Methods for the Construction of Membership Functions

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In almost every work on fuzzy sets, the existence of membership functions taking part in the considered model is assumed and it is not studied in depth whether or not such functions exist. On the other hand, generally the relationship between a certain studied characteristic and its referential set is not problematic since it is usually a matter of direct measurement. However, in a great variety of situations it is necessary to work with properties whose measurement is not obvious, but is an object of study in itself. In this work, we start by approaching the description of the cases in which the existence of a membership function can be guaranteed. Next, we consider the situations where one faces linguistic terms associated with attributes that cannot be directly measured; in such cases, the existence of a membership function cannot be assured. However, the conditions of existence and methods for the construction of those membership functions may be based on the psychological measurement theory. © 1999 John Wiley & Sons, Inc.

1. INTRODUCTION

Given a referential X, a fuzzy subset F of X is defined by its membership function

$$\mu_F : X \to [0, 1]$$

As a matter of fact, F can be identified with μ_F. Thus, for example, given a set of ages $X = [0, 100]$, the adjective "young" could have many different representations and interpretations depending on context in terms of membership functions, e.g., for humans, animals, etc.

The different interpretations of the concept of membership function have been dealt with by many authors[1–10] and have been recently systematized to a certain extent by Bilgiç and Türksen[11], and Sancho.[12] Under the framework given by Turksen[10] and generally accepted, those attributes whose measurement

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is "not obvious," but an object of analysis in itself cannot be found in those interpretations, as for example are the cases of quality, efficiency, etc., which can be applicable to different kinds of stimuli and/or objects.

It is worth clarifying as well that generally it is understood that stimulus denotes any agent or condition that by means of receptors can have an influence on a subject.[13] Here we will refer to those stimuli that bear the imputation of a certain characteristic, i.e., those that can be evaluated according to such a characteristic. For example, the set of stimuli can be a group of universities and the characteristic to be evaluated can be "quality."

As is evident, this last context does not allow, at least in an "immediate way," the use of the Turksen model for the construction of membership functions for those linguistic terms associated with attributes that cannot be directly measured. The psychological measurement theory, however, has developed many methods in this field. These are given the generic name *scaling methods*.

In fact, the methods that are most frequently used for the construction of membership functions start from the fact that there exists a function

$$V: \Theta \to X$$

of element set Θ to a referential set X, which is generally an interval in a nonrestrictive sense of the real line.

As Turksen[10] says on page 9, "the function $V: \Theta \to X$ is implicitly assumed and never written. This suppression is justified because the relation between an attribute and its referential set is unproblematic since it is usually just a matter of direct measurement."

In situations where we face linguistic terms associated with attributes that cannot be measured directly (comfort, satisfaction, ...), the methods mentioned for the construction of membership functions are not valid as they appear, but the methods of scaling peculiar to the psychological measurement theory can be valid.

Therefore, from now on we will analyze the cases where the existence of a membership function can be shown and we will describe the main methods for the construction of such a case. Next, in the third section we will introduce scaling methods peculiar to the psychological measurement theory and take them as a basis to establish methods for the construction of membership functions in the cases where linguistic terms associated with attributes that cannot be directly measured are considered. In the last section, we will introduce such methods for the construction of membership functions.

2. MEMBERSHIP FUNCTIONS. DIFFERENT INTERPRETATIONS

By identifying membership functions assigned to the fuzzy sets, most authors agree in interpreting them as representations of meanings[3] of linguistic variables' terms. Although these terms are key elements in any natural language, in reasoning and in human communication[10] there is no unanimous agreement on what is meant by representations of meaning, and it is for this reason that it is convenient to know the different interpretations of the concept. These interpretations can be included within two main approaches according to whether a syntactic or semantic approach is made.

Roughly speaking, the syntactic approach (other synonyms could be formal, axiomatic, theoretical) refers to the interpretations that do not set an agreement as a final objective or contrast with an observable reality (reasoning and/or human communication), but rather internal consistency. Therefore, this approach does not emphasize the attainment of membership functions from human reasoning. From such a point of view, we find few works related to experiments on the construction of membership functions. The semantic approach (also referred to as empirical or pragmatic) is, on the contrary, related to interpretations whose objective is the attainment of models that are contrastable with the observable reality and it can be divided in its turn into normative and descriptive methods. The former ones (also called prescriptive) try to show how people should organize their judgements in a particular situation by using fuzzy sets, while descriptive methods analyze how individuals organize their judgements in fact.

On the other hand, to suggest that the first approach—syntactic—is not concerned about a certain concordance with the observable would be as absurd as to think that the second one—semantic—does not aspire to internal consistency. Both interpretations are separated to emphasize their character with the main aim of establishing differences and classifying them for their study. In any case, the various and best known interpretations are those that appear in terms of likelihood, statistic intervals, similarity, utility theory, and measurement theory.[11]

Particularly, in the latter approach in terms of measurement theory, the degree of membership is considered to be a qualitative subjective phenomenon that can be measured by stressing the different kinds of scales required, and these obviously being in relation to the nature of referential set X. More specifically, according to this interpretation, a degree of membership $\mu_F(x) = p$ is understood as when x is compared to other elements of X, x is in a certain scale of membership to F, with value p. Thus, for instance, in the case of age considered above there could exist a measurement of youthfulness with a value of 0.5 in a 0 to 1 scale for a person aged 40.

