Stochastics and thermodynamics for equilibrium measures of holomorphic endomorphisms on complex projective spaces

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Abstract It was proved by Urbański and Zdunik (Fund Math 220:23–69, 2013) that for every holomorphic endomorphism $f : \mathbb{P}^k \to \mathbb{P}^k$ of a complex projective space $\mathbb{P}^k, k \geq 1$, there exists a positive number $\kappa_f > 0$ such that if J is the Julia set of f (i.e. the support of the maximal entropy measure) and $\phi : J \to \mathbb{R}$ is a Hölder continuous function with $\sup(\phi) - \inf(\phi) < \kappa_f$ (pressure gap), then ϕ admits a unique equilibrium state μ_ϕ on J. In this paper we prove that the dynamical system (f, μ_ϕ) enjoys exponential decay of correlations of Hölder continuous observables as well as the Central Limit Theorem and the Law of Iterated Logarithm for the class of these variables that, in addition, satisfy a natural co-boundary condition. We also show that the topological pressure function $t \mapsto P(t\phi)$ is real-analytic throughout the open set of all parameters t for which the potentials $t\phi$ have pressure gaps.

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1 Introduction

Fix an integer $k \geq 1$. Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism of degree $d \geq 2$ of the complex projective space $\mathbb{P}^k$. Denote by $J = J(f)$ the Julia set of the map $f : \mathbb{P}^k \to \mathbb{P}^k$, i.e. the topological support of the measure of maximal entropy. The map $f : \mathbb{P}^k \to \mathbb{P}^k$ is called regular if its exceptional set $E = E(f)$ does not intersect the Julia set $J = J(f)$. Recall that the exceptional set $E = E(f)$ is a proper algebraic, totally invariant subset contained in the critical set, such that, given $a \in \mathbb{P}^k$, the sequence of the measures $d^{-kn}(f^n)^* \delta_a$ equally distributed on the preimages of the point a converges to the measure of maximal entropy if and only if $a \notin E$. For $k = 1$ the set $E(f)$ is either empty, or of cardinality 2 ($z \mapsto z^{\pm d}$), or of cardinality 1 (polynomials). Obviously, for $k = 1$ the set $E(f)$ never intersects $J(f)$, so in dimension 1, every map is regular. In dimension $k > 1$, we put this as an additional assumption, although we do not know any example of a holomorphic map $f : \mathbb{P}^k \to \mathbb{P}^k$ for which the intersection $E(f) \cap J(f)$ would be nonempty. Moreover, it is known that the set $E(f)$ is empty for a generic holomorphic map $f : \mathbb{P}^k \to \mathbb{P}^k$ (see [3, Lemma 1.5.6]).

Let $\phi : J(f) \to \mathbb{R}$ be a continuous function, in the sequel frequently referred to as a potential. By $P(\phi)$ we denote the (classical) topological pressure of the potential ϕ with respect to the dynamical system $f : J(f) \to J(f)$. Its definition and a systematic account of properties can be found for example in [11]. If μ is a Borel probability f-invariant measure on $J(f)$, we denote by $h_\mu(f)$ its Kolmogorov–Sinai metric entropy. The relation between pressure and entropy is given by the following celebrated Variational Principle.

$$P(\varphi) = \sup \left\{ h_\mu(f) + \int \varphi \, d\mu \right\}, \tag{1.1}$$

where the supremum is taken over all Borel probability f-invariant measures μ, or equivalently, over all Borel probability f-invariant ergodic measures μ. The measures μ for which

$$h_\mu(f) + \int \phi \, d\mu = P(\phi)$$

are called equilibrium states for the potential ϕ. The main theorem proved in [15] is this.

