughly speaking, we want a two-dimensional pattern to be a bounded planar shape made of letters of a finite alphabet. We have already restricted ourselves to the alphabet $\{1,2,3\}$.

#### 2.1.1. Two-dimensional pointed letters

Let us first define the basic component of a pattern, that is, a letter located at a given position: a *two-dimensional pointed letter* denotes any pair $(\mathbf{x}, i)$, where $\mathbf{x} \in \mathbb{Z}^2$ is the location of the pointed letter in the plane, and $i \in \{1,2,3\}$ the letter itself. The set of two-dimensional pointed letters is denoted $\mathscr{L}$:

$$\mathscr{L} = \mathbb{Z}^2 \times \{1,2,3\}.$$

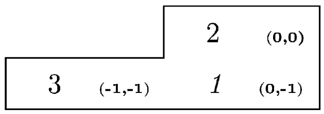
#### 2.1.2. Two-dimensional pointed patterns

A *pointed pattern* $W$ is a finite set of pointed letters with distinct location. It is represented as a word on the alphabet $\mathscr{L}$, and denoted $W = (\mathbf{x_1}, i_1)\ldots(\mathbf{x_j}, i_j)$, where $j$ is the number of pointed letters in this pattern. Such a definition is consistent and does not depend on the order of the pointed letters as soon as all $\mathbf{x}_k$'s are different. Hence, we define the set $\mathscr{L}^*$ of pointed patterns, called *two-dimensional language* as follows:

$$\mathscr{L}^* = \{(\mathbf{x_1}, i_1)\ldots(\mathbf{x_j}, i_j),\ j \in \mathbb{N},\ \forall 1 \leqslant k \leqslant j,\ (\mathbf{x}_k, i_k) \in \mathscr{L},$$
$$\mathbf{x}_k \neq \mathbf{x}_{k'} \text{ if } k \neq k'\}.$$

For instance, the pattern $((\mathbf{0},\mathbf{0}),\mathit{1})((\mathbf{0},\mathbf{1}),3)((\mathbf{0},\mathbf{2}),2)((\mathbf{0},\mathbf{3}),2)$ denotes the pattern $\boxed{\mathit{1}\ 3\ 2\ 2}$, where the italic character *1* is at position $(\mathbf{0},\mathbf{0})$.

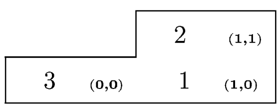
Both $((\mathbf{0},\mathbf{0}),2)((\mathbf{0},-\mathbf{1}),1)((-\mathbf{1},-\mathbf{1}),3)$ and $(((-\mathbf{1},-\mathbf{1})),3)((\mathbf{0},\mathbf{0}),2)((\mathbf{0},-\mathbf{1}),1)$ denote the following pattern, where the *1* is at position $(\mathbf{0},-\mathbf{1})$:



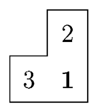

The *size* of a pattern $W=(\mathbf{x_1},i_1)\dots(\mathbf{x_j},i_j)$ is equal to $j$. Its *support* is the set $(\mathbf{x_1},\dots,\mathbf{x_j})\subset\mathbb{Z}^2$.

Observe that in the theory of one-dimensional words, there is no need for such a formalism since there is no confusion when one writes $w=w_1w_2\dots w_n$, but this is no more true if the pattern is not connected.

It will be convenient below to consider pointed patterns whose support contain a given point, for instance the point $(\mathbf{1},\mathbf{0})$; in that case, a short way to represent this pattern is to draw them as nonpointed patterns, with the letter at $(\mathbf{1},\mathbf{0})$ written in bold face, see the figure below.




A pointed pattern whose support contains $(\mathbf{1},\mathbf{0})$




Its short representation, with the letter at $(\mathbf{1},\mathbf{0})$ emphasized

*2.1.3. Nonpointed patterns*

The lattice $\mathbb{Z}^2$ acts by translation on pointed patterns: if $W=(\mathbf{x}_1,i_1)\dots(\mathbf{x}_j,i_j)\in\mathcal{L}^*$ is a pointed pattern and $\mathbf{y}\in\mathbb{Z}^2$ is a vector, let $W+\mathbf{y}=(\mathbf{x}_1+\mathbf{y},i_1)\dots(\mathbf{x}_j+\mathbf{y},i_j)$. We define a *nonpointed pattern* as a pointed pattern up to a translation; it is thus a pattern considered without a precise location in $\mathbb{Z}^2$.

Each pointed pattern represents a unique nonpointed pattern, called its *underlying nonpointed pattern*. Conversely, a pointed pattern which represents a nonpointed pattern is called a *representative*.

**Definition 2.1** (Substitution rule). A *two-dimensional substitution rule* $\Sigma$ on three letters is a map from $\{1,2,3\}$ on the set of finite two-dimensional nonpointed patterns on $\{1,2,3\}$.

An example is given by $\Sigma_0$: $1\mapsto \begin{matrix}2\\1\end{matrix}$, $2\mapsto 3$, $3\mapsto 1$.

This is what we represent usually as a two-dimensional morphism; note however that this definition is not complete: it tells us by what we must replace each letter, but not how to place the patterns we obtain. In dimension 1 (and if we consider only patterns whose support is an interval of $\mathbb{N}$, that is, usual words!) this problem does not occur, because it has an obvious solution, using the natural order on $\mathbb{N}$, or the monoid structure on the set of words.

We must now explain how to obtain the image, not of a letter, but of a pattern, an then iterate the morphism. Note that the morphism we obtain will only be defined in a meaningful way on some patterns, not all in general. Especially, we will need to extend the definition as a morphism on patterns. Let us introduce the most natural operation on pointed patterns, that is, union.

### 2.1.4. Union of pattern

One defines as follows an algebraic operation on pointed patterns which corresponds to the *union*. If $W=(\mathbf{x_1},i_1)\dots(\mathbf{x_{j_1}},i_{j_1})$ and $V=(\mathbf{y_1},k_1)\dots(\mathbf{y_{j_2}},k_{j_2})\in\mathscr{L}^*$ satisfy $\mathbf{x_l}\neq\mathbf{y_m}$ for every $l,m$, let

$$W.V=(\mathbf{x_1},i_1)\dots(\mathbf{x_{j_1}},i_{j_1})(\mathbf{y_1},k_1)\dots(\mathbf{y_{j_2}},k_{j_2}).$$

Notice that this operation provides a pointed pattern if and only if $\mathbf{x_l}\neq\mathbf{y_m}$ for every $l,m$. Such a pair of pointed patterns is called *disjoint pointed patterns*.

If $W_1,W_2,\dots,W_k$ are pointed patterns, their union $W_1.\dots.W_k$ is also denoted $\odot_{j\leqslant k}W_j$. Let us observe that the set $\mathscr{L}^*$ is not stable under this operation. Let $\mathscr{L}_w^*$ denote the set of *weighted pointed patterns*, that is, the set of all patterns on $\mathbb{Z}^2\times\mathscr{A}$ with no condition about the support: a letter $(\mathbf{x},i)$ may appear twice (or more) in a such a weighted pattern, as well as both the letters $(\mathbf{x},1)$ and $(\mathbf{x},2)$. Geometrically, weighted patterns have no real meaning but $\mathscr{L}_w^*$ endowed with the union becomes a monoid.

## 2.2. Two-dimensional morphisms with local rules

An example of a substitution rule is given by $\Sigma_0:1\mapsto \begin{smallmatrix}2\\1\end{smallmatrix}$, $2\mapsto 3$, $3\mapsto 1$. The aim of this section is to illustrate what we mean by defining a two-dimensional morphism from the substitution rule $\Sigma_0$.

Suppose one wants to iterate $\Sigma_0$ starting from 1 at the position $(\mathbf{1},\mathbf{0})$. A first problem occurs at the beginning: where will we place the nonpointed pattern $\Sigma_0(1)$? We define an *initial rule* to solve this problem.

**Definition 2.2** (Initial rule). Let $\Sigma$ be a two-dimensional substitution rule. An *initial rule* for $\Sigma$ is given by a map $\mathscr{I}$ which sends a given pointed letter $((\mathbf{m},\mathbf{n}),a)$ to a pointed pattern whose support contains $((\mathbf{m},\mathbf{n}),a)$ and which represents the pattern $\Sigma(a)$. The letter $((\mathbf{m},\mathbf{n}),a)$ is called the initial pointed letter of the initial rule $\mathscr{I}$.

For example, Fig. 4 shows the initial rule $\mathscr{I}:((\mathbf{1},\mathbf{0}),1)\mapsto((\mathbf{1},\mathbf{0}),1)((\mathbf{1},\mathbf{1}),2)$ for the two-dimensional substitution rule $\Sigma_0$; the initial letter $((\mathbf{1},\mathbf{0}),1)$ has been written in bold face.

A second problem occurs at the second iteration: what is the place of the image of 2 with respect to that of 1? Hence we need more information to iterate the process.

**Definition 2.3** (Local rules). Let $\Sigma$ be a two-dimensional substitution rule. A *local rule* is given by a map $W\mapsto\Sigma(W)$, where $W$ is a nonpointed pattern of size 2 (that

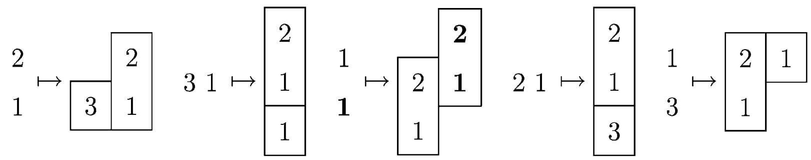


Fig. 5. Local rules for the substitution rule $\Sigma_0$.

underlies a pointed pattern denoted $(\mathbf{x}, a)(\mathbf{y}, b))$ and $\Sigma(W)$ is a nonpointed pattern that underlies the disjoint union of a pointed pattern that represents $\Sigma(a)$ and a pointed pattern that represents $\Sigma(b)$. The nonpointed pattern $W$ is called the initial pattern of the local rule.

The intuitive meaning of the rule will be that, if we know how to place the image of $a$, we will know how to place the image of $b$ when it is in a particular position with respect to $a$.

As an example, let us thus introduce the following set $\mathscr{S}$ of 5 local rules given in Fig. 5: if we know the place of the image of a given letter, we know the place of the images of the adjacent letters by using our 5 local rules. We use bold characters in Fig. 5 in the image of $\begin{smallmatrix}1\\1\end{smallmatrix}$ to indicate the place of the respective images. One can note that the image of the letter 1 which is in the lowest position is located above the image of the other 1. In fact, this produces a spiral movement which will allow one to cover all $\mathbb{Z}^2$ when iterating $\Sigma_0$.

**Definition 2.4** (Covered pattern). A pointed pattern $W$ is *covered* by a set of local rules if for every pair $(\mathbf{x}, a)$ and $(\mathbf{x}', b)$ of pointed letters in $W$ there exists a path of local rules from one letter to the other, that is, there exists $(\mathbf{y_1}, j_1), \ldots, (\mathbf{y_n}, j_n)$ pointed letters of the pattern such that $(\mathbf{y_1}, j_1) = (\mathbf{x}, a)$, $(\mathbf{y_n}, j_n) = (\mathbf{x}', b)$, and for $0 \leqslant k \leqslant n-1$, the pattern associated with $(\mathbf{y_k}, j_k)(\mathbf{y_{k+1}}, j_{k+1})$ is the initial nonpointed pattern of one of the local rules. In that case, we say that the path joins $(\mathbf{x}, a)$ and $(\mathbf{x}', b)$.

A nonpointed pattern is said covered if it admits a covered pointed representative.

One checks that the images of the initial nonpointed patterns of the 5 local rules given for the example $\Sigma_0$ are themselves covered. Hence, we are now able to extend the image of any pattern covered by these 5 local rules.

