Kasner Cosmological Models with Two-Fluid Source in General Relativity

V.G. Mete · V.M. Umarkar · A.M. Pund

Abstract In this paper we present Bianchi type-I metric of the Kasner form describing two-fluid source of the universe in general relativity. In Kasner cosmological models one fluid is a radiation field modeling the cosmic microwave background, while the other is a matter field, modeling material content of the universe. The radiation and matter content of the universe are in interactive phase. We have also presented anisotropic, homogeneous nature of Kasner cosmological models with two-fluid. The behavior of fluid parameters and kinematical parameters of the models are also discussed.

Keywords Kasner space-time · Two-fluids · Higher dimensions · Cosmological models of universe

1 Introduction

In recent years, there has been considerable interest in the study of Friedman-Robertson-Walker (FRW) spatially homogeneous isotropic models which are widely considered as good approximation of the present and early stages of the universe. Two-fluid FRW models of the universe have been investigated where one fluid is the radiation field corresponding to the observed cosmic background radiation, while a perfect fluid is chosen to represent the matter content of the universe Coley and Tupper [1, 2]. Coley and Dunn [3] examined Bianchi type-VI_0 model with two-fluid source. Two-fluid cosmological models using Bianchi type II space-time has been investigated by pant and Oli [4].

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The generalization of Kasner model were proposed by many authors [5–11] has defined an analytic expression for the anisotropy of the Kasner metric. When cosmological constant is zero and the model isotropizes in Einstein theory for a nonzero cosmological constant, Mothanty and Daud [12] have shown that the model reduces to Kasner [13] form, wherein they studied Bianchi type-I cosmological model with gauge function in vacuum. In Einstein general relativity, a viscous cosmological fluid does not permit the Kasner metric to be anisotropic which have been proved by [14–16].

The Kasner metric is one of the more widely studied metric. It has played a central role in the elucidation of the existence and structure of anisotropic cosmological models and their singularities in general relativity motivates authors to study this problems. The discovery of 2.73 K isotropic cosmic microwave background radiation (CMBR) motivated many authors to investigate FRW model with a two-fluid source [17, 18].

Several prominent results obtained in the development of the super-string theory therefore the study of higher dimensional physics is important. In the latest study of super-strings and super-gravity theories, Weinberg [19] studied the unification of the fundamental forces with gravity which reveals that the space-time should be different from four. The string theories are discussed in 10-dimension or 26-dimensions of space-time, since the concept of higher dimensions is not unphysical. For that reason, many researcher to enter into such field of study to explore the hidden knowledge of the universe to studies in higher dimensions. Several authors [20–27] have studied the multi-dimensional cosmological models in general relativity and in other alternative theories of gravitation.

In this paper, we have investigated two fluid Kasner cosmological models in four and five dimension; an analytic solution of general relativistic field equation is presented and also discussed the physical behavior of the corresponding models.

2 Field Equations

We consider an anisotropic Bianchi type-I metric of Kasner form as

$$ds^2 = dt^2 - t^{2p_1}dx^2 - t^{2p_2}dy^2 - t^{2p_3}dz^2, \tag{2.1}$$

where p_1, p_2 and p_3 are three parameters that we shall requires to be constant, satisfying the two relation $p_1 + p_2 + p_3 = 1$ and $p_1^2 + p_2^2 + p_3^2 = 1$.

The Einstein's field equations for a two fluid source in natural unit (gravitational units) are written as

$$R_{ij} - \frac{1}{2}g_{ij}R = -8\pi T_{ij}. \tag{2.2}$$

The energy momentum tensor for a two fluid source is given by

$$T_{ij} = T_{ij}^{(m)} + T_{ij}^{(r)}, \tag{2.3}$$

where $T_{ij}^{(m)}$ is the energy momentum tensor for matter field and $T_{ij}^{(r)}$ is the energy momentum tensor for radiation field [3] is given by

$$T_{ij}^{(m)} = (p_m + \rho_m)u_i^m u_j^m - p_m g_{ij}, \tag{2.4}$$

$$T_{ij}^{(r)} = \frac{4}{3}\rho_r u_i^r u_j^r - \frac{1}{3}\rho_r g_{ij} \tag{2.5}$$

with

$$g^{ij}u_i^m u_j^m = 1, \qquad g^{ij}u_i^r u_j^r = 1. \tag{2.6}$$

Thus the matter field $T_{ij}^{(m)}$ is described by a perfect fluid with density ρ_m, pressure p_m and the radiation field $T_{ij}^{(r)}$, has density ρ_r, pressure $p_r = \frac{1}{3}\rho_r$.