Theorem 1.1 *Let f be a regular holomorphic endomorphism of a complex projective space $\mathbb{P}^k, k \geq 1$, of algebraic degree d. Then there exists a positive number κ_f, where $\log d \geq \kappa_f > 0$, such that if $\phi : J(f) \to \mathbb{R}$ is a Hölder continuous function with $\sup(\phi) - \inf(\phi) < \kappa_f$ (we then say that ϕ has a pressure gap), then ϕ admits a unique (hence ergodic) equilibrium state μ_ϕ on J. This equilibrium state is equivalent to a fixed point of the normalized dual Perron–Frobenius operator. In addition the dynamical system (f, μ_ϕ) is K-mixing. In the case when the Julia set J does not intersect any periodic irreducible algebraic variety contained in the critical set of f, we have that $\kappa_f = \log d$.*

Remark 1.2 We would like to remark that in [2] the key assumption (pressure gap) was that

$$\sup(\phi) < P(\phi), \tag{1.2}$$

and this assumption was then repeated in [12]. It corresponds to the assumption

$$(k - 1) \log d + \sup(\phi) < P(\phi) \tag{1.3}$$

in the multidimensional (k is the dimension) case; it reduces to (1.2) if $k = 1$. In the present paper the weakest assumption is that $\sup(\phi) - \inf(\phi) < \log d$, which is somewhat stronger than (1.3). In fact we inherited it from [15], where it permitted us to estimate almost painlessly from below the eigenvalues of the Perron–Frobenius operator. With the assumption (1.3) it would be an additional hard technical issue to, the already technically heavy paper [15].

The main object of study in our paper will be the dynamical system (f, μ_ϕ). We shall show in Theorem 7.6 that this system enjoys exponential decay of correlations of Hölder continuous observables as well as the Central Limit Theorem and the Law of Iterated Logarithm for the class of these variables that, in addition, satisfy a natural co-boundary condition. We also show in Theorem 6.1 that the topological pressure function $t \mapsto P(t\phi)$ is real-analytic throughout the open set of all parameters t for which the potentials $t\phi$ have pressure gaps.

This paper is self-contained in the sense that all notions used are introduced and all the steps leading to the main theorems are explained. Many proofs, however, are close to those for the 1-dimensional case dealt with in [12], and pointing out the particular fragment of [12] we refer the reader to this paper for some proofs.

Another class of invariant measures was considered by Dupont in [5]. These are images (under certain natural coding) of Gibbs measures for some Hölder continuous potentials, on an appropriate symbolic space. Stochastic properties were proved for these measures in [5] (see also [4]). However, the approach used in that paper cannot be applied for our class of the most natural invariant measures. In our case, the potential is a regular (Hölder continuous) function defined directly in the dynamical space $\mathbb{P}^k$. But its composition with the coding map is (usually) highly irregular.

2 Good holomorphic inverse branches

Assumptions. Throughout the paper, we keep the following two assumptions:

(1) $f : \mathbb{P}^k \to \mathbb{P}^k$ is a regular holomorphic map.

(2) $\phi : J(f) \to J(f)$ is a Hölder continuous potential satisfying $\sup \phi - \inf \phi < \kappa_f$.

Under these assumptions, the main theorem from [15] holds (see Theorem 1.1 above). First, we recall that in [15] the equilibrium measure was constructed as a fixed point of the corresponding Perron–Frobenius operator. More precisely, assuming (which is always possible) that $P(\phi) = 0$, we finally obtain the invariant density ρ_ϕ as a unique fixed point of the operator $\mathcal{L}_\phi : C(J(f)) \to C(J(f))$:

$$\mathcal{L}_\phi(\rho)(x) = \sum_{y \in f^{-1}(x)} \exp \phi(y) \rho(y),$$

where the inverse images of critical values of f are counted with multiplicities. As usually, we shall denote by $S_n \phi$ the sum

$$S_n \phi(y) = \phi(y) + \cdots + \phi(f^{n-1}(y)).$$

The unique conformal measure m_ϕ is obtained as a fixed point of the dual operator $\mathcal{L}_\phi^*$. The measure μ_ϕ is then equal to

$$d\mu_\phi = \rho_\phi \, dm_\phi.$$

Note that $\exp(-\phi)$ is the Jacobian of the measure m_ϕ while

$$\exp(-\tilde{\phi}) = \frac{\rho_\phi \circ f}{\rho_\phi} \exp(-\phi) \tag{2.1}$$

is the Jacobian of the measure μ_ϕ, where

$$\tilde{\phi} = \log \rho_\phi - \log \rho_\phi \circ f + \phi.$$

Note also that for the normalized Perron–Frobenius operator $\mathcal{L}_{\tilde{\phi}}$, defined analogously to $\mathcal{L}_\phi$, with ϕ replaced by $\tilde{\phi}$, we have

$$\mathcal{L}_{\tilde{\phi}}(\mathbb{1}) = \mathbb{1}.$$

From now on, we assume that $P(\phi) = 0$.