**Definition 2.5** (Morphism defined by local rules). Let $\Sigma$ be a two-dimensional substitution rule. A *two-dimensional morphism defined by local rules* $(\Sigma, \mathscr{I}, \mathscr{S})$ is given by the substitution rule $\Sigma$, an initial rule $\mathscr{I}$, and a finite collection $\mathscr{S}$ of local rules associated to this substitution rule.

The morphism $(\Sigma, \mathscr{I}, \mathscr{S})$ acts on covered patterns whose support contains the initial pointed letter $((\mathbf{m}, \mathbf{n}), a)$ of $\mathscr{I}$. One first locates the image of $((\mathbf{m}, \mathbf{n}), a)$ and then apply

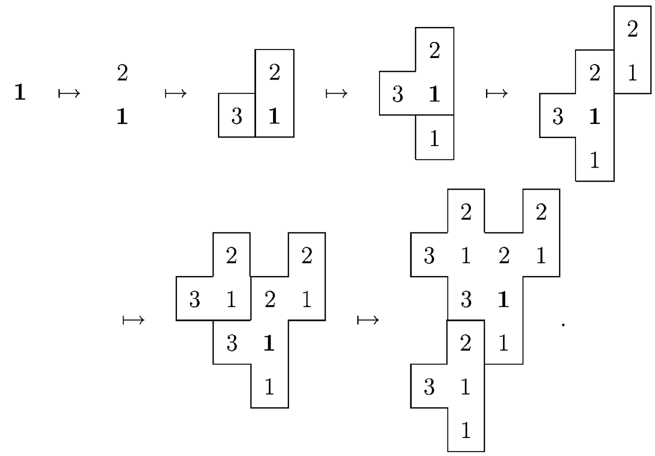


Fig. 6. Iteration of the local rules associated with $\Sigma_0$.

the local rules in $\mathscr{S}$. Nevertheless, the image of a pointed letter should not depend on the path used to join it to $((\mathbf{m},\mathbf{n}),a)$.

**Definition 2.6** (Consistency). A pointed pattern is said to be *consistent* for a morphism with local rules if:
(1) it contains the initial letter $((\mathbf{m},\mathbf{n}),a)$ of the initial rule $\mathscr{I}$;
(2) it is covered by the set of local rules;
(3) the image of a pointed letter is well defined, that is, it does not depend on the path of local rules used to join it $((\mathbf{m},\mathbf{n}),a)$;
(4) the images of two different pointed letters are disjoint.

We can now build the image of some particular consistent patterns. One can find in Fig. 6 the first iterations of $(\Sigma_0,\mathscr{I},\mathscr{S})$, that we denote for short $\Sigma_{0,l}$. Each iteration ${\Sigma_{0,l}}^n((\mathbf{1},\mathbf{0}),1)$ is split into two parts, one part corresponding to ${\Sigma_{0,l}}^{n-1}((\mathbf{1},\mathbf{0}),1)$ whereas the other one is the image of the corresponding part in ${\Sigma_{0,l}}^{n-1}((\mathbf{1},\mathbf{0}),1)$ (that is, the complement of ${\Sigma_{0,l}}^{n-2}((\mathbf{1},\mathbf{0}),1)$ in ${\Sigma_{0,l}}^{n-1}((\mathbf{1},\mathbf{0}),1)$). These first iterations are pointed patterns: the bold symbol **1** denotes the initial letter $((\mathbf{1},\mathbf{0}),1)$.

We can already make a first observation: the pointed pattern ${\Sigma_{0,l}}^n((\mathbf{1},\mathbf{0}),1)$ is a subpattern of ${\Sigma_{0,l}}^{n+1}((\mathbf{1},\mathbf{0}),1)$. Furthermore if one iterates the process, one can note that the images of the letters do not depend of the set of rules defined to link them to the $(\mathbf{1},\mathbf{0})$ (this can be made in different ways), and there are no overlaps by placing the images of the different letters, that is, the rules are consistent.

Hence one gets larger and larger nested pointed patterns. Moreover, the patterns grow with no "holes", and eventually cover any point in $\mathbb{Z}^2$ (see Section 6.2 for a proof). More precisely the sequence of finite patterns $({\Sigma_{0,l}}^n((\mathbf{1},\mathbf{0}),1))$ converges in $\{1,2,3\}^{\mathbb{Z}^2}$ for the topology over $\{1,2,3\}^{\mathbb{Z}^2}$ endowed with the product topology (which coincides with the natural topology on tilings). Hence, we will prove the following theorem, which is a more precise version of the theorem stated in the introduction:

**Theorem 2.7** (Local rules). *The two-dimensional substitution rule $\Sigma_0$, endowed with the initial local rule $\mathscr{I}:((\mathbf{1},\mathbf{0}),1)\mapsto((\mathbf{1},\mathbf{0}),1)((\mathbf{1},\mathbf{1}),2)$ and the set of local rules $\mathscr{S}$ given in Fig.* 5, *defines a two-dimensional morphism with local rules $(\Sigma_0,\mathscr{I},\mathscr{S})$, that we denote $\Sigma_{0,l}$. The pointed patterns ${\Sigma_{0,l}}^n((\mathbf{1},\mathbf{0}),1)$ are all consistent. Moreover, the sequence of pointed patterns $({\Sigma_{0,l}}^n((\mathbf{1},\mathbf{0}),1))_{n\in\mathbb{N}}$ converges in $\{1,2,3\}^{\mathbb{Z}^2}$ to the following infinite sequence $U=(U(\mathbf{m},\mathbf{n}))_{\mathbb{Z}^2}$, where $\alpha>1$ denotes the largest root of $x^3-x^2-1$:*

$$U(\mathbf{m},\mathbf{n})=1 \quad \textit{if} \ (\alpha^2 m+\alpha n)\mathrm{mod}(\alpha^2+\alpha+1)\in\,]0,\alpha^2],$$

$$U(\mathbf{m},\mathbf{n})=2 \quad \textit{if} \ (\alpha^2 m+\alpha n)\mathrm{mod}(\alpha^2+\alpha+1)\in\,]\alpha^2,\alpha^2+\alpha],$$

$$U(\mathbf{m},\mathbf{n})=3 \quad \textit{if} \ (\alpha^2 m+\alpha n)\mathrm{mod}(\alpha^2+\alpha+1)\in\,]\alpha^2+\alpha,\alpha^2+\alpha+1].$$

Remark that $\alpha\approx 1,46557$ is the second smallest Pisot number.

### 2.3. Two-dimensional morphisms with global rules

For the proof of the preceding theorem we will need to endow $\Sigma_0$ with global rules.

**Definition 2.8** (Morphism defined by global rules). Let $\Sigma$ be a two-dimensional substitution rule; for any pattern $\Sigma(a)$, choose a particular representative $\overline{\Sigma(a)}$. A global rule is a map $\mathbf{v}$ from the set of all pointed letters to $\mathbb{Z}^2$. A *two-dimensional morphism with global rules* is defined as a map $(\Sigma,\mathbf{v}):(\mathbf{x},a)\mapsto\overline{\Sigma(a)}+\mathbf{v}(\mathbf{x},a)$.

It is unclear that a global morphism can be applied to a large pointed pattern (there could be overlaps); in fact, we have the definition:

**Definition 2.9** (Consistency). We say that a pointed pattern $W$ is *consistent* for a morphism with global rules $(\Sigma,\mathbf{v})$ if, for any two distinct pointed letters contained in $W$, their images by $(\Sigma,\mathbf{v})$ are disjoint pointed patterns.

The following result will be proved in Section 6.2.

**Theorem 2.10** (Global placing rules). *For all $(\mathbf{m},\mathbf{n})\in\mathbb{Z}^2$, define $r=-\lceil(\alpha^2 m+\alpha n)/(\alpha^2+\alpha+1)\rceil+1$, and define $(\mathbf{m}',\mathbf{n}')=\mathbf{v}(\mathbf{m},\mathbf{n})=(\mathbf{1}-\mathbf{n},\mathbf{m}-\mathbf{n}-\mathbf{r})$.*

*Using the two-dimensional substitution rule $\Sigma_0$, define a two-dimensional morphism with global rules $(\Sigma_0,\mathbf{v})$ by*

- *the image under $\Sigma_0$ of the letter* 1 *located at $(\mathbf{m},\mathbf{n})$ in the sequence $U$ is the pattern $\boxed{\begin{smallmatrix}2\\ \mathbf{1}\end{smallmatrix}}$, where **1** is located at $(\mathbf{m}',\mathbf{n}')$;*

• *the image of the letter* 2 *located at* $(\mathbf{m},\mathbf{n})$ *is the pattern* $\boxed{3}$ *located at* $(\mathbf{m}',\mathbf{n}')$,
• *the image of the letter* 3 *located at* $(\mathbf{m},\mathbf{n})$ *is the pattern* $\boxed{1}$ *located at* $(\mathbf{m}',\mathbf{n}')$.
*The sequence* $(U(\mathbf{m},\mathbf{n}))_{\mathbb{Z}^2}$ *is consistent for the two-dimensional morphism with global rules* $(\Sigma_0,\mathbf{v})$; *moreover, it is a fixed point of this morphism.*

The sequence $U$ is a *two-dimensional Sturmian word* following [6,7]. Two-dimensional Sturmian words have many interesting combinatorial properties which allow us to consider them as a higher-dimensional generalization of Sturmian words. Classic one-dimensional Sturmian words code the approximation of a line by a discrete line made of horizontal and vertical segments with integral vertices (for more details, see for instance [15,16]). We will recall a proof of the fact that these multidimensional sequences code discrete plane approximations in Section 4. In our example, the sequence $U$ is a discrete approximation of the plane $\alpha^2 x+\alpha y+z=0$ in $\mathbb{R}^3$.

These multidimensional sequences are also generated by two-dimensional morphisms governed by the Jacobi–Perron algorithm. Namely, a geometric interpretation of the Jacobi–Perron algorithm is given in [2] as an induction process. Consequently, one can associate with the Jacobi–Perron algorithm a sequence of two-dimensional morphisms which generates the two-dimensional Sturmian sequences mentioned above. This is the process we want to extend here to a class of morphisms. Indeed, this paper aims mainly at explaining the process that allows one to deduce the local rules above from a one-dimensional iterated morphism, and more generally from any iterated morphism that satisfies the Pisot unimodular property on a three-letter alphabet. Observe again that we have no direct combinatorial proof of Theorems 2.7 and 2.10 and that we would be very interested in getting one.

## 3. Discrete plane associated with an irrational plane

The aim of this section is to introduce the notion of a discrete approximation of a plane following [2,7,12,13,21]. Let $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$ denote the canonical basis of $\mathbb{R}^3$.

We call *integral cube* any translate of the fundamental cube with integral vertices, that is, any set $(\mathbf{p},\mathbf{q},\mathbf{r})+\mathscr{C}$ where $(\mathbf{p},\mathbf{q},\mathbf{r})\in\mathbb{Z}^3$ and $\mathscr{C}$ denotes the fundamental unit cube (see Fig. 7):

$$\mathscr{C}=\{\lambda\mathbf{e_1}+\mu\mathbf{e_2}+v\mathbf{e_3},\ (\lambda,\mu,v)\in[0,1]^2\}.$$

Let $\mathscr{P}\subset\mathbb{R}^3$ be a plane with equation $ax+by+cz+h=0$. We suppose that the plane has totally irrational direction, that is, the triple $(a,b,c)$ satisfies no rational relation. We also suppose that $a,b,c>0$.

We will approximate the plane $\mathscr{P}$ by selecting points with integral coordinates above and within a bounded distance of the plane. The discrete plane is defined as a union of faces of integral cubes that connect these points.