In most methods for the construction of membership functions, information is obtained from a set of subjects and this information has an outstanding cognitive character so, as we said, the use of theories and methods from the psychological field seems reasonable and adequate. The aim of the psychological measurement theory is to guarantee in which situations we can assign quantities to properties of the objects of study and how and when operations among these quantities show combinations of any kind among the subjects. In relation to the interpretation as measurements we are talking about, the results from the psychological measurement theory are essential[14,15] in terms of representation, uniqueness, and meaningfulness theorems; that is:

(1) Representation: If a given empirical relational structure satisfies certain axioms, then a homomorphism between this one and a certain numerical relational structure can be constructed.
(2) Uniqueness: If the numerical assignments based on (1) admit certain basic transformations, then the kind of uniqueness of the numerical assignments determines the scale of the numerical relationship.
(3) Meaningfulness: If the scale of the numerical assignments determined in (2) accepts only a certain kind of transformation, then only a certain kind of combinations of numerical assignments are acceptable as specific transformations that produce significant results in the sense of the psychological measurement theory.

On the other side, it is well known that any representation of fuzzy sets requires a basic understanding of the relationships among five different but related conceptual symbols: (a) a set of elements $\delta \in \Theta$, (b) a linguistic variable V, an attribute of elements $\delta \in \Theta$, (c) a linguistic term A of a linguistic variable in a set of linguistic terms, in a context, (d) an interval of measurable numerical assignment $X \in [-\infty, \infty]$, the referential set for V, of Θ, and (e) a subjective numerical assignment $\mu_A(\delta)$, which represents the degree to which δ belongs to the set of elements identified by the linguistic term A.[10]

In this framework, the empirical relational membership structure (the empirical relational structure) is $(\Theta, \geq_A)$, where $\delta \geq_A \delta'$ if δ shows the attribute A at least as much as δ', i.e., $\geq_A$ is a relational (not numerical) order. The numerical relational membership structure (the numerical relational structure) is $[\mu_A(\delta), \geq]$, with $\geq$ in its classical sense.

Within the various scales, which can be constructed in the psychological measurement theory, we basically find five[15]: nominal, ordinal, interval, ratio, and absolute. In the first ones, only the objects of study are distinguished from each other and the valid transformations are bijections. In the second ones, the objects are put in order with at least a weak order and the acceptable transformations are the monotonic ones. In the third ones, the differences among the measures of the objects have to be comparable and verify certain axioms. In this case, the acceptable transformations are lineal with a positive homothetic coefficient. This is known as positive linear transformation. In the fourth ones, the quotient of these differences must be able to be compared and there are more axioms to be satisfied. Moreover, the acceptable transformations are positive coefficient similarities. The fifth is the absolute scale with identity transformation. Within these five kinds of constructible scales, the ordinal and interval ones are outstanding due to their applicability to the context we are interested in here.

Thus, when the considered situation implies that membership functions are based on ordinal scales, the starting point for their construction is the work by Norwich and Turksen.[16] To establish an ordinal scale it is essential that the empirical relational membership structure $(\Theta, \geq_A)$ have at least a weak order. That is, that for every $\delta_i, \delta_j, \delta_k \in \Theta$, the following axioms are satisfied:

(1) Connectedness: Either $\delta_i \geq_A \delta_j$ or $\delta_j \geq_A \delta_i$.
(2) Transitivity: If $\delta_i \geq_A \delta_j$ and $\delta_j \geq_A \delta_k$, then $\delta_i \geq_A \delta_k$.

Then it is shown that if a weak order exists in an empirical structure, then there exists a membership function $\mu_A \in [0,1]$ on Θ so that

$$\forall \delta_i, \delta_j \in \Theta, \qquad \delta_i \geq_A \delta_j \quad \Leftrightarrow \quad \mu_A(\delta_i) \geq \mu_A(\delta_j)$$

and also if μ'_A is another membership function for A on Θ, it has the same property if and only if there exists a transformation function f, monotonic on $[0,1]$, such that for every $\delta \in \Theta$, $\mu'_A(\delta) = f[\mu_A(\delta)]$. That is, μ_A is an ordinal scale for transformation functions f if and only if (1) and (2) are satisfied.

On the other hand, if the situation we suppose assumes membership functions to be based on interval scales, then let $\Theta' = \Theta \times \Theta$, and let us note by $\delta_i \delta_j$ the pair (δ_i, δ_j) of Θ'. A bounded empirical structure $(\Theta, \geq_A)$ for which intervals $\delta_i \delta_j$ in Θ can be weakly put in order by an order $\geq'_A$ is known as a difference-comparable structure and is denoted by $(\Theta', \geq'_A)$. Then we say that

$$\delta_s \delta_t \approx'_A \delta_r \delta_p \quad \Leftrightarrow \quad \delta_s \delta_t \geq'_A \delta_r \delta_p \text{ and } \delta_r \delta_p \geq'_A \delta_s \delta_t$$

and that

$$\delta_s \delta_t >_A \delta_r \delta_p \quad \Leftrightarrow \quad \delta_s \delta_t \geq'_A \delta_r \delta_p \text{ and not } \delta_s \delta_t \approx_A \delta_r \delta_p$$