The off diagonal equations of (2.2) together with energy conditions imply that the matter and radiation are both co-moving, we get,

$$u_i^{(m)} = (0, \ 0, \ 0, \ 1), \qquad u_i^{(r)} = (0, \ 0, \ 0, \ 1). \tag{2.7}$$

Using (2.1), (2.3), (2.4), (2.5) and (2.6), the field equations (2.2) reduces to:

$$\left[p_1(s-1) - \frac{1}{2}(s^2 - 2s + \phi) \right] t^{-2} = 8\pi \left(p_m + \frac{\rho_r}{3} \right), \tag{2.8}$$

$$\left[p_2(s-1) - \frac{1}{2}(s^2 - 2s + \phi) \right] t^{-2} = 8\pi \left(p_m + \frac{\rho_r}{3} \right), \tag{2.9}$$

$$\left[p_3(s-1) - \frac{1}{2}(s^2 - 2s + \phi) \right] t^{-2} = 8\pi \left(p_m + \frac{\rho_r}{3} \right), \tag{2.10}$$

$$\frac{1}{2}(\phi - s^2)t^{-2} = -8\pi(\rho_m + \rho_r), \tag{2.11}$$

where $s = p_1 + p_2 + p_3$, $\phi = p_1^2 + p_2^2 + p_3^2$.

By comparing Eqs. (2.8), (2.9) and (2.10), we get

$$p_1 = p_2 = p_3 = p, \tag{2.12}$$

where p is arbitrary constant.

Using above, Bianchi type-I metric of Kasner form can be written as

$$ds^2 = dt^2 - t^{2p}\left[dx^2 + dy^2 + dz^2 \right].$$

3 Some Physical and Kinematical Properties

We assume the relation between pressure and energy density of matter field through the "gamma-law" equation of state which is given by

$$p_m = (\gamma - 1)\rho_m, \quad 1 \leq \gamma \leq 2.$$

We get energy density of matter, energy density of radiation and total energy density as

$$\rho_m = \left[\frac{s(2s-3) - 3p(s-1) + \phi}{(4-3\gamma)t^2} \right], \tag{3.1}$$

$$\rho_r = \left[\frac{6p(s-1) - 3\phi(2-\gamma) + 3s(2-\gamma s)}{2t^2(4-3\gamma)} \right], \tag{3.2}$$

$$\rho = \rho_m + \rho_r = \frac{4(s^2 - \phi) + 3\gamma(\phi - s^2)}{2t^2(4-3\gamma)}. \tag{3.3}$$

The generalized mean Hubble parameter H is given by

$$H = \frac{1}{3}(H_1 + H_2 + H_3),$$

where H_1, H_2 and H_3 are the directional Hubble Parameter in the direction of x, y and z axes respectively.

Case I: Dust model In order to investigate the physical behavior of the fluid parameters we consider the particular case of dust i.e. when $\gamma = 1$.

The scalar of expansion, shear scalar and deceleration parameter are given by

$$\theta = 3H = 3p,$$
$$\sigma^2 = \frac{3p^2}{2}, \tag{3.4}$$
$$q = -1.$$

The density parameters are

$$\Omega_m = \left[\frac{-3p(s-1) + s(2s-3) + \phi}{3p^2t^2} \right],$$
$$\Omega_r = \left[\frac{6p(s-1) - 3\phi + 3s(2-s)}{6p^2t^2} \right],$$
$$\Omega_0 = \Omega_m + \Omega_r$$
$$= \frac{s^2 - \phi}{6p^2t^2},$$

where Ω_0 is total density parameter.

Case II: Radiation universe (when $\gamma = \frac{4}{3}$) On substituting $\gamma = \frac{4}{3}$, we get, the scalar of expansion, shear scalar and deceleration parameter as

$$\theta = 3H = 3p,$$
$$\sigma^2 = \frac{3p^2}{2}, \tag{3.5}$$
$$q = -1.$$

And the density parameters using are $\gamma = \frac{4}{3}$ in (3.1) and (3.2) are given by

$$\Omega_m = \infty,$$
$$\Omega_r = \infty.$$

Case III: Hard Universe ($\gamma \in (\frac{4}{3}, 2)$ let $\gamma = \frac{5}{3}$) The scalar of expansion, shear scalar and deceleration parameter in Hard Universe are

$$\theta = 3H = 3p,$$
$$\sigma^2 = \frac{3p^2}{2}, \tag{3.6}$$

$$q = -1.$$

For $\gamma = \frac{5}{3}$, (3.1) and (3.2) imply density parameters as

$$\Omega_m = \left[\frac{3p(s-1) - s(2s-3) - \phi}{3p^2t^2} \right],$$

$$\Omega_r = -\left[\frac{6p(s-1) + s(6-5s) - \phi}{6p^2t^2} \right],$$

$$\Omega_0 = \Omega_m + \Omega_r = \frac{s^2 - \phi}{6p^2t^2}.$$

Here ρ_m and total density ρ are positive whereas ρ_r is negative.