**Definition 3.1** (Arnoux et al. [2], Berthé and Vuillon [6]). Let $\mathscr{S}$ be the set of integral cubes that intersect the lower closed half-space $ax+by+cz+h\leqslant 0$. The *discrete plane* associated with $\mathscr{P}$ is the boundary of the set $\mathscr{S}$. This discrete plane is denoted

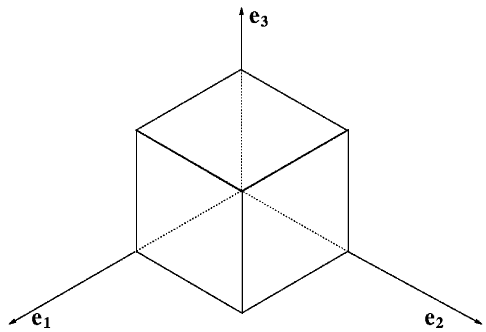



Fig. 7. The fundamental cube $\mathscr{C}$.

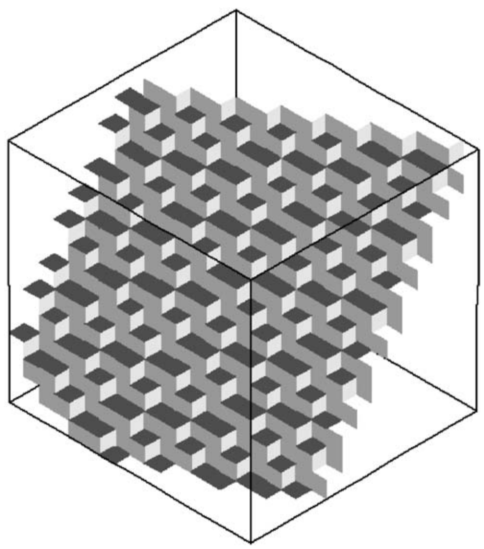


Fig. 8. A part of the discrete plane $\mathfrak{P}$ for the plane $\alpha^2 x + \alpha y + z = 0$, where $\alpha^3 = \alpha^2 + 1$.

$\mathfrak{P}$. A *vertex* of the discrete plane $\mathfrak{P}$ is an integral point that belongs to the discrete plane. Let $\mathscr{V}$ denote the set of vertices of $\mathfrak{P}$ (see Fig. 8).

### *3.1. Vertices in the discrete plane*

In this section we give a numerical characterization of the vertices of the discrete plane $\mathfrak{P}$. We recover the results of [2] by giving a more detailed proof for the sake of clarity.

**Proposition 3.2** (Arnoux et al. [2]). *An integral point* $(\mathbf{p}, \mathbf{q}, \mathbf{r})$ *is a vertex of the discrete plane* $\mathfrak{P}$ *if and only if* $0 < ap + bq + cr + h \leqslant a + b + c$.

**Proof.** The proof needs some intermediate steps.

(1) *The integral cube* $(\mathbf{p},\mathbf{q},\mathbf{r})+\mathscr{C}$ *is included in the set* $\mathscr{S}$ *if and only if* $ap+bq+cr+h\leqslant 0$. Indeed, the cube $(\mathbf{p},\mathbf{q},\mathbf{r})+\mathscr{C}$ is included in $\mathscr{S}$ if it intersects the lower half-space $ax+by+cz+h\leqslant 0$. Since $a,b,c$ are positive, if a point belongs to the lower half-space, then every other point that is "below" also belongs to the half-space. Consequently, the cube $(\mathbf{p},\mathbf{q},\mathbf{r})+\mathscr{C}$ intersects this half-space if and only its lowest point $(\mathbf{p},\mathbf{q},\mathbf{r})$ belongs to it, that is, $ap+bq+cr+h\leqslant 0$.

(2) *An integral point* $(\mathbf{p},\mathbf{q},\mathbf{r})\in\mathbb{Z}^3$ *belongs to* $\mathscr{S}$ *if and only if* $ap+bq+cr+h\leqslant a+b+c$.

Indeed the point $(\mathbf{p},\mathbf{q},\mathbf{r})$ belongs to $\mathscr{S}$ if it belongs to an integral cube $(\mathbf{p_1},\mathbf{q_1},\mathbf{r_1})+\mathscr{C}$ included in $\mathscr{S}$, that is, $ap_1+bq_1+cr_1+h\leqslant 0$. Since we know that $(\mathbf{p},\mathbf{q},\mathbf{r})\in(\mathbf{p_1},\mathbf{q_1},\mathbf{r_1})+\mathscr{C}$, that is, $p-p_1,q-q_1,r-r_1\in\{0,1\}$, a characterization for integral vertices in $\mathscr{S}$ is given by $ap+bq+cr+h\leqslant a+b+c$.

In other words, an integral point belongs to the set $\mathscr{S}$ if and only if its translate by the vector $(-1,-1,-1)$ belongs to the lower half-space $ax+by+cz+h\leqslant 0$.

(3) Observe that, since the set $\mathscr{S}$ is covered by integral cubes, an integral point $(\mathbf{p},\mathbf{q},\mathbf{r})$ belongs to the interior of $\mathscr{S}$ if and only if all the cubes containing $(\mathbf{p},\mathbf{q},\mathbf{r})$ also belong to $\mathscr{S}$.

Consequently, the point $(\mathbf{p},\mathbf{q},\mathbf{r})\in\mathbb{Z}^3$ belongs to $\mathfrak{P}$ if it is on its boundary, that is, if the following conditions are satisfied:

- the point $(\mathbf{p},\mathbf{q},\mathbf{r})\in\mathscr{S}$, that is, $ap+bq+cr+h\leqslant a+b+c$;
- the point $(\mathbf{p},\mathbf{q},\mathbf{r})$ does not belong to the interior of $\mathscr{S}$, that is, there exists a cube $(\mathbf{p_1},\mathbf{q_1},\mathbf{r_1})+\mathscr{C}$ that contains $(\mathbf{p},\mathbf{q},\mathbf{r})$ and that is not included in $\mathscr{S}$. Hence, there must exist $(\mathbf{p_1},\mathbf{q_1},\mathbf{r_1})$ such that $p-p_1,q-q_1,r-r_1\in\{0,1\}$ and $ap_1+bq_1+cr_1+h>0$. An equivalent condition is $ap+bq+cr+h>0$. □

### 3.2. Partition of the discrete plane by pointed faces

By construction, a discrete plane is a union of faces of integral cubes. We would like to define a true partition by these faces; this is impossible if we take closed faces, since edges will then belong to two faces, or if we take open faces, since vertices and edges will then belong to no face. We need to introduce a convention to define a notion of canonical faces that are neither open nor closed. (The main difference between the notation here and that of [2] lies in our choice of the distinguished vertices of the faces of type 1, 2, 3 (see Fig. 9); this choice is motivated by Proposition 3.3 where the successive lengths of intervals are *a*, *b*, *c* (and not *c*, *b*, *a*, as in [2]).)

Let $E_1$, $E_2$, and $E_3$ be the three following basic faces for the discrete plane:

$$E_1=\{\lambda\mathbf{e}_2+\mu\mathbf{e}_3,\ (\lambda,\mu)\in[0,1[^2\},$$

$$E_2=\{-\lambda\mathbf{e}_1+\mu\mathbf{e}_3,\ (\lambda,\mu)\in[0,1[^2\},$$

$$E_3=\{-\lambda\mathbf{e}_1-\mu\mathbf{e}_2,\ (\lambda,\mu)\in[0,1[^2\}.$$

We call *face of type i pointed on* $(\mathbf{p},\mathbf{q},\mathbf{r})$ or shortly *pointed face* the set $(\mathbf{p},\mathbf{q},\mathbf{r})+E_i$.

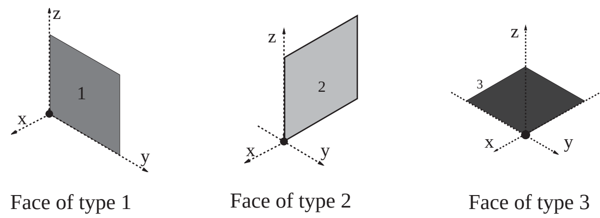



Fig. 9. The three different kinds of pointed faces in $\mathbb{R}^3$.

Notice that each face contains exactly one integral point. We call it the *distinguished vertex* of the face. Hence, the point $(\mathbf{p},\mathbf{q},\mathbf{r})$ is the distinguished vertex of the face $(\mathbf{p},\mathbf{q},\mathbf{r})+E_i$.

The reason for the presence of the semi-open intervals and of the signs in the definition of the faces is that such a choice of faces provides a true partition for the discrete plane. This result is not immediate since problems may occur on edges and integral vertices. The proof requires the following intermediate result, whose proof is given in the Appendix for the sake of clarity.

**Proposition 3.3.** *A point* $(\mathbf{p},\mathbf{q},\mathbf{r})\in\mathbb{Z}^3$ *is the distinguished vertex of a face of type* 1 (*resp*. 2 *or* 3) *in the discrete plane* $\mathfrak{P}$ *if and only if* $ap+bq+cr+h\in\,]0,a]$ (*resp*. $]a,a+b]$ *or* $]a+b,a+b+c]$). *If so*, $(\mathbf{p},\mathbf{q},\mathbf{r})$ *belongs to no other face in the discrete plane*.

**Proof.** See Appendix A. □

Geometrically, this proposition means that an integral point $\mathbf{x}=(\mathbf{p},\mathbf{q},\mathbf{r})$ is the distinguished vertex of a face of type 1 in the discrete plane $\mathfrak{P}$ if and only if $\mathbf{x}$ is strictly above the plane $\mathscr{P}$ whereas $(\mathbf{p}-\mathbf{1},\mathbf{q},\mathbf{r})=\mathbf{x}-\mathbf{e_1}$ is below the plane. Similarly, it is the distinguished vertex of a face of type 2 if $\mathbf{x}-\mathbf{e_1}$ is above the plane and $\mathbf{x}-\mathbf{e_1}-\mathbf{e_2}$ is below. The type is 3 if $\mathbf{x}-\mathbf{e_1}-\mathbf{e_2}$ is above the plane and $\mathbf{x}-\mathbf{e_1}-\mathbf{e_2}-\mathbf{e_3}$ is below.

**Theorem 3.4.** *The pointed faces form a partition of the discrete plane* $\mathfrak{P}$.

**Proof.** See Appendix A. □

## 4. Two-dimensional sequence associated with an irrational plane

We can now introduce a symbolic coding of the discrete plane as a two-dimensional sequence with values in the three-letter alphabet $\{1,2,3\}$. We first project the vertices of the discrete plane in the diagonal plane $x+y+z=0$; we thus obtain a bijection between $\mathscr{V}$ and the lattice $\mathbb{Z}^2$.

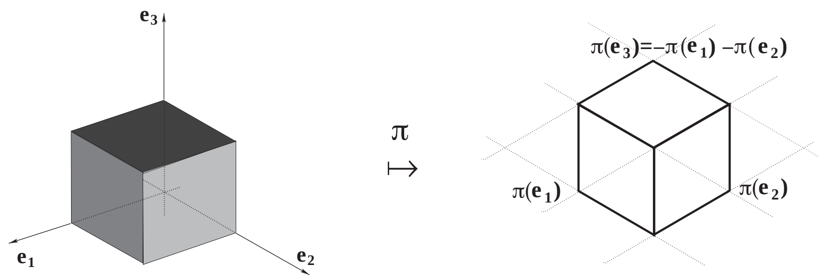



Fig. 10. The projection of the faces $E_1$, $E_2$ and $E_3$ in the diagonal plane endowed with the lattice generated by $\pi(\mathbf{e}_1)$ and $\pi(\mathbf{e}_2)$.

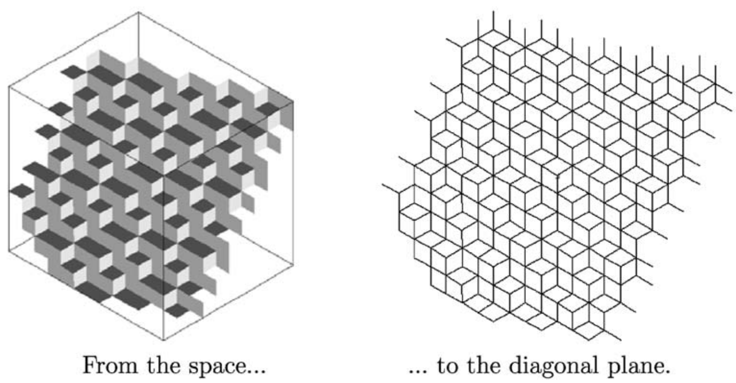



Fig. 11. Vertices in the discrete plane project onto a regular lattice.