A comparable differences structure constitutes an algebraic difference structure[15] if the following axioms are satisfied:

(1) Weak order:
Weak connection: Either $\delta_s \delta_t \geq'_A \delta_r \delta_p$ or $\delta_r \delta_p \geq'_A \delta_s \delta_t$.
Transitivity: If $\delta_s \delta_t \geq'_A \delta_r \delta_p$ and $\delta_r \delta_p \geq'_A \delta_q \delta_u$, then $\delta_s \delta_t \geq'_A \delta_q \delta_u$.
(2) Sign inversion: If $\delta_r \delta_s \geq'_A \delta_p \delta_q$, then $\delta_q \delta_p \geq'_A \delta_s \delta_r$.
(3) Weak monotony: If $\delta_r \delta_s \geq'_A \delta'_r \delta'_s$ and $\delta_s \delta_t \geq'_A \delta'_s \delta'_t$, then $\delta_r \delta_t \geq'_A \delta'_r \delta'_t$.
(4) Resolution: If $\delta_r \delta_s \geq'_A \delta_p \delta_q \geq'_A \delta_s \delta_s$, then there exist $\delta'_p, \delta''_p \in \Theta$ such that $\delta'_p \delta_s \geq'_A \delta_p \delta_q \geq'_A \delta_r \delta''_p$.
(5) Archimedean condition: If $\delta_1, \delta_2, \ldots, \delta_i, \ldots$ is a strictly bounded standard succession, that is, $\delta_{i+1} \delta_i \geq'_A \delta_2 \delta_1$ for each δ_i, δ_{i+1} in the succession and not $\delta_2 \delta_1 \approx'_A \delta_1 \delta_1$, and there exists $\delta'_p, \delta''_p \in \Theta$ such that $\delta''_p \delta'_p >'_A \delta_i \delta_1 >'_A \delta'_p \delta''_p$ for every a_i in the succession, then said standard succession is finite.

For a set Θ with a dense order, if an empirical membership structure $(\Theta, \geq_A)$ is of comparable differences and bounded in such a way that $(\Theta', \geq'_A)$ is an algebraic difference structure, then there exists real value bounded membership structure called $\mu_A \in [0,1]$ on Θ, such that for every $\delta_i, \delta_j, \delta_k, \delta_l \in \Theta$,

$$\delta_j \geq_A \delta_i \quad \Leftrightarrow \quad \mu_A(\delta_j) \geq \mu_A(\delta_i)$$

and

$$\delta_j \delta_i \geq'_A \delta_l \delta_k \quad \Leftrightarrow \quad \mu_A(\delta_j) - \mu_A(\delta_i) \geq \mu_A(\delta_l) - \mu_A(\delta_k)$$

On the other hand, μ_A is unique except for bounded and positive linear transformations; that is, if μ'_A has the same property as μ_A, there exist two constants real α, β, with $\alpha > 0$, such that $\mu'_A(\delta) = \min\{1, \alpha \cdot \mu_A(\delta) + \beta\}$, $\forall\delta \in \Theta$; therefore, we can say that μ_A is an interval scale.

Seeing that the existence of such membership functions has been demonstrated in this first case, for their effective construction there exist different methods, among which the following are outstanding.

2.1. Direct Rating

Direct rating supposes that vagueness arises from individual subjective uncertainty. In this procedure elements $\delta \in \Theta$, with values $V(\delta) \in X$, are randomly selected. They are shown to the evaluator and he or she is asked to answer the question: "How A is δ?" where A is the term of a linguistic variable for which we want to construct a membership function.[16] δ is shown to the subject, but $V(\delta)$ is known only by the analyst. The answer given by the subject is a value $y \in [\mu_L, \mu_U]$, generally from 0 to 10.

2.2. Polling

Polling supposes that the vagueness arises from interpersonal disagreements; that is, lack of a precise common meaning. The first authors to use this procedure were Labov[17] and Hersch and Carmazza.[18] Keeping the latter notation, this method starts by asking the following question to the subject set: "Do you agree that δ is A?," allowing just one yes or no dichotomous answer. Value $\mu(x)$ [where $x = V(\delta)$] is directly obtained as the proportion of positive answers over the total number of answers. Therefore, this method as well as interval estimation are especially adequate for constructing collective membership functions.

2.3. Interval Estimation

Interval estimation subscribes to an interpretation in terms of statistical intervals of a membership function. The subject is asked to give an interval that describes the fuzzy property studied given an x, e.g., "From what age and until what age do you consider anybody to be young?" This is also a direct valuation method, but the membership function is obtained by taking into account the whole answer interval. Since each evaluator provides a value set, this is a valuated statistic set method equivalent to the one proposed by Wang, Lin, and Sánchez.[19] With the aim of obtaining membership function values, various forms can be supposed a priori.

2.4. Reverse Rating

The subject is shown a membership degree and is asked to identify the object to which such a membership degree would correspond in relation to the fuzzy term at issue.[10,20] It also requires the evaluation to be made at least on an interval scale. In a way similar to direct valuation, this method provides membership values several times and they have to be presented in such a way that answer memorization is avoided. For each δ an answer distribution $V(\delta) \in X$ is obtained.