Given an open connected subset W of $\mathbb{P}^k$ and given an integer $n \geq 1$, we denote by $I_n(W)$ the collection of all connected components of $f^{-n}(W)$. If $V \in I_n(W)$ and $f|_V^n : V \to W$ is a bijection (equivalently an injection), we set

$$f_V^{-n} := (f|_V^n)^{-1} : W \to V.$$

We denote the collection of all such components by $\mathrm{PG}_n(W)$ and refer to them as pre-good components. Of course, if $V \in \mathrm{PG}_n(W)$, then the map $f_V^{-n} : W \to V$

is a holomorphic homeomorphism from W to V. We call it the holomorphic inverse branch of f^n from W to V.

It follows from our assumptions that there exists $\delta > 0$ such that

$$\inf(\phi) \geq \sup(\phi) - (1 - \delta) \log d.$$

Using the above notation, we have

$$1 = \int \mathcal{L}_\phi \mathbb{1} dm_\phi = \int \sum_{w \in f^{-1}(z)} \exp \phi(w) dm_\phi \geq d^k \exp(\inf \phi)$$
$$= \exp(k \log d + \inf(\phi))$$
$$\geq \exp((k - 1 + \delta) \log d + \sup \phi)).$$

Using this, we get

$$\exp(\phi(z)) \leq \exp(-(k - 1 + \delta) \log d) = d^{-(k-1+\delta)}.$$

Thus,

$$\exp(S_n \phi(z)) \leq d^{-n(k-1+\delta)}$$

and

$$\exp(S_n \tilde{\phi}(z)) \leq M d^{-n(k-1+\delta)} \tag{2.2}$$

where $M = \sup(\rho_\phi) / \inf(\rho_\phi)$. The proof of the following proposition is now immediate.

Proposition 2.1 *If $V \in PG_n(W)$, i.e. if V is a pre-good component of $f^{-n}W$, then*

$$\mu_\phi(V) \leq M d^{-(k-1+\delta)} \mu_\phi(W).$$

Given $l \in \mathbb{N}$, denote by PC_l the postcritical set of the map f^l, i.e.

$$PC_l = f^l(\mathrm{Crit}(f^l)).$$

Our main technical tool in this section is the following.

Lemma 2.2 *Fix δ as above and some $0 < \delta' < \delta$. Let $\theta = d^{-\frac{\delta'}{2}} < 1$. Then there exists $\eta > 0$ such that, for every $\varepsilon > 0$ there exist $l \in \mathbb{N}$ and $C < \infty$ such that if B is an arbitrary open ball centered at a point from the Julia set J, and if B is disjoint from the set PC_l then, with $B'' = \eta B$, for every integer $n \geq l$ there exists a subset of components of $G_n(B'') \subset PG_n(B'')$ with the following properties.*

(a) *If $V \in G_{n+1}(B'')$, then $f(V) \in G_n(B'')$.*
(b) *If $V \in G_n(B'')$, then $\mathrm{diam}(V) \leq C\theta^n$.*
(c) $\mu_\phi \left(\bigcup G_n(B'') \right) \geq (1 - \varepsilon) \mu_\phi(B)$, *where* $\bigcup G_n(B'') := \bigcup \{V : V \in G_n(B'')\}$

(d) *There exists $\beta > 0$ such that*

$$\mu_\phi \left(\bigcup B_n(B'') \right) \leq \exp(-\beta n),$$

where $B_n(B'')$ consists of all connected components of the sets $f^{-1}(V), V \in G_{n-1}(B'')$ that do not belong to $G_n(B'')$ and $\bigcup B_n(B'') := \bigcup \{V : V \in B_n(B'')\}$.
(e) $\sup_z B_n(z) \leq \exp(-\beta n)$, *where* $B_n(z) = \sum_{w \in f^{-n}(z) \cap \bigcup B_n(B'')} \exp S_n \tilde{\phi}(w)$.