### 4.1. Definition of the projection onto the diagonal plane

Let $\pi$ be the affine projection on the plane $x+y+z=0$ along the direction $(1,1,1)$.

Geometrically, this projection simply means that we look at the plane from the diagonal direction towards the origin. In particular, let us notice that the image by this projection of the unit cube is nothing else than a regular hexagon (see Fig. 10).

A simple computation gives: $\pi(\mathbf{p},\mathbf{q},\mathbf{r}) = (p-r)\pi(\mathbf{e}_1) + (q-r)\pi(\mathbf{e}_2)$.

Hence, the projection $\Gamma$ of the lattice $\mathbb{Z}^3$ is a lattice in $x+y+z=0$:

$$\Gamma = \{(p-r)\pi(\mathbf{e}_1) + (q-r)\pi(\mathbf{e}_2), (\mathbf{p},\mathbf{q},\mathbf{r}) \in \mathbb{Z}^3\} = \mathbb{Z}\pi(\mathbf{e}_1) + \mathbb{Z}\pi(\mathbf{e}_2).$$

### 4.2. Projection of the discrete plane onto the diagonal plane

A quite unexpected result is that a discrete plane can be recoded on a regular lattice, despite its three-dimensional structure. An illustration of this result is given in Fig. 11. This is no longer true for instance when considering a discrete plane approximation in $\mathbb{R}^4$.

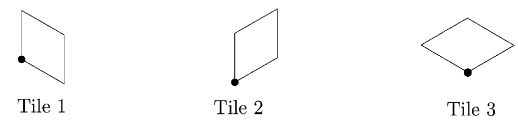


Fig. 12. The three basic tiles in the plane $x + y + z = 0$.

**Proposition 4.1** (Arnoux et al. [2]). *The projection $\pi$ is a bijection from the set of vertices $\mathscr{V}$ in the discrete plane $\mathfrak{P}$ onto the lattice $\Gamma$.*

**Proof.** Let $\mathbf{g} = m\pi(\mathbf{e}_1) + n\pi(\mathbf{e}_2)$ be a point in the lattice $\Gamma$. A point $(\mathbf{p}, \mathbf{q}, \mathbf{r}) \in \mathscr{V}$ projects on $\mathbf{g}$ if and only if $\pi(\mathbf{p}, \mathbf{q}, \mathbf{r}) = (p - r)\pi(\mathbf{e}_1) + (q - r)\pi(\mathbf{e}_2) = \mathbf{g} = m\pi(\mathbf{e}_1) + n\pi(\mathbf{e}_2)$. Hence $m = p - r$ and $n = q - r$.

Since $(\mathbf{p}, \mathbf{q}, \mathbf{r}) \in \mathscr{V}$, the coordinates satisfy $0 < ap + bq + cr + h \leqslant a + b + c$. Hence, $0 < am + bn + r(a + b + c) + h \leqslant a + b + c$ which implies $r = -\lceil (am + bn + h)/(a + b + c) \rceil + 1$. Consequently, $(\mathbf{p}, \mathbf{q}, \mathbf{r})$ exists and is uniquely determined, so that $\pi$ is a bijection from the set of vertices onto $\Gamma$. □

Explicit formulas are deduced from the proof:

$$\pi : (\mathbf{p}, \mathbf{q}, \mathbf{r}) \in \mathscr{V} \mapsto (p - r)\pi(\mathbf{e}_1) + (q - r)\pi(\mathbf{e}_2) \in \Gamma, \tag{4.1}$$

$$\pi^{-1} : m\pi(\mathbf{e}_1) + n\pi(\mathbf{e}_2) \in \Gamma \mapsto \begin{pmatrix} m \\ n \\ 0 \end{pmatrix} + \left( - \left\lceil \frac{am + bn + h}{a + b + c} \right\rceil + 1 \right) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in \mathscr{V}.$$

### *4.3. Two-dimensional sequence associated with an irrational plane*

Fig. 11 illustrates that Proposition 4.1 directly provides a tiling for the diagonal plane with the three diamonds: $\mathscr{T}_1 = \pi(E_1)$, $\mathscr{T}_2 = \pi(E_2)$, and $\mathscr{T}_3 = \pi(E_3)$, shown in Fig. 12.

Let us summarize the results obtained in the preceding sections:

- Let $(\mathbf{m}, \mathbf{n}) \in \mathbb{Z}^2$. There exists a unique corresponding point in the lattice $\Gamma$, namely, $\mathbf{g} = m\pi(\mathbf{e}_1) + n\pi(\mathbf{e}_2) \in \Gamma$.
- There exists a unique vertex with coordinates $(\mathbf{p}, \mathbf{q}, \mathbf{r})$ in the discrete plane $\mathfrak{P}$ such that the point $\mathbf{g} \in \Gamma$ is the projection through $\pi$ of this point, that is, $(\mathbf{p}, \mathbf{q}, \mathbf{r}) = \pi^{-1}(\mathbf{g}) \in \mathfrak{P}$.
- The vertex $(\mathbf{p}, \mathbf{q}, \mathbf{r}) \in \mathfrak{P}$ is the distinguished vertex of a unique face. The type of this face is completely determined by $ap + bq + cr + h \bmod (a + b + c)$.

Hence, it becomes natural to associate with each pair $(\mathbf{m}, \mathbf{n}) \in \mathbb{Z}^2$ the type of the corresponding face in the discrete plane $\mathfrak{P}$. This is shown in Fig. 13.

The results proved above imply that this coding is one-to-one, meaning that such a coding is sufficient to rebuild the whole discrete plane $\mathfrak{P}$, as shown in Fig. 14.

**Theorem 4.2.** *Let $(\mathbf{m}, \mathbf{n}) \in \mathbb{Z}^2$ and $\mathbf{g} = m\pi(\mathbf{e}_1) + n\pi(\mathbf{e}_2)$ in the lattice $\Gamma$. There exists a unique integer $U(\mathbf{m}, \mathbf{n}) \in \{1, 2, 3\}$ such that $\pi^{-1}(\mathbf{g})$ is the distinguished vertex of a*

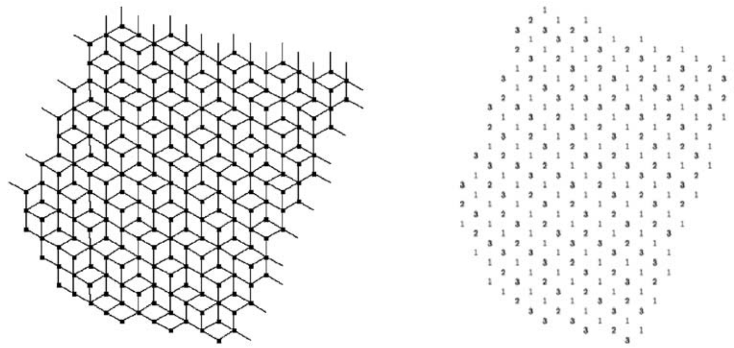



Fig. 13.

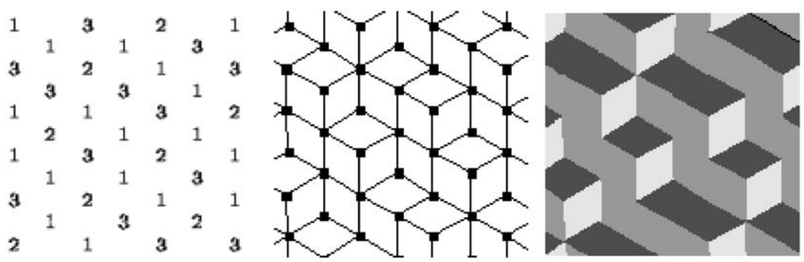



Fig. 14. From a two-dimensional coding to a discrete plane.

*face of type* $U(\mathbf{m},\mathbf{n})$ *in the discrete plane* $\mathfrak{P}$:

$$U(\mathbf{m},\mathbf{n}) = 1 \quad \textit{if} \ \ (am+bn+h)\mathrm{mod}(a+b+c) \in ]0,a],$$

$$U(\mathbf{m},\mathbf{n}) = 2 \quad \textit{if} \ \ (am+bn+h)\mathrm{mod}(a+b+c) \in ]a,a+b],$$

$$U(\mathbf{m},\mathbf{n}) = 3 \quad \textit{if} \ \ (am+bn+h)\mathrm{mod}(a+b+c) \in ]a+b,a+b+c].$$

*The sequence* $(U(\mathbf{m},\mathbf{n}))_{\mathbb{Z}^2}$ *is called the* two-dimensional coding *associated with the plane* $ax+by+cz+h=0$.

## 5. A three-dimensional morphism acting on the faces of a discrete plane

The aim of this section is to recall the formalism of [3] which introduces *generalized substitutions* acting on faces of a discrete plane $\mathfrak{P}$, and replacing them by finite unions of faces included in $\mathfrak{P}$. For more details on generalized substitutions, see [3,4] and Chapter 8 in [16].

### 5.1. Discrete plane associated with Pisot unimodular morphisms

#### 5.1.1. One-dimensional iterated morphisms

Let $\mathscr{A}$ be the finite alphabet $\{1,2,3\}$ and $\mathscr{A}^*$ the set of finite words defined over $\mathscr{A}$. The empty word is denoted $\varepsilon$. A one-dimensional *iterated morphism* $\sigma$ is an endomorphism of the free-monoid $\mathscr{A}^*$ such that the image of a letter of $\mathscr{A}$ is never empty; we also require that for at least one letter $a$, we have $|\sigma^n(a)| \to +\infty$, where $|w|$ denotes the length of the word $w$. It extends in a natural way to infinite or biinfinite sequences in $\mathscr{A}^{\mathbb{N}}$ and $\mathscr{A}^{\mathbb{Z}}$.

#### 5.1.2. Abelianization

Let $\mathbf{l}: \mathscr{A}^* \mapsto \mathbb{N}^3$ be the natural homomorphism obtained by abelianization of the free monoid: if $|W|_a$ denotes the number of occurrences of the letter $a \in \mathscr{A}$ in a finite word $W$, then we have $\mathbf{l}(W) = (|W|_1, |W|_2, |W|_3) \in \mathbb{N}^3$.

With each one-dimensional iterated morphism $\sigma$ on $\mathscr{A}$ is canonically associated its *incidence matrix* $\mathbf{M} = (m_{i,j})_{1\leqslant i,j\leqslant 3}$ defined by $m_{i,j} = |\sigma(j)|_i$ ($|W|_i$, which counts the number of occurrences of the letter $i$ in $W$), so that we have $\mathbf{l}(\sigma(W)) = \mathbf{M}\mathbf{l}(W)$ for every $W \in \mathscr{A}^*$.

#### 5.1.3. Iterated morphism of Pisot type

A morphism $\sigma$ on three letters is *of Pisot type* if its eigenvalues satisfy $\alpha > 1 > |\lambda_1| \geqslant |\lambda_2| > 0$. In particular, the dominant eigenvalue $\alpha$ is a Pisot number. Furthermore, its incidence matrix $\mathbf{M}$ is primitive [8,16], that is, it admits a power with strictly positive entries.

An iterated morphism $\sigma$ [1] is *unimodular* if $\det \mathbf{M} = \pm 1$.

#### 5.1.4. Discrete plane associated with a one-dimensional Pisot iterated morphism

Let $\alpha > 1 > |\lambda_1| \geqslant |\lambda_2|$ denote the eigenvalues of the iterated morphism $\sigma$. Let $\mathscr{P}$ be the *contracting plane* (that is, the real plane generated by the eigenvectors associated with $\lambda_1, \lambda_2$) of the incidence matrix $\mathbf{M}$ of $\sigma$. In particular, $\mathscr{P}$ is stable under the action of $\mathbf{M}$ and its inverse $\mathbf{M}^{-1}$. Similarly, the lower half-space and the upper half-space are stable under the action of $\mathbf{M}$ and $\mathbf{M}^{-1}$.