2.5. Membership Function Exemplification

Membership function exemplification, also called continuous direct valuation, has been studied by Norwich and Turksen[16] and Chameau and Santamarina.[21] It comprises making membership value evaluations on a point set (generally equidistributed) of the reference variable X interval. Whereas there are no repetitions, answers probably will show a great variability among subjects. The use of computer interactive graphics improves this process greatly. Those graphics can suggest to the evaluator various forms of membership functions and, in those contexts where this can be applied, they can provide consequences for the selection. This method requires knowledge of fuzzy sets topics by experimental subjects, and for this reason it is more suitable for knowledge obtaining methods on trained subjects.

2.6. Pairwise Comparison

The evaluator is asked to choose from among a couple of objects the one that best exemplifies the fuzzy term and to what degree it does so. This method, as well as the latter one, seems to be more appropriate for obtaining individual membership functions.[11] A practical difficulty with this method lies in the considerable number of comparisons necessary for a relatively small number of stimuli or objects of the referential set. The highest theoretical difficulty is the treatment of inconsistencies and intransitivities in subject's answers. However, an advantage is its easy applicability to individuals untrained and without knowledge of fuzzy sets topics.

Besides these methods, there exist other methods called neuro-fuzzy[22,23] which are based on neuron simulation techniques, but are less developed than the latter.

3. SCALING METHODS FOR THE CONSTRUCTION OF MEMBERSHIP FUNCTIONS (EVALUATIONS)

In this section the question of extracting, starting from the classical scaling methods, fuzzy characteristic or attribute membership functions from a stimuli set is approached. In general, information gathering tasks can be classified in two types: judgement tasks and answer tasks. In the former, the subjects express an opinion about a stimulus; in the latter, the subjects express an opinion about him or herself or about his or her agreement with a phrase. Methods for obtaining information for psychological scaling, called tasks, can be classified[24] in a general way according to the following characteristics, of which the most usual in the construction of scales are presented here.[25]

3.1. Simple Stimulus Tasks

A simple stimulus task is that on which a subject is required to give a dichotomous answer in relation to a stimulus. Let us give two examples.

(1) Relative to a judgement task, the stimulus is a university and the question posed to the subject is “Is that a demanding university?” There will be two possible answers: yes or no.
(2) Relative to an answer task, the stimulus would be the question “Habitually, are you an absentminded professor?” As before, there will be two possible answers: yes or no.

Here we find the task proposed by the polling method described above. The first authors 1976[17,18] to use it within the fuzzy sets framework did not mention the use of scales. However, Turksen[10] noted it, although he did not include its use for characteristics unrelated to a continuum (an interval, in general) in a natural way.

3.2. Choice Selection Tasks

In this case, the task to be carried out is the selection of an answer among a small number of choices, generally linguistic expressions set in an increasing order and, generally, in an odd number.[25,26,30] With respect to the latter example, the difference lies in the number of answer choices. For example, for judgement tasks, once a university report has been shown, the subject would be asked to answer the question, “How do you consider the university that has been shown?” The following answers could be allowed: not demanding at all, a little demanding, fairly demanding, quite demanding, or very demanding. On the other hand, for answer tasks, the following example is possible. The subject is asked to remember how much he or she has used the following ways to cope with a stressful situation: “I tried to analyze the causes of the problem in order to face them: (1) never, (2) rarely, (3) sometimes, (4) frequently, (5) nearly always.” As can be imagined, various sets of choices can occur. As well, Likert[30] scale type assessment of “consumer preferences” can be determined for products with linguistic scales from very poor, poor, fairly poor, neutral, fairly good, good, and very good.[10,27,29]

3.3. Paired Stimuli Comparison

Supposing that all possible pairs among a limited stimulus set are made, they are shown to the subjects, who are required to point out which one has a quality to a higher degree. Application to the latter example is clear: the subject would be asked to choose from each pair of universities presented that which he or she considered to be of higher quality. Paired stimuli comparison also can occur in answer tasks, although it is less common. For example, the subject is asked to answer the question “Which of the following two sentences do you most agree with? (a) On certain occasions I feel like taking part in conventions; (b) I cannot stand other colleagues to ask me.

3.4. Grouped Stimuli Comparison

This method can be considered to be a generalization of the latter one. In this case, the task is to choose among a group of stimuli that which best represents the characteristic to be evaluated. This task requires the stimuli to be presented in equally distributed groups; that is, each stimulus must appear the same number of times in the groups presented.

3.5. Grouped Stimuli Arrangement

The task consists of arranging groups from the set of stimuli according to the characteristic to be evaluated. An arrangement task is just a successive comparison task. The best is chosen, then the best of the remaining ones, and so on. Let us suppose that the set of stimuli is universities A, B, C, D, and E. We offer the subject the possible three-stimulus groups and we ask him or her to arrange them in a decreasing way according to "quality" characteristic; that is, the first one will be the one with the highest quality, the second one will be the following one, and so on. This is done with each of the possible triplets of three different universities. For the arrangement of stimuli in judgement tasks we can put forth the following example. Arrange the following sentences according to your degree of agreement with each of them (first, the one you agree with most, etc.): (a) I like correspondence teaching; (b) I read my notes before giving a class; (c) Chatting with my colleagues is the most pleasant thing; (d) I dedicate all the time I can to my students.