Proof It can be deduced from the proof of the similar result for the measure of maximal entropy, see [6, p.1599]. The difference is that, here, we have to estimate not only the number of "bad" components B_n, which is sufficient for the case of maximal measure, but also the value of the measure μ_ϕ of their union. The estimate (d) can be easily deduced from [6], using Proposition 2.1 above (the estimate of the measure of a pre-good component), as we will sketch briefly below. Of course (d) follows from (e), and (c) follows from (d) for l large enough.

We put some details below.

In the first step—exactly in the same way as Guedj—we show that, the family $f^{-1}(PG_n(\rho_{n-1}B) \setminus PG_n(\rho_n B))$ (meaning: all components in $f^{-1}PG_n(\rho_{n-1}B)$, which are not components in $PG_n(\rho_n B)$ and $\rho_n = \sum_l^n j^{-2}$) consists of at most $Dn^{4(k-1)}d^{(k-1)n}$ components, where D is a constant depending on the dimension k and on the degree of the map f, but independent of n. In the light of Proposition 2.1 this gives estimates on the number and measure of components in $PG_n(\rho B)$ where $\rho = \lim \rho_n = \sum_l^\infty j^{-2}$.

In the second step we define $G_n(B'')$, $B'' = a\rho B$, to be:

$$G_n(B'') := \{V \in PG_n(B'') : \text{diam}(V) \leq C\theta^n = d^{-\delta'/2}\}$$

where a and C are constants which will be specified below.

If we prove that $PG_n(B'') \setminus G_n(B'')$ consists of at most $d^{(k-1+\delta')n}$ components, then we will be able to estimate the function $B_n(z)$ by $Md^{-(\delta-\delta')n}(Dn^{4(k-1)} + 1)$. The latter is smaller than $\exp(-\beta n)$ (with β equal, say $(\delta - \delta')/2$) for all $n \geq l$ if l has been chosen large enough.

To show that, first—again, in the same way as Guedj—we notice that

$$\sum_{V \in PG_n(B')} \int_{\mathbb{P}^{k-1}} \text{Area}(f_V^{-n}(\Delta_\gamma)) \, d\nu(\gamma) \leq d^{(k-1)n},$$

where ν denotes Fubini-Study measure on $\mathbb{P}^{k-1}$, $\Delta_\gamma := L_\gamma \cap \rho B$ and L_γ is the complex line passing through the center of B in the direction γ. Thus, there are at most $d^{(k-1+\delta')n}$ components $V \in PG_n(\rho B)$, for which the inequality

$$\int \text{Area}(f_V^{-n}(\Delta_\gamma)) \, d\nu(\gamma) \leq d^{-\delta' n} \tag{2.3}$$

does not hold. If on some $V = f_V^{-n}(\rho B)$ it does hold, then the diameter of $f_V^{-n}(1/4\rho B)$ is small. To prove it, we shall use the same argument as in [6], which we include for the convenience of the reader. Consider $V \in PG_n(\rho B)$ on which (2.3) does hold. Then on a set $A \in \mathbb{P}^{k-1}$ of measure bigger than $1/2$ we have $\text{Area}(f_V^{-n}\Delta_\gamma) \leq 2d^{-\delta'n}$. Now, an argument using extremal length (see lemma in Appendix of [1]) we get that $\text{diam}(f_V^{-n}1/2\Delta_\gamma) \leq 2cd^{-\delta'n/2}$ for $\gamma \in A$ where c is a constant depending on the dimension k only. Then, a theorem in [14] (see [3], Theorem 1.4.10 for a convenient formulation) gives us that $\text{diam}(f_V^{-n}a\rho B) \leq 2cd^{-\delta'n/2}$, where, again a is a constant depending only on the dimension of the space but independent of f and of n. This gives the statement of the Lemma with $\eta = a\rho$. $\qquad\square$

For η as in the statement of Lemma 2.2, put $B' = \frac{1}{2}\eta B = \frac{1}{2}B''$. Applying Cauchy's formulas for partial derivatives, we directly obtain from Lemma 2.2 the following.