Numerically, the equation of $\mathscr{P}$ is $ax + by + cz = 0$, where $(a,b,c) = \mathbf{v}$ is a normalized expanding left eigenvector associated with the expanding eigenvalue $\alpha$. The Perron–Frobenius theorem ensures that $(a,b,c)$ is a strictly positive vector with no rational relationship since $\mathbf{M}$ is primitive. Observe that the vector $(a,b,c)$ has algebraic coordinates. Let $\mathfrak{P}$ be the discrete plane for the plane $\mathscr{P}$.

[1] Till the end of the paper, $\sigma$ denotes an iterated morphism on three letters that is unimodular and of Pisot type.

*5.1.5. Example*

Let $\Sigma_0$ be the following iterated morphism $\Sigma_0 : 1 \mapsto 13,\ 2 \mapsto 1,\ 3 \mapsto 2$.

Its incidence matrix is $\mathbf{M}$ with

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{M}^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Its characteristic polynomial is $X^3 - X^2 - 1$, hence it admits one eigenvalue $\alpha > 1$, which is the second smallest Pisot number, and two complex conjugate eigenvalues of modulus strictly smaller than 1. The contracting plane of this matrix has equation $\alpha^2 x + \alpha y + z = 0$ (it is perpendicular to the left eigenvector associated with $\alpha$ (that is, $(\alpha^2, \alpha, 1)$) whereas its expanding direction is given by the right eigenvector associated with $\alpha$ (that is, $(\alpha^2, 1, \alpha)$).

Notice that this iterated morphism belongs to the class of *modified Jacobi–Perron substitutions* following [12] (see also [16, Chapter 8]). This class of morphisms is deduced from the generalized modified Jacobi–Perron continuous fraction algorithm (which is a two-point extension of Brun's algorithm). Their matrices of incidence describe this algorithm in its linear additive form.

## *5.2. Generalization of the one-dimensional iterated morphism to the discrete plane*

In [3,4], the definition of the iterated morphism $\sigma$ is extended by duality to the faces $(\mathbf{p}, \mathbf{q}, \mathbf{r}) + E_i$ on the discrete plane $\mathfrak{P}$.

Let $\mathfrak{P}^*$ denote the set of finite pointed patterns on the discrete plane, that is, the set of finite disjoint unions of faces in the discrete plane.

Remark that the generalized substitutions introduced in [3,4] (and denoted $E_1^*(\sigma)$ in these papers) act on faces which for technical reasons do not correspond exactly to the faces $(E_1, E_2, E_3)$. The faces that are considered in [3,4] are $(E_1^*, E_2^*, E_3^*)$ such that $\mathbf{x} + E_i^* = \mathbf{x} + E_i + \mathbf{e}_1 + \cdots + \mathbf{e}_i$. The formalism of generalized substitution that is introduced in [3] provides the following formula with our notation.

**Definition 5.1.** Let $\sigma$ be a one-dimensional iterated morphism on three letters that is unimodular and of Pisot type. We call *generalized substitution acting on faces* the following transformation, denoted $\Sigma_{\mathfrak{P}}$, that maps any face of the discrete plane $\mathfrak{P}$ on a pattern in $\mathfrak{P}$. For every face $\mathbf{x} + E_i \subset \mathfrak{P}$, let

$$\Sigma_{\mathfrak{P}}(\mathbf{x} + E_i) = \bigcup_{k \in \{1,2,3\}} \bigcup_{P, \sigma(k) = PiS} (\mathbf{M}^{-1}[\mathbf{x} - \mathbf{l}(P) - (\mathbf{e}_1 + \cdots + \mathbf{e}_i)] + (\mathbf{e}_1 + \cdots + \mathbf{e}_k)) + E_k \subset \mathfrak{P}^*. \tag{5.1}$$

The faces that occur in the image of a face of type $i$ are associated with all the occurrences of the letter $i$ in the images of the letters of $\{1,2,3\}$. The incidence matrix of $\Sigma_{\mathscr{P}}$ is hence the dual of that of $\sigma$. (see Fig. 15)

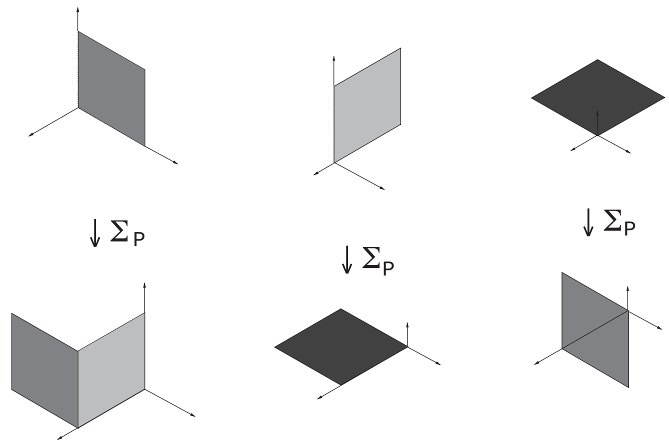



Fig. 15. The generalized substitution acting on faces associated with $\Sigma_0$.

### 5.3. Example

Let $\Sigma_0$ denote our example $1 \mapsto 13$, $2 \mapsto 1$, $3 \mapsto 2$. Then

$$\begin{aligned}
\Sigma_{\mathfrak{P}}:\ \mathbf{x} + E_1 &\mapsto (\mathbf{M}^{-1}[\mathbf{x} - \mathbf{e_1}] + \mathbf{e_1} + E_1) \cup (\mathbf{M}^{-1}[\mathbf{x} - \mathbf{e_1}] + \mathbf{e_1} + \mathbf{e_2} + E_2) \\
&= (\mathbf{M}^{-1}\mathbf{x} + \mathbf{e_1} - \mathbf{e_2} + E_1) \cup (\mathbf{M}^{-1}\mathbf{x} + \mathbf{e_1} + E_2), \\
\mathbf{x} + E_2 &\mapsto \mathbf{M}^{-1}[\mathbf{x} - \mathbf{e_1} - \mathbf{e_2}] + \mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3} + E_3 \\
&= \mathbf{M}^{-1}\mathbf{x} + \mathbf{e_1} + E_3, \\
\mathbf{x} + E_3 &\mapsto \mathbf{M}^{-1}[\mathbf{x} - 2\mathbf{e_1} - \mathbf{e_2} - \mathbf{e_3}] + \mathbf{e_1} + E_1 \\
&= \mathbf{M}^{-1}\mathbf{x} - \mathbf{e_2} - \mathbf{e_3} + E_1.
\end{aligned}$$

### 5.4. Iteration of the substitution $\Sigma_{\mathfrak{P}}$

Formula (5.1) defines rules that allow one to replace a single face in the discrete plane $\mathfrak{P}$ by a union of faces. Many points must be checked to be able to iterate these rules:

- First, we must be sure that the rule maps a face of the discrete plane into a union of faces included in the discrete plane.
- For consistency, the union in formula (5.1) must be disjoint.
- If one wants to iterate $\Sigma_{\mathfrak{P}}$, one needs to be able to extend the definition of $\Sigma_{\mathfrak{P}}$ to patterns of $\mathfrak{P}^*$, that is, to finite disjoint unions of faces in $\mathfrak{P}$. This is possible as soon as two different faces map to disjoint unions of faces.

The following theorem is given in [3]. We detail its proof in the appendix to make the paper self-contained.

**Theorem 5.2.** *Any one-dimensional iterated morphism that is unimodular and of Pisot type over a three-letter alphabet can be extended to a generalized substitution acting on faces of the discrete plane associated with the contracting plane of the incidence matrix of the iterated morphism. This generalized substitution maps any pattern* (*that is*, *a finite disjoint union of faces in the discrete plane*) *on a pattern of the discrete plane*.

*The discrete plane is invariant under the action of the substitution on faces. Furthermore two distinct faces have images which do not intersect*.

**Proof.** See Appendix A.

**Remark** (Extension of $\Sigma_{\mathfrak{P}}$ to $\mathbb{Z}^3$). Nothing prevents us to extend the definition of the substitution on faces $\Sigma_{\mathfrak{P}}$ to the whole set of faces in the space (and not only to the faces that are included in the discrete plane): Formula (5.1) is available for any $\mathbf{x} \in \mathbb{Z}^3$ and any type $k$, providing that the image of such a face may contain the same face more than once. Hence, the substitution $\Sigma_{\mathfrak{P}}$ extends to $\mathbb{Z}^3 \times \{1,2,3\}$ as a weighted substitution. For a more precise formalism, see [3,4].

## 6. Two-dimensional morphisms

### *6.1. Two-dimensional morphism with global rules associated with a one-dimensional iterated morphism*

Let $\sigma$ denote a one-dimensional iterated morphism that is unimodular and of Pisot type. Let $\mathscr{P}$ stand for the contracting plane for its incidence matrix and let $\mathfrak{P}$ denote the discrete plane associated with $\mathscr{P}$. In Section 5 was explained how to extend the definition of $\sigma$ to the faces of the discrete plane, by introducing a generalized substitution $\Sigma_{\mathfrak{P}}$. In Section 3 was proved that faces in the discrete plane project canonically onto $\mathbb{Z}^2 \times \{1,2,3\}$. In this section the results of Sections 3 and 5 are mixed together to extend the definition of $\Sigma_{\mathfrak{P}}$ as a two-dimensional morphism with global rules: we give a symbolic translation of (5.1).

**Definition 6.1.** Let $\sigma$ be a one-dimensional iterated morphism that is unimodular and of Pisot type and $\mathbf{M}$ its incidence matrix. The *global placing rules* associated with $\sigma$ are defined from $\mathscr{L}$ to $\mathscr{L}_w^*$ as follows:

$$\begin{aligned}(\Sigma,\mathbf{v}):((\mathbf{m},\mathbf{n}),i) \mapsto \bigodot_{\sigma(k)=PiS} (&\mathbf{N}(\mathbf{e}_1+\cdots+\mathbf{e}_k) - \mathbf{N}\mathbf{M}^{-1}(\mathbf{e}_1+\cdots+\mathbf{e}_i)\\ &-\mathbf{N}\mathbf{M}^{-1}\mathbf{l}(P) + \mathbf{N}\mathbf{M}^{-1}(m+r(\mathbf{m},\mathbf{n}), n+r(\mathbf{m},\mathbf{n}), r(\mathbf{m},\mathbf{n})), k),\end{aligned} \tag{6.1}$$

where

$$r(\mathbf{m},\mathbf{n}) = -\lceil (am+bn)/(a+b+c)\rceil + 1 \quad \text{and} \quad \mathbf{N} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

Indeed, by using the maps $\pi$ and $\pi^{-1}$, one can give a formulation for (5.1) in the lattice $\mathbb{Z}^2$. More precisely, the point $(\mathbf{m},\mathbf{n}) \in \mathbb{Z}^2$ admits a unique preimage according to $\pi$ in the discrete plane $\mathfrak{P}$ as $\pi^{-1}(\mathbf{m},\mathbf{n})$ (see (4.1)). We then apply $\Sigma_{\mathfrak{P}}$ to the geometric face $\pi^{-1}(\mathbf{m},\mathbf{n}) + E_i$; its image shall be decomposed into a finite union of geometric faces following (5.1) that we do project according to $\pi$.