3.6. Total Group of Stimuli Arrangement

We will refer to total group of stimuli arrangement when the group of stimuli to be arranged is the total set, that is, here we would offer the subject the latter five universities and would ask him or her to arrange them according to the characteristic pointed out; first the one with the highest quality, etc.

Given a finite set of stimuli, any scaling method gives us a scale E that can be an ordinal or an interval (we are interested here in interval scales, which are the ones that produce unique membership functions) so that E, in interval scales, has a range on an interval I and associates to each stimulus a real value on said interval. Scale E can be understood as the valuation (of a subject or set of subjects) of a certain characteristic of the stimulus. In answer tasks, the characteristic that is imputable to the stimulus is not explicit, but the scale can be understood as the valuation of the agreement between subject and stimulus. In this way, the formal framework for judgement and answer tasks is the same.

On the other hand, a measurement interval by itself does not guarantee the existence of a fuzzy set on the interval. In this sense, as has been said, the work by Turksen[9,10] has laid the foundation for the construction of fuzzy set membership functions based on psychological measurement theory. In the case that an empirical membership function for the set of stimuli can be established, and from this, a difference-comparable structure on the stimuli pair set can be established, the scale obtained can be interpreted as a fuzzy set on X just by making a transformation of I on $[0, 1]$.

As we have seen before, the acceptable transformations on interval scales are linear positive, therefore, if the scale is an interval one and said membership function exists, the associated membership function must be unique, because there is only one positive linear transformation of $I = [m, M]$ to $[0, 1]$. It does not happen like this if the scale is ordinal because any monotonic transformation of I in $[0, 1]$ provides us with a membership function on the set of stimuli. From this we can deduce that in the psychometrical scaling methods that can produce membership functions through interval scales, the membership functions are unique just by adding a positive linear transformation to the scale interval. As a matter of fact, a great part of the methods for the construction of membership functions described above are classical scaling methods (for example, pair comparison, interval estimation, polling, ...). However, these methods are generally reduced to those cases where the subjacent referential set is continuous in a natural way, as noted above.[10] For the other cases, we must make use of psychological scaling methods.

Once we have described the various most fundamental or significant tasks upon which scaling methods are based, it is our intention to determine the conditions under which it is possible to associate a model for the construction of membership functions to these methods. As we already know from the result by Norwich and Turksen,[16] for this to be done necessitates the existence of an empirical structure to which at least one unique ordinal scale can be associated, except for monotonic transformations. However, this is not enough if we want the membership function obtained through the studied task to be unique. To do this it is necessary to extend the empirical structure to the stimuli product set (stimuli pair), constructing a difference-comparable structure.[15] Then, according to the result mentioned by Norwich and Turksen,[16] this structure will generate a unique interval scale except for positive linear transformations, and so the scale obtained in $[0, 1]$, which is the membership function sought, is unique.

When we state here the uniqueness of the membership function, what we want to state is that no other membership function associated with the empirical membership structure or the difference-comparable structure exists. There possibly may be more than one way of defining, given an information-gathering task, a membership function. However, the ones that will be proposed here for each task are those that are understood to be simplest and most intuitive.

Nevertheless, and as a previous step, from now on it is shown how the structures referred to can be constructed for these classical scaling methods.

4. EMPIRICAL STRUCTURES ASSOCIATED WITH THE SET OF STIMULI

Structures associated with simple stimulus tasks, choice selection, paired stimuli comparison, grouped stimuli comparison, grouped stimuli arrangement, and arrangement of total group of stimuli, are constructed, respectively, on the following bases.

4.1. Simple Stimulus Tasks

Let $\Theta = \{x_1, \dots, x_m\}$ be a finite set of m stimuli. Let A be a property capable of being evaluated for these stimuli (for example, "participating"). Let $E = \{e_1, \dots, e_n\}$ be a set of evaluators. Each evaluator is asked for each x_j to answer the question, "Is x_j A ('participating')?" It is worth recalling here that all the following constructions and their subsequent treatment are valid for answer tasks just by agreeing that the attribute evaluated is the "agreement" of the subject to the stimulus. This attribute would, for example, correspond to the question, "Do you agree with x_j?"

Henceforth we suppose that either judgement in answer tasks is treated in each of the information-gathering methods studied.

Let l_{ij} ($i = 1, \dots, n$ and $j = 1, \dots, m$) be the answer given by evaluator i to the question relative to stimulus j, codified with 1 the affirmative and 0 the negative. We define on Θ a relationship

$$x_s \geq_A x_r \quad \Leftrightarrow \quad \sum_{i=1 \cdots n} l_{is} \geq \sum_{i=1 \cdots n} l_{ir}$$

Lemma 4.1. $\geq_A$ *is a weak order on* Θ.

Proof.

(1) $\geq_A$ is connected because $\geq$ is connected in N.
(2) $\geq_A$ is transitive. In fact, let x_s, x_r, and x_t are elements of Θ. We suppose $x_s \geq_A x_r$ and $x_r \geq_A x_t$. Then

$$\sum_{i=1 \cdots n} l_{is} \geq \sum_{i=1 \cdots n} l_{ir}$$

and

$$\sum_{i=1 \cdots n} l_{ir} \geq \sum_{i=1 \cdots n} l_{it}$$

We immediately see that

$$\sum_{i=1 \cdots n} l_{is} \geq \sum_{i=1 \cdots n} l_{it}$$

that is, $x_s \geq_A x_t$.