Corollary 2.3 *With the hypotheses of Lemma 2.2 we have that there exists a constant $C_\eta > 0$ such that*

$$||Df_V^{-n}(z)|| \leq C_\eta \theta^n$$

for all $n \geq 0$, all $V \in G_n(B'')$ and all $z \in B' = \frac{1}{2}\eta B$.

Fixing l. We now keep the setting of Lemma 2.2 with $\varepsilon := 1/2$. This gives the value of iterate l, which we fix at this step of the construction.

Notation. From now on, we frequently write G_n, B_n, instead of $G_n(B')$, $B_n(B')$. As in Lemma 2.2, item (e), we denote

$$B_n(z) = \sum_{w \in f^{-n}(z) \cap \bigcup B_n(B')} \exp S_n\tilde{\phi}(w).$$

Since we will need to pass several times to an iterate, we now introduce the following.

Notation. $G_n^{f^m}$ denotes the family of good components G_{mn}. Next, $B_n^{f^m}$ equals, roughly speaking $f^{-m}(G_{n-1}^{f^m}) \setminus G_n^{f^m}$, or, more precisely, $B_n^{f^m}$ consists of those components of the sets $f^{-m}V$ (where $V \in G_{n-1}^{f^m}$), which are not in $G_n^{f^m}$. Accordingly, we put

$$B_n^{f^m}(z) = \sum_{w \in f^{-nm}(z) \cap \bigcup B_n^{f^m}(B')} \exp S_{nm}\tilde{\varphi}(w).$$

Lemma 2.4 *There exists a constant $C = C(\beta)$ such that, for all $q \geq l, n \geq 2$, and all $z \in B'$, we have that*

$$B_n^{f^q}(z) \leq C \exp(-\beta(n-1)q).$$

Proof First, notice that

$$B_n^{f^q} = f^{-q}\left(\bigcup G_{q(n-1)}\right) \setminus \bigcup G_{qn} = \bigcup_{(n-1)q < k \leq nq} f^{-(nq-k)}\left(\bigcup B_k\right).$$

Thus, remembering that $\mathcal{L}_{\tilde{\phi}}(\mathbb{1}) = \mathbb{1}$, and, consequently, for every set A, and every m, $\mathcal{L}_{\tilde{\phi}}(\mathbb{1}_{f^{-m}(A)}) = \mathbb{1}_A!$, we get that

$$B_n^{f^q}(z) = \mathcal{L}_{\tilde{\phi}}^{nq} \mathbb{1}_{B_n^{f^q}}(z) = \sum_{(n-1)q < k \leq nq} \mathcal{L}_{\tilde{\phi}}^{nq} \mathbb{1}_{f^{-(nq-k)}(B_k)}(z) = \sum_{(n-1)q < k \leq nq} \mathcal{L}_{\tilde{\phi}}^{k} \mathbb{1}_{B_k}(z)$$

$$= \sum_{(n-1)q < k \leq nq} B_k(z) \leq \sum_{(n-1)q < k \leq nq} \exp(-\beta k)$$

$$\leq \frac{\exp(-\beta)}{1 - \exp(-\beta)} \exp(-\beta(n-1)q).$$

We are done.   □

Corollary 2.5 *Let $q \geq l$; put $g = f^q$. There exists a constant C_q such that, for all $n \geq 1$ and all $z \in B'$,*

$$B_n^g(z) := B_n^{f^q}(z) \leq C_q \exp(-\beta nq).$$

Proof This directly follows from Lemma 2.4, by putting

$$C_q := \exp(\beta q) \max \left\{ C(\beta), \sup_z B_1^g(z) \right\}.$$

□

In what follows, we shall need to pass to yet another iterate $h = g^r$. The corollary below gives an explicit bound for $B_n^h(z)$.