Let us note that if $\pi^{-1}(\mathbf{m},\mathbf{n})+E_i$ is not a face on the discrete plane, then the pointed letters in the union might be weighted: for any $(\mathbf{x}, i)$, nothing allows one to state that the image $(\Sigma,\mathbf{v})(\mathbf{x}, i)$ is a pointed pattern. We just know that it is a weighted pointed pattern. However, from Theorem 5.2, we know that pointed patterns map to pointed patterns in the case where the pattern corresponds to a finite part of the discrete plane $\mathfrak{P}$. Indeed, since $\pi$ is a bijection (Proposition 4.1), a direct consequence of Theorem 5.2 is the following:

**Theorem 6.2.** *Let $\sigma$ be a one-dimensional iterated morphism that is unimodular and of Pisot type on a three-letter alphabet. Let $U = (U(\mathbf{m},\mathbf{n}))_{\mathbb{Z}^2}$ be the coding of the discrete plane associated with $\sigma$ following Theorem* 4.2. *Let $\mathscr{L}_U$ be the set of pointed patterns that appear in $U$.*

*Then, the global placing rules given by* (6.1) *define a two-dimensional morphism, denoted $(\Sigma,\mathbf{v})$, acting on $\mathscr{L}_U$.*

*Moreover, the coding $U$ of the discrete plane is a fixed point for $(\Sigma,\mathbf{v})$ which satisfies*

(1) *for all $(\mathbf{m},\mathbf{n})$, $(\Sigma,\mathbf{v})((\mathbf{m},\mathbf{n}), U(\mathbf{m},\mathbf{n}))$ is a pointed pattern that occurs in $U$;*
(2) *for all $(\mathbf{m},\mathbf{n})$, there exists a unique $(\mathbf{s},\mathbf{t})$ such that $(\Sigma,\mathbf{v})((\mathbf{s},\mathbf{t}), U(\mathbf{s},\mathbf{t}))$ contains the pointed letter $((\mathbf{m},\mathbf{n}), U(\mathbf{m},\mathbf{n}))$;*
(3) *if $((\mathbf{m},\mathbf{n}), U(\mathbf{m},\mathbf{n}))$ and $((\mathbf{s},\mathbf{t}), U(\mathbf{s},\mathbf{t}))$ are two distinct pointed letters in $U$, then their images under the action of $(\Sigma,\mathbf{v})$ are distinct.*

The only point which remains to check here is the second assertion. It is a direct consequence of the fact if $(\mathbf{p},\mathbf{q},\mathbf{r})$ (resp. $(\mathbf{u},\mathbf{v},\mathbf{w})$) is the point of the discrete plane $\mathfrak{P}$ in bijection with $\mathbf{g} = \mathbf{m}\pi(\mathbf{e_1}) + \mathbf{n}\pi(\mathbf{e}_2)$ (resp. $\mathbf{h} = \mathbf{s}\pi(\mathbf{e_1}) + \mathbf{t}\pi(\mathbf{e}_2)$), then by definition of $(\Sigma,\mathbf{v})$, $(\mathbf{u},\mathbf{v},\mathbf{w}) = \mathbf{M}^{-1}(\mathbf{p},\mathbf{q},\mathbf{r})$. In this sense, $(\Sigma,\mathbf{v})$ can be considered as an "invertible" map over $U$.

### 6.2. *Proofs of Theorems 2.7 and 2.10*

We now have gathered all the elements and tools we need to prove Theorem 2.7, and in particular, the consistency for the local rules of Fig. 5.

#### 6.2.1. *Computation of the global rules*

Let $\sigma_0 : 1 \mapsto 13,\ 2 \mapsto 1,\ 3 \mapsto 2$ and $a = \alpha^2$, $b = \alpha$ and $c = 1$, $\alpha^3 = \alpha^2 + 1$. Let $r = r(\mathbf{m},\mathbf{n}) = -\lceil (am + bn)/(a + b + c) \rceil + 1$. Then one computes Formula (6.1) in this case, with $\mathbf{v}((\mathbf{m},\mathbf{n}), i) = (\mathbf{1} - \mathbf{n}, \mathbf{m} - \mathbf{n} - \mathbf{r}(\mathbf{m},\mathbf{n}))$:

$$\begin{aligned}(\Sigma_0,\mathbf{v}) : ((\mathbf{m},\mathbf{n}),1) &\mapsto ((\mathbf{0},-\mathbf{1}),1)((\mathbf{0},\mathbf{0}),2) + \mathbf{v}((\mathbf{m},\mathbf{n}),1)\\ ((\mathbf{m},\mathbf{n}),2) &\mapsto ((\mathbf{0},\mathbf{0}),3) + \mathbf{v}((\mathbf{m},\mathbf{n}),2)\\ ((\mathbf{m},\mathbf{n}),3) &\mapsto ((\mathbf{0},\mathbf{0}),1) + \mathbf{v}((\mathbf{m},\mathbf{n}),3).\end{aligned} \tag{6.2}$$

Hence the two-dimensional substitution rule associated with $\sigma_0$ is given by:

$$\Sigma_0\colon\quad 1 \mapsto \begin{matrix}2\\1\end{matrix}\quad 2 \mapsto 3 \quad 3 \mapsto 1.$$

The two-dimensional morphism with global rules $(\Sigma_0, \mathbf{v})$ acts on the set of pointed patterns $\mathscr{L}_U$ of the two-dimensional coding $U$ of the plane $ax + by + c = 0$ following Theorem 6.2. The explicit expression for $U$ is given by Theorem 4.2. The sequence $U$ is a fixed point for $(\Sigma_0, \mathbf{v})$ with the placing rule of (6.2).

From the two-dimensional morphism with global rules $(\Sigma_0, \mathbf{v})$, let us deduce the expression for the two-dimensional morphism with local rules $(\Sigma_0, \mathscr{I}, \mathscr{S})$, that we denote $\Sigma_{0,l}$.

#### 6.2.2. Computation of the initial rule

One has $U(\mathbf{1},\mathbf{0}) = 1$ and $(\Sigma_0, \mathbf{v})((\mathbf{1},\mathbf{0}),1) = ((\mathbf{1},\mathbf{0}),1)((\mathbf{1},\mathbf{1}),2)$. The initial letter $((\mathbf{1},\mathbf{0}),1)$ of the initial rule $\mathscr{I}$ is sent on $((\mathbf{1},\mathbf{0}),1)((\mathbf{1},\mathbf{1}),2)$.

#### 6.2.3. Computation of the local rules

Let us suppose that the pointed pattern $\boxed{\mathbf{3}\ 1}$ occurs at $(\mathbf{m},\mathbf{n})$ in the fixed point $U$, that is, the pattern $W = ((\mathbf{m},\mathbf{n}),3)((\mathbf{m}+\mathbf{1},\mathbf{n}),1)$ belongs to $\mathscr{L}_U$. Hence,

- $ma + nb \in\, ]a + b, a + b + c]$ modulo $(a + b + c)$,
- $(m + 1)a + nb \in\, ]0, a]$ modulo $(a + b + c)$.

Observe that this pattern indeed occurs somewhere in $U$ by density of the set of points $ma + nb$ modulo $(a + b + c)$.

We know that $ma + nb \in\, ]a + b, a + b + c]$ modulo $(a + b + c)$ and if we add $a$ to this quantity, then we obtain the number $(m + 1)a + nb$ that belongs to the interval $]0, a]$ modulo $(a + b + c)$. Hence $\lceil (am + bn)/(a + b + c)\rceil$ and $\lceil (am + bn + a)/(a + b + c)\rceil$ differ by exactly 1, so that

- $r(\mathbf{m}+\mathbf{1},\mathbf{n}) = r(\mathbf{m},\mathbf{n}) - 1$,
- $m'(\mathbf{m}+\mathbf{1},\mathbf{n}) = m'(\mathbf{m},\mathbf{n})$,
- $n'(\mathbf{m}+\mathbf{1},\mathbf{n}) = n'(\mathbf{m},\mathbf{n}) + 2$.

Consequently,

$$(\Sigma_0, \mathbf{v})((\mathbf{m},\mathbf{n}) + ((\mathbf{0},\mathbf{0}),3)(\mathbf{1},\mathbf{0}),1)) = (\mathbf{m}',\mathbf{n}') + ((\mathbf{0},\mathbf{0}),1)((\mathbf{0},\mathbf{1}),2)((\mathbf{0},\mathbf{0}),2).$$

Symbolically this gives

$$\boxed{\mathbf{3}\ 1} \mapsto \begin{array}{|c|}\hline 2\\ 1\\ \hline \mathbf{1}\\ \hline\end{array}$$

The proof of the computation for the other local rules in Fig. 5 follows the same scheme.

### 6.2.4. Consistency and convergence towards the two-dimensional coding $U$

One first checks that the images of the initial nonpointed patterns of the 5 local rules, as well as $\mathscr{I}((\mathbf{1},\mathbf{0}),1)$, are themselves covered. Hence, we are now able to extend the image of any pattern covered by these 5 local rules.

All the pointed patterns ${\Sigma_{0,l}}^n((\mathbf{1},\mathbf{0}),1)$ occur in $U$, since $U(\mathbf{1},\mathbf{0})=1$, and $\Sigma_{0,l}((\mathbf{1},\mathbf{0}),1)=((\mathbf{1},\mathbf{0}),1)((\mathbf{1},\mathbf{1}),2)$, which underlies a local rule. Observe that the iteration using local rules does not produce overlaps following Theorem 6.2. Furthermore, the image of a pointed letter does not depend on the path of local rules which joins it to the initial letter $((\mathbf{1},\mathbf{0}),1)$, since this value is given by $U$. Hence the pointed patterns ${\Sigma_{0,l}}^n((\mathbf{1},\mathbf{0}),1)$ are consistent. In order to prove that the sequence of pointed patterns ${\Sigma_{0,l}}^n((\mathbf{1},\mathbf{0}),1)$ admits a limit sequence defined everywhere in $\mathbb{Z}^2$, it remains to check that we cover all $\mathbb{Z}^2$.

### 6.2.5. Covering of the full lattice $\mathbb{Z}^2$

This is equivalent with covering all the discrete plane $\mathfrak{P}$ when we iterate $\Sigma_{\mathfrak{P}}$ starting from $\mathbf{e}_1+E_1$. The idea of the proof is to strongly use the "inversibility" of $\Sigma_{\mathfrak{P}}$: a face of the discrete plane $\mathbf{x}+E_i$ is said to be the *direct ancestor* of the face $\mathbf{y}+E_j$ if $\mathbf{y}+E_j$ occurs in $\Sigma_{\mathfrak{P}}(\mathbf{x}+E_i)$ in formula (5.1). Indeed the unicity of the direct ancestor comes from Theorem 5.2.

The main point is the following lemma, meaning that the direct ancestor of a given face that is far enough from the origin is nearer from the origin than this face. The Euclidean norm in $\mathbb{R}^3$ is denoted $||\ ||$.

**Lemma 6.3.** *Let $\lambda$ denote one of the two complex conjugate contracting eigenvalues of the incidence matrix of $\Sigma_0$. Let $\mathbf{x}+E_i$ be the direct ancestor of the face $\mathbf{y}+E_j$. Let*

$$C=\frac{1}{\sqrt{|\lambda|}-|\lambda|}\left(3+(\alpha+|\lambda|)\,\frac{\alpha^3}{\alpha^3+2}\,\sqrt{\alpha^4+\alpha^2+1}\right)\quad \textit{and}\quad \mu=\sqrt{|\lambda|}.$$

*If* $||\mathbf{y}||\geqslant C$, *then* $||\mathbf{x}||\leqslant\mu||\mathbf{y}||$.

**Proof.** See Appendix A.

As a consequence, let $\mathbf{y}+E_j$ be a face in the discrete plane. Let $\mathbf{y}_0+E_{j_0}=\mathbf{y}+E_j$, $\mathbf{y}_1+E_{j_1},\ldots,\mathbf{y}_k+E_{j_k}$ be a finite sequence of ancestors of $\mathbf{y}+E_j$, that is, each face $\mathbf{y}_i+E_{j_i}$ occurs in $\Sigma_{\mathfrak{P}}(\mathbf{y}_{i+1}+E_{j_{i+1}})$. Then Lemma 6.3 means that $||\mathbf{y}_i||\leqslant\max\{C,\mu^i||\mathbf{y}||\}$. This implies that there exists an integer $N$ such that $\mathbf{y}+E_j$ occurs in $\Sigma_{\mathfrak{P}}^N(\mathbf{y}_N+E_{j_N})$, and $\mathbf{y}_N$ belongs to the ball of radius $C$ in $\mathbb{Z}^3$.