From this lemma it is deduced that $(\Theta, \geq_A)$ is an empirical structure to which at least an ordinal scale can be associated, unique except for monotonic transformations.[28]

Now we extend the structure to the product set. As we pointed out above, the fulfillment of the extended axiomatic structure (difference-comparable structure) is analyzed next, DEFINITION 4.1. *Let $\Theta' = \Theta \times \Theta$. We denote by $x_r x_s$ the pair $(x_r, x_s) \in \Theta'$, and designate $\geq'_A$ as the relationship defined on Θ', given by*

$$x_r x_s \geq'_A x_p x_q \quad \Leftrightarrow \quad \sum_{i=1\cdots n} (l_{is} - l_{ir}) \geq \sum_{i=1\cdots n} (l_{iq} - l_{ip})$$

LEMMA 4.2. *The relationship defined on Θ', $\geq'_A$ verifies the following properties,*

(i) *It is a weak order on Θ'.*
(ii) *Weak monotonicity: If $x_r x_s \geq_A x_{r'} x_{s'}$ and $x_s x_t \geq_A x_{s'} x_{t'}$, then $x_r x_t \geq_A x_{r'} x_{t'}$.*
(iii) *Sign reversal: If $x_r x_s \geq'_A x_p x_q$, then $x_q x_p \geq'_A x_s x_r$.*

Proof.

(i) $\geq'_A$ is obviously connected, because $\geq$ is connected in R, by the same reason $\geq'_A$ is transitive.
(ii) From $x_r x_s \geq_A x_{r'} x_{s'}$ and $x_s x_t \geq_A x_{s'} x_{t'}$, we have

$$\sum_{i=1\cdots n} (l_{is} - l_{ir}) \geq \sum_{i=1\cdots n} (l_{is'} - l_{ir'})$$

and

$$\sum_{i=1\cdots n} (l_{it} - l_{is}) \geq \sum_{i=1\cdots n} (l_{it'} - l_{is'})$$

and adding and cancelling,

$$\sum_{i=1\cdots n} (l_{it} - l_{ir}) \geq \sum_{i=1\cdots n} (l_{it'} - l_{ir'})$$

From this,

$$x_r x_t \geq_A x_{r'} x_{t'}$$

(iii) From $x_r x_s \geq'_A x_p x_q$, we have

$$\sum_{i=1\cdots n} (l_{is} - l_{ir}) \geq \sum_{i=1\cdots n} (l_{iq} - l_{ip})$$

and changing sign and order,

$$\sum_{i=1\cdots n} (l_{ir} - l_{is}) \leq \sum_{i=1\cdots n} (l_{ip} - l_{iq})$$

that is,

$$x_q x_p \geq'_A x_s x_r$$

To associate a scale interval to a difference structure, in addition to the three conditions demonstrated above, the two following conditions are necessary,[15]

(iv) Solvability: If $x_r x_s \geq'_A x_p x_q \geq'_A x_s x_s$, then there exist $x_{p'}$ and $x_{p''} \in \Theta$ such that $x_{p'} x_s \geq'_A x_p x_q \geq x_r x_p$.
(v) Archimedean condition: If $d_1, d_2, \ldots, d_i \in \Theta$ is a strictly bounded standard succession, then said standard succession is finite.

It can be demonstrated easily that these conditions are verified within the context we are considering.[12] Therefore, we can state that $(\Theta', \geq'_A)$ is an empirical structure to which at least an interval scale μ_A, unique except for positive linear transformations, can be associated.[16]

4.2. Choice Selection Tasks

With the same notation as in the latter section, let answer choices now be $\{a_0, a_1, \ldots, a_h\}$ (generally h is not more than 7).[25,26] Each evaluator is asked for each x_j to answer which choice he or she considers to best represent a: "x_j is A," or in answer tasks, "I agree with x_j." Let l_{ij} be the answer of evaluator i to the question relative to stimulus j, codified as 0 for answer a_0 and[1] for $a_1, \ldots, h$ for a_h.

Keeping the same framework as in the latter section, we can conclude that it is possible to construct an empirical structure associated with the set of stimuli such that an interval scale is obtained. To do this, we define a relationship on Θ as

$$x_s \geq_A x_r \quad \Leftrightarrow \quad \sum_{i=1\cdots n} l_{is} \geq \sum_{i=1\cdots n} l_{ir}$$

where it is verified in a trivial way that $\geq_A$ is a weak order on Θ and, therefore, that $(\Theta, \geq_A)$ is an empirical structure.

Now we extend the structure to the product set.

DEFINITION 4.2. *Let $\Theta' = \Theta \times \Theta$. We denote by $x_r x_s$ the pair $(x_r, x_s) \in \Theta'$ and designate $\geq'_A$ as the relationship defined on Θ' given by*

$$x_r x_s \geq'_A x_p x_q \quad \Leftrightarrow \quad \sum_{i=1\cdots n} (l_{is} - l_{ir}) \geq \sum_{i=1\cdots n} (l_{iq} - l_{ip})$$

Then it can be demonstrated[12] that the relationship defined on Θ', $\geq'_A$ verifies the properties of being a weak order on Θ', weak monotonicity, sign reversal, solvability, and the Archimedean condition. Therefore, for this second choice selection task, structure $(\Theta', \geq'_A)$ is an empirical difference-comparable structure to which at least one interval scale μ_A can be associated.