Corollary 2.6 *Let, as before, $g = f^q$, $q \geq l$. Now, let $r \geq 1$ and $n \geq 2$. Then*

$$B_n^h(z) := B_n^{f^{qr}}(z) \leq C_q \exp\left(\frac{-n\beta qr}{2}\right)$$

where C_q is another constant depending on q, but independent of r and n.

Proof Writing

$$\bigcup B_n^h = \bigcup_{(n-1)r < k \leq nr} g^{-(nr-k)} \left(\bigcup B_k^g\right)$$

and using the same reasoning as in the proof of Lemma 2.4, we get

$$B_n^h(z) \leq \sum_{(n-1)r < k \leq nr} B_k^g(z) \leq \sum_{(n-1)r < k \leq nr} C_q \exp(-\beta kq)$$

$$\leq C_q (1 - \exp(-\beta q))^{-1} \exp(-\beta(n-1)qr).$$

Since $n \geq 2$, we have $n - 1 \geq n/2$, and the result follows.   □

We fix here the constant C_q, for which both Corollaries 2.5 and 2.6 hold.

The most significant consequence for a component to belong to G_n is this.

Corollary 2.7 *There exists a constant C_l such that, for all n, for every good component V of the set $f^{-n}(B')$, i.e. $V \in G_n$, and for all $x, y \in V$, we have that*

$$\frac{\exp(S_n\phi(x))}{\exp(S_n\phi(y))} \le C_l \quad \text{and} \quad \frac{\exp\left(S_n\tilde{\phi}(x)\right)}{\exp\left(S_n\tilde{\phi}(y)\right)} \le C_l. \tag{2.4}$$

Proof This follows directly from the expanding property along good branches (see item (b) in Lemma 2.2) and Hölder continuity of ϕ. □

3 Selection of the root ball

First, note that the topological support of the measure μ_ϕ is equal to $J(f)$. This means that every open ball intersecting $J(f)$ has a positive measure μ_ϕ.

The proof of the following lemma is rather standard. A slightly different version can be found in [12, Lemma 9].

Lemma 3.1 *Let $a < b$ be two real numbers. If μ is a Borel finite measure on $[a, b]$, then for every $\lambda > 1$ large enough there exists a point $c \in (a, b)$, in fact a measurable set of positive Lebesgue measure of such points c, such that*

$$\mu\left((c - \lambda^{-n}, c + \lambda^{-n})\right) \le \tilde{\lambda}^{-n}$$

for all integers $n \ge 1$, where $\lambda = \tilde{\lambda}^3$.

As a straightforward consequence of the abstract Lemma 3.1, we shall prove the following.

Lemma 3.2 *For every point $w \in J \setminus PC_l$ there exists a ball B centered at w such that $\frac{2}{\eta} B \cap PC_l = \emptyset$ and*

$$\mu_\phi\left(B(\partial B, \lambda^{-n})\right) \le \tilde{\lambda}^{-n}.$$

with $\lambda, \tilde{\lambda}$ taken from Lemma 3.1, and η, taken from Lemma 2.2.

Proof Take any $R > 0$ so small that $\frac{2}{\eta} B(w, R) \cap PC_l = \emptyset$ and consider the map $P : \overline{B}(w, R) \to [0, R]$ given by the formula $P(z) = ||z - w||$. Applying Lemma 3.1 to the measure $\mu_\phi \circ P^{-1}$ the assertion of Lemma 3.2 immediately follows. □

4 Fine inducing scheme

In this section, starting from the ball selected above, we define good and very good pullbacks defined on B. By pullbacks we mean the holomorphic branches of f^{-qr} (where r will be precisely determined below). Using these selected "very good" pullbacks we shall define, in Sect. 5, the induced map (an infinite Iterated Function System).