It suffices now to check that there exists an iteration starting from 1 which covers the ball of radius $C$ in $\mathbb{Z}^3$ to complete the proof.

In our case, one computes $\alpha = 1,46557$ and $|\lambda| = 0,826031$, so that $75 \leqslant C \leqslant 80$. One checks by computation that ${\Sigma_{\mathfrak{P}}}^{100}(\mathbf{e}_1 + E_1)$ covers the ball of size 80 in $\mathbb{R}^3$, which concludes the proof of Theorems 2.7 and 2.10. □

### 6.3. Generalization of this proof to other two-dimensional morphisms

Lemma 6.3 holds for general Pisot unimodular one-dimensional iterated morphisms. Hence, local rules for a two-dimensional morphism arising from a Pisot unimodular one-dimensional iterated morphism will allow one to generate the full plane $\mathbb{Z}^2$ as soon as the three following properties are satisfied:

- The images of initial patterns of local rules are themselves covered.
- The image of the initial letter of the initial rule is covered.
- There exists an iteration of $(\Sigma, \mathscr{I}, \mathscr{S})$ on the initial letter which covers the ball of radius $C$ given in Lemma 6.3.

Some examples satisfy these conditions while other do not. It remains to understand which classes of examples do satisfy these conditions.

## 7. Additional remarks

This study is only a starting point, and many open problems remain. In particular, one would like to know the largest set on which the two-dimensional morphism with local rules $(\Sigma_0, \mathscr{I}, \mathscr{S})$ is defined. One would also like to have a general (algebraic) framework that allows to compute, for a given set of local rules, the set of patterns and of sequences which is consistent for these rules.

Let us focus on the fact that the local rules correspond to a set of nonpointed patterns. Nevertheless, if the image of a given letter is fixed, then one can put the images of the adjacent letters, thus producing a pointed pattern. We have no general statement yet on the existence of the local rules. But we conjecture that there always exists a set of finite local rules for any Pisot unimodular iterated morphism. These rules need not be necessarily connected (as the images of the letters under the two-dimensional morphism). Observe furthermore that the local rules or the global placing rules depend on the one-dimensional morphism $(\Sigma_0, \mathscr{I}, \mathscr{S})$ and not only on the matrix $\mathbf{M}$.

Another question is the “seed” problem: what is the minimal pattern that generates the complete sequence? Examples show that this question is not trivial. As a related example, observe that if one wants to generate the above iterations of $(\Sigma_0, \mathscr{I}, \mathscr{S})$ starting from $((\mathbf{0}, \mathbf{0}), 1)$ (and not $((\mathbf{1}, \mathbf{0}), 1)$ as previously), then one needs further local rules for the beginning. One then can generate any sequence with the same set of factors as $U$ using the extended finite set of local rules.

If we understand reasonably well the theory of one given two-dimensional morphism, it would remain to study the theory of an infinite sequence of morphisms following the $S$-adic approach (see for instance [9,16, Chapter 12]). In the setting of the paper, this amounts to go from the study of the contracting plane of a matrix (given by an equation with algebraic coefficients) to a general plane, using some kind of multidimensional continued fraction algorithm. This study has already been started in [2].

## Appendix A. Technical proofs

**Proof of Proposition 3.3.** Let us notice first that *if four integral vertices that make a unit square all belong to the discrete plane* $\mathfrak{P}$, *then the associated face is all included in* $\mathfrak{P}$. Hence, the discrete plane $\mathfrak{P}$ contains no square hole.

Indeed, when four integral vertices make a square, one of their coordinates is constant. Suppose that it is the third coordinate. Hence the four vertices are of the form $(\mathbf{p},\mathbf{q},\mathbf{r})$, $(\mathbf{p}+\mathbf{1},\mathbf{q},\mathbf{r})$, $(\mathbf{p},\mathbf{q}+\mathbf{1},\mathbf{r})$, $(\mathbf{p}+\mathbf{1},\mathbf{q}+\mathbf{1},\mathbf{r})$. Since $(\mathbf{p},\mathbf{q},\mathbf{r})\in\mathfrak{P}$, we have $ap+bq+cr+h>0$, so that the cube $(\mathbf{p},\mathbf{q},\mathbf{r})+\mathscr{C}$ is not included in $\mathscr{S}$. But $(\mathbf{p}+\mathbf{1},\mathbf{q}+\mathbf{1},\mathbf{r})\in\mathfrak{P}$, which means $a(p+1)+b(q+1)+cr+h\leqslant a+b+c$, that is, $ap+bq+c(r-1)+h\leqslant 0$. Hence the cube $(\mathbf{p},\mathbf{q},\mathbf{r}-\mathbf{1})+\mathscr{C}$ is included in $\mathscr{S}$. Consequently, the unit square with vertices $(\mathbf{p},\mathbf{q},\mathbf{r})$, $(\mathbf{p}+\mathbf{1},\mathbf{q},\mathbf{r})$, $(\mathbf{p},\mathbf{q}+\mathbf{1},\mathbf{r})$, $(\mathbf{p}+\mathbf{1},\mathbf{q}+\mathbf{1},\mathbf{r})$ is at the intersection of a cube in $\mathscr{S}$ and another cube outside $\mathscr{S}$. This means that the full square is at the boundary of $\mathscr{S}$.

If $ap+bq+cr+h\in]0,a]$, then $(\mathbf{p},\mathbf{q},\mathbf{r})$, as well as $(\mathbf{p},\mathbf{q},\mathbf{r}+\mathbf{1})$, $(\mathbf{p},\mathbf{q}+\mathbf{1},\mathbf{r})$ and $(\mathbf{p},\mathbf{q}+\mathbf{1},\mathbf{r}+\mathbf{1})$ satisfy the relation $0<ax+by+cz+h\leqslant a+b+c$, so that they are vertices in $\mathfrak{P}$ and they are the corners of the face $(\mathbf{p},\mathbf{q},\mathbf{r})+E_1$, whose distinguished vertex is $(\mathbf{p},\mathbf{q},\mathbf{r})$. Hence, this face is included in the discrete plane.

If $ap+bq+cr+h\in]a,a+b]$, then $(\mathbf{p},\mathbf{q},\mathbf{r})$ as well as $(\mathbf{p}-\mathbf{1},\mathbf{q},\mathbf{r})$, $(\mathbf{p},\mathbf{q},\mathbf{r}+\mathbf{1})$ and $(\mathbf{p}-\mathbf{1},\mathbf{q},\mathbf{r}+\mathbf{1})$ are vertices of $\mathfrak{P}$ and they are the corners of the face $(\mathbf{p},\mathbf{q},\mathbf{r})+E_2$.

If $ap+bq+cr+h\in]a+b,a+b+c]$ then $(\mathbf{p},\mathbf{q},\mathbf{r})$ as well as $(\mathbf{p}-\mathbf{1},\mathbf{q},\mathbf{r})$, $(\mathbf{p},\mathbf{q}-\mathbf{1},\mathbf{r})$ and $(\mathbf{p}-\mathbf{1},\mathbf{q}-\mathbf{1},\mathbf{r})$ are vertices of the $\mathfrak{P}$ and they are the corners of the face $(\mathbf{p},\mathbf{q},\mathbf{r})+E_3$.

Hence $(\mathbf{p},\mathbf{q},\mathbf{r})$ is the distinguished vertex of at least one face in the discrete plane $\mathfrak{P}$.

Conversely, if $(\mathbf{p},\mathbf{q},\mathbf{r})$ belongs to two faces in $\mathfrak{P}$, then these faces must have different types since two faces with the same type are disjoint by construction. However, $(\mathbf{p},\mathbf{q},\mathbf{r})+E_1$ and $(\mathbf{p},\mathbf{q},\mathbf{r})+E_2$ cannot be included simultaneously in $\mathfrak{P}$: if so, their closure is also included in $\mathfrak{P}$, so that the points $(\mathbf{p},\mathbf{q}+\mathbf{1},\mathbf{r}+\mathbf{1})$ and $(\mathbf{p}-\mathbf{1},\mathbf{q},\mathbf{r})$ belong simultaneously to $\mathfrak{P}$, while they cannot both satisfy the relationship $0<ax+by+cz+h\leqslant a+b+c$. Similarly, $(\mathbf{p},\mathbf{q},\mathbf{r})+E_1$ and $(\mathbf{p},\mathbf{q},\mathbf{r})+E_3$ cannot both be included in $\mathfrak{P}$ (both $(\mathbf{p},\mathbf{q},\mathbf{r}+\mathbf{1})$ and $(\mathbf{p}-\mathbf{1},\mathbf{q}-\mathbf{1},\mathbf{r})$ cannot appear to be an integral vertex). The same holds for $(\mathbf{p},\mathbf{q},\mathbf{r})+E_2$ and $(\mathbf{p},\mathbf{q},\mathbf{r})+E_3$, where $(\mathbf{p},\mathbf{q},\mathbf{r}+\mathbf{1})$ and $(\mathbf{p}-\mathbf{1},\mathbf{q}-\mathbf{1},\mathbf{r})$ would both belong to $\mathscr{V}$. An illustration is given in Fig. 16. This proves the unicity. □

**Proof of Theorem 3.4.** The faces partition the points in $\mathfrak{P}$ that are neither vertices nor on edges. We proved above that vertices are included into exactly one face. Hence the faces of type 1,2 and 3 provide a partition of $\mathfrak{P}$ as soon as every edge is also included into exactly one face. It thus remains to prove that *every edge in the discrete plane* $\mathfrak{P}$ *is included in one and only one face of* $\mathfrak{P}$ *and does not intersect any other face*.

Let us observe that if an edge is included in $\mathfrak{P}$, then so does its closure since $\mathfrak{P}$ is closed. Let $[(\mathbf{p},\mathbf{q},\mathbf{r}),(\mathbf{p},\mathbf{q},\mathbf{r}+\mathbf{1})]\subset\mathfrak{P}$ be a vertical edge included in the discrete plane (the proofs for the edges parallel to the other directions are similar).

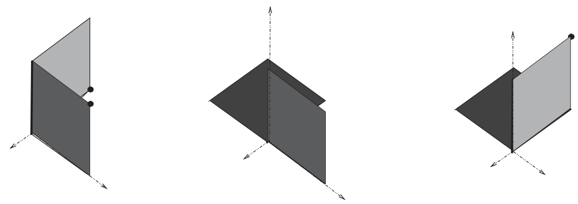


Fig. 16. Configurations that are forbidden in the discrete plane $\mathfrak{P}$.

From the proof of Proposition 3.2, the following holds:

- We know that $(\mathbf{p},\mathbf{q},\mathbf{r})$ belongs to $\mathfrak{P}$, that is, $a\mathbf{p}+b\mathbf{q}+c\mathbf{r}+h>0$. This implies $(\mathbf{p},\mathbf{q},\mathbf{r})+\mathscr{C}\not\subset\mathscr{S}$.
- We also know that $(\mathbf{p},\mathbf{q},\mathbf{r}+\mathbf{1})\in\mathfrak{P}$. This implies that the following inclusion holds $(\mathbf{p}-\mathbf{1},\mathbf{q}-\mathbf{1},\mathbf{r})+\mathscr{C}\subset\mathscr{S}$.