4.3. Paired and Grouped Stimuli Comparison

Let $\Theta = \{x_1, \ldots, x_m\}$ a finite set of m stimuli. Let A be a property capable of being evaluated for these stimuli. Let $\sigma_1, \ldots, \sigma_k$ be tuples $(t \leq m)$ of stimuli of Θ. The subject is asked to choose the stimulus from each tuple that best represents the property evaluated or for answer tasks, that with which he or she agrees most. If we suppose $m > 2$ and $t \geq 2$, then for $t = 2$ we have the task designated above as paired stimuli comparison. Otherwise, $t > 2$, we are in the task designated as grouped stimuli comparison. For case $t = m$ (only one tuple), in reality we choose only one stimulus from Θ.

Let $l_{ij} = 1$ if stimulus x_j is chosen in tuple σ_j and 0 in another case ($i = 1, \ldots, k$ and $j = 1, \ldots, m$). We define on Θ a relationship in the manner

$$x_s \geq_A x_r \quad \Leftrightarrow \quad \sum_{i=1 \cdots n} l_{is} \geq \sum_{i=1 \cdots n} l_{ir}$$

From which the following result is verified:

LEMMA 4.3. *$\geq_A$ is a weak order on Θ.*

The proof is obvious in the same way as Lemma 4.1.

We can extend $\geq_A$ as above to pair set Θ' without any difficulty, obtaining again an empirical difference-comparable structure to which an interval scale is associated.

4.4. Grouped Stimuli or Full Set of Stimuli Arrangement

Let Θ be a finite set of m stimuli. Let A be an evaluatable property of stimuli from Θ. Let $\sigma_1, \ldots, \sigma_k$, be tuples ($t \leq m$) of elements from Θ. The subject is asked to arrange these tuples from a higher to a lower degree in relation to the property evaluated or with his or her approval or agreement. We can suppose $m > 2$ and $t \geq 2$. For $t = 2$, we find that an arrangement task coincides with the comparison already studied. In case $t = m$ (only one tuple), in reality we arrange all the stimuli of Θ, what we called above the full set of stimuli arrangement. Otherwise, ($t < m$) we would be arranging grouped stimuli.

Let then l_{ij} ($i = 1, \ldots, k$ and $j = 1, \ldots, m$) be the order x_j takes in tuple σ_i. We suppose $l_{ij} = 0$ if x_j does not appears in σ_i. Hence $l_{ij} \in \{0, \ldots, t\}$. We define on Θ a relationship in the manner

$$x_s \geq_A x_r \quad \Leftrightarrow \quad \sum_{i=1 \cdots k} l_{is} \leq \sum_{i=1 \cdots k} l_{ir}$$

We must see that since the arrangement is done from a higher to a lower degree and the numerical assignment is opposed, order signs are interchanged. It seems more natural to keep the numeration increasing and the order of the elements of the tuple decreasing, although it makes it necessary to change order signs. Thus, as above, it is verified that $\geq_A$ is a weak order on Θ, which we can easily extend to product space.

Therefore, as in the cases above, we obtain an empirical structure to which an interval scale is associated.

With this, for the described tasks, we have laid the foundations for the construction of a set of stimuli membership functions that are significant from the point of view of psychological measurement theory. In the following section, we will construct the associated membership function for each of these tasks.

5. CONSTRUCTION OF MEMBERSHIP FUNCTIONS (EVALUATIONS)

Norwich and Turksen[28] established the nexus between the latter structures and membership functions in the following results.

REPRESENTATION THEOREM. *Let $\Theta = \{x_1, \ldots, x_m\}$ be the finite set that represents m stimuli such that $(\Theta, \geq_A)$ is an empirical membership structure and such that $(\Theta \times \Theta, \geq'_A)$ is an algebraic difference structure. Then there exists a bounded real function μ_A: $\Theta \to [0,1]$ such that*

$$x_s \geq_A x_r \quad \Leftrightarrow \quad \mu_A(x_s) \geq \mu_A(x_r)$$

and

$$x_r x_s \geq'_A x_p x_q \quad \Leftrightarrow \quad \mu_A(x_s) - \mu_A(x_r) \geq \mu_A(x_q) - \mu_A(x_p)$$

UNIQUENESS THEOREM. *If μ'_A is another function that satisfies the two conditions of the theorem above, then*

$$\mu'_A(x_i) = c_1\, \mu_A(x_i) + c_2$$

where c_1 and c_2 are real numbers, $c_1 > 0$.

As we will see next, membership function definitions for each of the tasks dealt with here fulfill this condition. It is then a matter of constructing, in each case, a function μ_A: $\Theta \to [0,1]$ such that it turns the defined empirical structure into a membership structure.