Case IV: Zeldovich Universe ($\gamma = 2$)   In this case, we get, the scalar of expansion, shear scalar and deceleration parameter as

$$\theta = 3H = 3p,$$

$$\sigma^2 = \frac{3p^2}{2}, \tag{3.7}$$

$$q = -1.$$

We get, the energy density of matter, energy density of radiation and total energy density as

$$\Omega_m = \left[\frac{3p(s-1) - s(2s-3) - \phi}{6p^2t^2} \right],$$

$$\Omega_r = -\left[\frac{3(s-1)(p-s)}{6p^2t^2} \right],$$

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where p_1, p_2, p_3 and p_4 are four parameters that we shall require to be constant, satisfying the two relation

$$p_1 + p_2 + p_3 + p_4 = 1 \quad \text{and} \quad p_1^2 + p_2^2 + p_3^2 + p_4^2 = 1. \tag{4.2}$$

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Using (4.1), (4.4), (4.5), (4.6) and (4.7) the field equations (4.3) reduces to:

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$$\left[p_4(s-1) - \frac{1}{2}(s^2 - 2s + \phi)\right]t^{-2} = 8\pi\left(p_m + \frac{\rho_r}{3}\right), \tag{4.12}$$

$$\frac{1}{2}(\phi - s^2)t^{-2} = -8\pi(\rho_m + \rho_r), \tag{4.13}$$

where $s = p_1 + p_2 + p_3 + p_4$, $\phi = p_1^2 + p_2^2 + p_3^2 + p_4^2$.

By comparing Eqs. (4.9), (4.10), (4.11) and (4.12) we get

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The generalized mean Hubble parameter H is given by

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$$\Omega_m = \left[\frac{-3p(s-1) + s(2s-3) + \phi}{3p^2t^2} \right],$$

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$$\Omega_m = \left[\frac{3p(s-1) - s(2s-3) - \phi}{3p^2t^2} \right],$$
$$\Omega_r = -\left[\frac{6p(s-1) + s(6-5s) - \phi}{6p^2t^2} \right],$$
$$\Omega_0 = \Omega_m + \Omega_r = \frac{s^2 - \phi}{6p^2t^2}.$$

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Case IV: Zeldovich Universe ($\gamma = 2$)   In this case, we get, the scalar of expansion, shear scalar and deceleration parameter as

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$$\sigma^2 = \frac{32p^2}{9}, \tag{5.7}$$
$$q = -1.$$

We get, the energy density of matter, energy density of radiation and total energy density as

$$\Omega_m = \left[\frac{3p(s-1) - s(2s-3) - \phi}{6p^2t^2} \right],$$
$$\Omega_r = -\left[\frac{3(s-1)(p-s)}{6p^2t^2} \right],$$
$$\Omega_0 = \frac{s^2 - \phi}{6p^2t^2}.$$

Here ρ_m and total density ρ is positive and ρ_r is negative.

6 Conclusion

In this paper, we have provided a detailed Kasner Cosmologies dominating by two relativistic cosmic fluids. Both the cases can be exactly solved in the framework of Einstein field equations.

The positive sign of ($q > 1$) corresponds to decelerating model whereas the negative sign ($-1 < q < 0$) indicates acceleration and $q = 0$ corresponds to expansion with constant velocity.

In Sects. 2 and 3, here, in all cases, we get $q = -1$ indicating that the universe is accelerating. This observation is consistent with the present day observation. the ratio $(\frac{\sigma}{\theta})^2 = \frac{1}{6} \neq 0$. Therefore, these models do not approach isotropy for large value of t.

In Sects. 4 and 5, here we consider five-dimensional anisotropic Bianchi type-I space time of Kasner from.

Here, in all cases, we get negative deceleration parameter indicating that the universe is accelerating. This observation is consistent with the present day observation.

The ratio $(\frac{\sigma}{\theta})^2 = \frac{2}{9} \neq 0$. Therefore, these models do not approach isotropy for large value of t.

It is observed that the results in four dimension retained in higher dimension too.

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