There are two other cubes in $\mathscr{S}$ which might contain this edge, that is, $(\mathbf{p}-\mathbf{1},\mathbf{q},\mathbf{r})+\mathscr{C}$ and $(\mathbf{p},\mathbf{q}-\mathbf{1},\mathbf{r})+\mathscr{C}$:

(1) Assume $(\mathbf{p}-\mathbf{1},\mathbf{q},\mathbf{r})\in\mathfrak{P}$. Then $(\mathbf{p}-\mathbf{1},\mathbf{q},\mathbf{r})+\mathscr{C}\not\subset\mathscr{S}$, so that $(\mathbf{p},\mathbf{q},\mathbf{r})+\overline{E_2}$ is included in the boundary of $\mathscr{S}$. This implies that this face contains the edge $[(\mathbf{p},\mathbf{q},\mathbf{r}),(\mathbf{p},\mathbf{q},\mathbf{r}+\mathbf{1})]$.

(2) Assume $(\mathbf{p}-\mathbf{1},\mathbf{q},\mathbf{r})\notin\mathfrak{P}$. Then $(\mathbf{p}-\mathbf{1},\mathbf{q},\mathbf{r})+\mathscr{C}\subset\mathscr{S}$ so that the face $(\mathbf{p},\mathbf{q},\mathbf{r})+\overline{E_1}$ is included in the boundary of the discrete plane, and contains the edge $[(\mathbf{p},\mathbf{q},\mathbf{r}),(\mathbf{p},\mathbf{q},\mathbf{r}+\mathbf{1})]$.

(3) Assume $(\mathbf{p},\mathbf{q}-\mathbf{1},\mathbf{r})\in\mathfrak{P}$, then $(\mathbf{p},\mathbf{q}-\mathbf{1},\mathbf{r})+\mathscr{C}\not\subset\mathscr{S}$, and the face $(\mathbf{p},\mathbf{q}-\mathbf{1},\mathbf{r})+\overline{E_1}$ is included in the discrete plane and contains the edge $[(\mathbf{p},\mathbf{q},\mathbf{r}),(\mathbf{p},\mathbf{q},\mathbf{r}+\mathbf{1})]$.

(4) Assume $(\mathbf{p},\mathbf{q}-\mathbf{1},\mathbf{r})\notin\mathfrak{P}$, then $(\mathbf{p},\mathbf{q}-\mathbf{1},\mathbf{r})+\mathscr{C}\subset\mathscr{S}$, and the face $(\mathbf{p}+\mathbf{1},\mathbf{q},\mathbf{r})+\overline{E_2}$ is included in the discrete plane and contains the edge $[(\mathbf{p},\mathbf{q},\mathbf{r}),(\mathbf{p},\mathbf{q},\mathbf{r}+\mathbf{1})]$.

In the last two cases, the edge $[(\mathbf{p},\mathbf{q},\mathbf{r}),(\mathbf{p},\mathbf{q},\mathbf{r}+\mathbf{1})[$ is included in the closure of a face but not in this face itself. Hence there exists exactly one face included in $\mathfrak{P}$ which contains $[(\mathbf{p},\mathbf{q},\mathbf{r}),(\mathbf{p},\mathbf{q},\mathbf{r}+\mathbf{1})[$ which corresponds to one of the first two cases. □

**Proof of Theorem 5.2.** Two points need to be checked to state the theorem.

(1) *The morphism $\Sigma_{\mathfrak{P}}$ is well defined from $\mathfrak{P}$ to $\mathfrak{P}^*$: The image of a face in the discrete plane is the union of distinct faces in the discrete plane.*

First, let us check that the union in formula (5.1) is disjoint. Suppose that two faces in this formula are equal. They must have the same type $k$. Hence $\sigma(k)$ shall be decomposed under the forms $\sigma(k)=P_1jS_1=P_2jS_2$ with $\mathbf{l}(P_1)=\mathbf{l}(P_2)$. But, with no loss of generality one can suppose $|P_1|<|P_2|$, so that $P_1$ is a strict prefix of $P_2$, leading to a contradiction.

Let $\mathbf{x}+E_i\subset\mathfrak{P}$. Let us prove that all the pieces in $\Sigma_{\mathfrak{P}}(\mathbf{x}+E_i)$ are faces in $\mathfrak{P}$. Let $\sigma(k)=PiS$, $\mathbf{y}=\mathbf{x}-\mathbf{l}(P)-(\mathbf{e}_1+\cdots+\mathbf{e}_i)$ and $\mathbf{M}^{-1}\mathbf{y}+(\mathbf{e}_1+\cdots+\mathbf{e}_k)+E_k$ be a face in $\Sigma_{\mathfrak{P}}(\mathbf{x}+E_i)$. From the geometrical interpretation of Proposition 3.3, this face is included in the discrete plane if and only if $\mathbf{M}^{-1}\mathbf{y}+\mathbf{e}_k$ is above the plane $\mathscr{P}$ and $\mathbf{M}^{-1}\mathbf{y}$ is below.

Since $\mathbf{x}+E_i\subset\mathfrak{P}$, we know that $\mathbf{x}-(\mathbf{e_1}+\cdots+\mathbf{e_i})=\mathbf{y}+\mathbf{l}(P)$ is below the plane. Hence $\mathbf{y}$ is below $\mathscr{P}$ and so is $\mathbf{M}^{-1}\mathbf{y}$.

We also know that $\mathbf{e_k} = \mathbf{M}^{-1}\mathbf{M}\mathbf{e}_k = \mathbf{M}^{-1}\mathbf{l}\sigma(k) = \mathbf{M}^{-1}[\mathbf{l}(P) + \mathbf{e}_i + \mathbf{l}(S)]$, so that $\mathbf{M}^{-1}\mathbf{y} + \mathbf{e}_k = \mathbf{M}^{-1}(\mathbf{x} - (\mathbf{e}_1 + \cdots + \mathbf{e}_{i-1}))$. By definition of $\mathfrak{P}$, $\mathbf{x} - (\mathbf{e}_1 + \cdots + \mathbf{e}_{i-1})$ is above the plane $\mathscr{P}$, and so does $\mathbf{M}^{-1}\mathbf{y} + \mathbf{e}_k$.

(2) *The morphism $\Sigma_{\mathfrak{P}}$ can be iterated on $\mathfrak{P}^*$: Two distinct faces on the discrete plane have images which do not intersect*. Let $\mathbf{y} + E_k$ be in the intersection of $\Sigma_{\mathfrak{P}}(\mathbf{x}_1 + E_{i_1})$ and $\Sigma_{\mathfrak{P}}(\mathbf{x}_2 + E_{i_2})$. Then there exists $\sigma(k) = P_1 i_1 S_1 = P_2 i_2 S_2$ such that $\mathbf{x}_1 - \mathbf{l}(P_1) - (\mathbf{e_1} + \cdots + \mathbf{e}_{i_1}) = \mathbf{x}_2 - \mathbf{l}(P_2) - (\mathbf{e_1} + \cdots + \mathbf{e}_{i_2})$. If $|P_1| = |P_2|$ then $i_1 = i_2$, which implies $\mathbf{x}_1 = \mathbf{x}_2$ so that $\mathbf{x}_1 + E_{i_1}$ and $\mathbf{x}_2 + E_{i_2}$ are the same. If $|P_1| \neq |P_2|$ we can suppose that $|P_1| < |P_2|$. Then $\mathbf{l}(P_2) = \mathbf{l}(P_1) + \mathbf{e}_{i_1} + \mathbf{z}$ with $\mathbf{z}$ a nonnegative vector. Hence the vector $\mathbf{x}_2 - (\mathbf{e_1} + \cdots + \mathbf{e}_{i_2}) = \mathbf{z} + \mathbf{x}_1 - (\mathbf{e_1} + \cdots + \mathbf{e}_{i_1 - 1})$ is the sum of the nonnegative vector $\mathbf{z}$ with a vector that is known to be above the plane $\mathscr{P}$. But $\mathbf{x}_2 - (\mathbf{e_1} + \cdots + \mathbf{e}_{i_2})$ is also below the plane since $\mathbf{x}_2 + E_{i_2}$ is in the discrete plane, that is, a contradiction. □

**Proof of Lemma 6.3.** Let $(\mathbf{x} + E_i)$ be the direct ancestor of $\mathbf{y} + E_j$, that is, $\mathbf{y} + E_j$ occurs in $\Sigma_{\mathfrak{P}}(\mathbf{x} + E_i)$.

Let $\pi_s$ denote the projection onto the contracting plane $\mathscr{P}$ of $\mathbf{M}$ along its expanding direction $(\alpha^2, 1, \alpha)$, and let $\pi_u$ denote the projection onto the expanding plane $\mathscr{P}$ along the contracting direction. Let $(\mathbf{p}, \mathbf{q}, \mathbf{r}) = \mathbf{x}$ denote the coordinates of $\mathbf{x}$. One has

$$\pi_u(\mathbf{x}) = \frac{\alpha^2 p + \alpha q + r}{\alpha^4 + 2\alpha}\,(\alpha^2, 1, \alpha).$$

Since $(\mathbf{p}, \mathbf{q}, \mathbf{r})$ belongs to the discrete plane, then $\alpha^2 p + \alpha q + r \leqslant \alpha^2 + \alpha + 1 = \alpha^4$,

$$||\pi_u(\mathbf{x})|| \leqslant K_\alpha = \frac{\alpha^4}{\alpha^4 + 2\alpha}\sqrt{\alpha^4 + 1 + \alpha^2}$$

and

$$||\mathbf{x}|| \leqslant ||\pi_s(\mathbf{x})|| + K_\alpha.$$

Since they have the same eigenspaces, the maps $\pi_u$ and $\mathbf{M}^{-1}$ commute, and $\pi_u(\mathbf{M}^{-1}\mathbf{x}) = \mathbf{M}^{-1}\pi_u(\mathbf{x}) = \alpha^{-1}\pi_u(\mathbf{x})$. We thus get

$$||\pi_s(\mathbf{M}^{-1}\mathbf{x})|| \leqslant ||\mathbf{M}^{-1}\mathbf{x}|| + ||\pi_u(\mathbf{M}^{-1}\mathbf{x})|| \leqslant ||\mathbf{M}^{-1}\mathbf{x}|| + \alpha^{-1}K_\alpha.$$

Let $\lambda$ denote one the two conjugate complex eigenvalues of $\mathbf{M}$ ($|\lambda| < 1$); then

$$||\pi_s(\mathbf{x})|| = ||\mathbf{M}\pi_s(\mathbf{M}^{-1}\mathbf{x})|| \leqslant |\lambda|\,||\pi_s(\mathbf{M}^{-1}\mathbf{x})|| \leqslant |\lambda|(||\mathbf{M}^{-1}\mathbf{x}|| + \alpha^{-1}K_\alpha)$$

From formula (5.1), the points $\mathbf{y}$ and $\mathbf{M}^{-1}\mathbf{x}$ differ at most by the sum of the three basic vectors, hence $||\mathbf{M}^{-1}\mathbf{x}|| \leqslant ||\mathbf{y}|| + 3$, so that

$$\begin{aligned} ||\mathbf{x}|| &\leqslant ||\pi_s(\mathbf{x})|| + K_\alpha \leqslant |\lambda|(||\mathbf{M}^{-1}\mathbf{x}|| + \alpha^{-1}K_\alpha) + K_\alpha \\ &\leqslant |\lambda|(||\mathbf{y}|| + 3 + \alpha^{-1}K_\alpha) + K_\alpha \leqslant |\lambda|(||\mathbf{y}|| + K'_\alpha), \end{aligned}$$

with $K'_\alpha = (3 + K_\alpha)/|\lambda| + \alpha^{-1}K_\alpha$.

Now we have to notice that the real function $a \mapsto |\lambda|(a + K'_\alpha)$ is an affine function that is above any affine function with a greater slope as soon as $a$ is large enough. For instance, let $|\lambda| < \mu = \sqrt{|\lambda|} < 1$. Then, if $a \geqslant K'_\alpha \mu/(1-\mu)$, one has $|\lambda|(a + K'_\alpha) \leqslant \mu a$. This implies

$$\text{if } ||\mathbf{y}|| \geqslant C = K'_\alpha \frac{\mu}{1-\mu}, \text{ then } ||\mathbf{x}|| \leqslant \mu ||\mathbf{y}||. \qquad \square$$

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