5.1. Simple Stimulus Tasks

Let $E = \{e_1, \ldots, e_n\}$ be a set of evaluators. Let l_{ij} ($i = 1, \ldots, n$, and $j = 1, \ldots, m$) be the answer of evaluator i to the question relative to stimulus j, codified with 1 the affirmative and 0 the negative. We define

$$\mu_A(x_j) = \left(\sum_{i=1 \cdots n} l_{ij} \right) \Big/ n$$

Obviously, $\mu_A(x_j) \in [0,1]$ for every j. Let us see that with μ_A, $(\Theta, \geq_A)$ is an empirical membership structure. We had defined for this task

$$x_s \geq_A x_r \quad \Leftrightarrow \quad \sum_{i=1 \cdots n} l_{is} \leq \sum_{i=1 \cdots n} l_{ir}$$

which is equivalent to

$$\mu_A(x_s) \geq \mu_A(x_r).$$

Note that $\mu_A(x_j) = 0$ if and only if no evaluator answers affirmatively to the simple question, while $\mu_A(x_j) = 1$ if and only if every evaluator does so.

In a similar way, we defined above

$$x_r x_s \geq'_A x_p x_q \quad \Leftrightarrow \quad \sum_{i=1\cdots n} (l_{is} - l_{ir}) \geq \sum_{i=1\cdots n} (l_{iq} - l_{ip})$$

which is obviously equivalent to

$$\mu_A(x_s) - \mu_A(x_r) \geq \mu_A(x_q) - \mu_A(x_p)$$

From this definition it is interesting to think that, for this task, the membership function obtained is collective and not individual. To obtain with this task a membership function related to the judgements of just one subject, it will be necessary to repeat the task, avoiding as much as possible answer recalling on the part of the subject. In practice, this basically can be obtained in two ways: by separating data gathering sessions in time or, if the set of stimuli is wide enough, by avoiding tasks where a given stimulus and attribute are in proximity to each other in each presentation. In the case that the number of stimuli and attributes is small and answer memory effect cannot be assured, the matter can be resolved by adding neutral and spurious stimuli and attributes for the task at hand.

5.2. Choice Selection Tasks

With the same notation as in the latter section, let now $\{a_0, a_1, \ldots, a_h\}$ be the answer choices. Let l_{ij} be the answer of evaluator i to the question related to stimulus j, codified as 0 for answer a_0 and 1 for $a_1, \ldots, h$ for a_h. We define

$$\mu_A(x_i) = \left(\sum_{i=1\cdots n} l_{ij} \right) \Big/ nh$$

Obviously, $\mu_A(x_j) \in [0,1]$ for every j. Note that $\mu_A(x_j) = 0$ if and only if all evaluators answer alternative a_0 for stimulus j, while $\mu_A(x_j) = 1$ if and only if all evaluators answer a_h for this stimulus.

Verifications on the good definition and compatibility with membership structure and difference-comparable structure are almost identical to the latter case. In the same way, it is possible to do the same final reflection here as in the latter section in relation to the production of individual membership functions.

5.3. Paired and Grouped Stimuli Comparison

Let $\sigma_1, \ldots, \sigma_k$ be tuples $(t \leq m)$ of stimuli of Θ. Let $l_{ij} = 1$ if stimulus x_j is chosen in tuple σ_i and 0 otherwise ($i = 1, \ldots, k$ and $j = 1, \ldots, m$). We define

$$\mu_A(x_j) = \left(\sum_{i=1\cdots k} l_{ij} \right) \Big/ k$$

Although with this definition, in the same ways as in the latter tasks it can be verified that μ_A is a well-defined membership function and compatible with membership structures, in the two latter tasks, membership functions for the stimuli were obtained given a set of evaluators. Now we obtain membership functions for the stimuli for just one evaluator. The role (in the summation) evaluators played in the latter tasks is now played by tuples. By construction, the following situation appears now: $\mu_A(x_j) = 0$ if and only if stimulus x_j is chosen in none of the groups, while $\mu_A(x_j) = 1$ if and only if in all groups said stimulus is chosen.

5.4. Grouped Stimuli or Full Set of Stimuli Arrangement

Let $\sigma_1, \ldots, \sigma_k$ be tuples $(t \leq m)$ of stimuli of Θ. The evaluator is asked to arrange these tuples to form a higher to a lower degree in relation to the characteristic evaluated. Let l_{ij} $(i = 1, \ldots, k$ and $j = 1, \ldots, m)$ be the order x_j occupies in tuple σ_i. We suppose $l_{ij} = 0$ if x_j does not appear in σ_i. Let $\Gamma(x) = 1$ if $x \neq 0$ and 0 otherwise. We define

$$\mu_A(x_j) = \left[\left(\sum_{i=1\cdots k} l_{ij} \right) \Big/ \left(\sum_{i=1\cdots k} \Gamma(l_{ij}) \right) - t \right] \Big/ (1 - t)$$

It is clear that $\mu_A(x_j)$ is well defined as, by construction

$$\left(\sum_{i=1\cdots k} \Gamma(l_{ij}) \right) \leq \sum_{i=1\cdots k} l_{ij} \leq \left(\sum_{i=1\cdots k} \Gamma(l_{ij}) \right) t$$

from which $1 \leq \sum_{i=1\cdots k} l_{ij} / \sum_{i=1\cdots k} \Gamma(l_{ij}) \leq t$ and, therefore, $\mu_A(x_j) \in [0, 1]$.

The compatibility with defined membership structures easily can be demonstrated.[12]

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