We assume that an integer $r \geq 1$ is so large that

$$C_\eta \theta^r \leq 1/4, \tag{4.1}$$

where the constant C_η comes from Corollary 2.3. More requirements on r will be imposed later. Also, q will be fixed later on. Having q and r chosen, we put, as before:

$$h = f^{qr} : \mathbb{P}^k \to \mathbb{P}^k.$$

and, to simplify the notation,

$$\phi_0 := \sum_{j=0}^{qr-1} \phi \circ f^j,$$

$$\tilde{\phi}_0 := \sum_{j=0}^{qr-1} \tilde{\phi} \circ f^j = \sum_{j=0}^{qr-1} \phi \circ f^j - \log \rho_\phi \circ f^{qr-1} + \log \rho_\phi.$$

Denote by $\mathcal{L}_0$ the normalized Perron–Frobenius operator associated to the map $h : J \to J$. It is given by the formula

$$\mathcal{L}_0(\zeta)(x) = \sum_{y \in h^{-1}(x)} \zeta(y) \exp(\tilde{\phi}_0)(y) = \frac{1}{\rho_\phi(x)} \mathcal{L}_\phi^{qr} (\zeta \cdot \rho_\phi)(x).$$

Note that $\mathcal{L}_0(\mathbb{1}) = \mathbb{1}$. Again, this is important in the forthcoming proofs.

Remark 4.1 Actually, since ρ_ϕ is a continuous function, $\mathcal{L}_0$ acts (as well as $\mathcal{L}_\phi$) as a continuous linear operator in the space $C(J)$. However, we do not use the continuity of ρ_ϕ. The only fact we need is that ρ_ϕ is bounded away from 0 and ∞.

The following, quite general Lemma was proved in [12].

Lemma 4.2 *Assume that $Q \subset \mathbb{P}^k$ is a set for which there exists $\gamma > 0$ such that*

$$\mathcal{L}_0(\mathbb{1}_Q)(x) > \gamma \tag{4.2}$$

for every $x \in J(f)$. Then there exist $\alpha \in (0, 1)$, an integer $n_0 \geq 0$, and $\delta > 0$, all three depending on γ only (in particular independent of $r \geq 1$), such that for all $n \geq n_0$ we have that,

$$\mu_\phi \left(\left\{ x \in J(f) : \# \left\{ 0 \leq i \leq n : h^i(x) \in Q \right\} \leq \alpha n \right\} \right) < \exp(-\delta n). \tag{4.3}$$

Remark 4.3 In order to make this lemma useful for us, we need some preparation. Note that, given h (recall that $h = f^{rq}$) and an open set Q, it is easy to find γ for which (4.2) is satisfied (the existence of such γ follows directly from the topological exactness of the map). However, at this point we need to show that, for our particular

choice of $Q = Q_r$ (see Lemma 4.4 below), γ can be chosen to be independent of r, as r will be modified in the course of the proof.

Fixing q. We choose q at this point; it will not be modified in the rest of the proof. On the contrary, r has not been fixed yet, and some additional conditions on r will appear later on. Recall that l has been already chosen in Lemma 2.2, applied for $\varepsilon = \frac{1}{2}$. The ball B, and, consequently $B' = (\eta/2)B$, has been fixed in Lemma 3.2. Since $f_{|J(f)}$ is topologically exact, there exists N such that $f^N(B') \supset J(f)$. Finally, we put $q = \max(N, l)$.

Lemma 4.4 *For every integer $q \geq l$ there exists $\gamma > 0$ such that for every integer $r \geq 1$, we have for every $z \in J(f)$ that,*

$$\mathcal{L}_0(\mathbb{1}_Q)(z) \geq \gamma,$$

where

$$Q := Q_r := \bigcup \{V : V \in G_{(r-1)q}(B')\}.$$

Proof As a straightforward consequence of Lemma 2.2, particularly its item (c), and (2.4), we get the following.

$$\mathcal{L}_{\tilde{\phi}}^{q(r-1)} \mathbb{1}_Q(z) \geq (2C_l)^{-1} \tag{4.4}$$

for all $z \in B$. By the choice of q, we have $f^q(B) \supset J(f)$. Thus, for every $x \in J$ there then exists $y \in B$ such that $f^q(y) = x$. Applying (4.4) we can then write

$$\begin{aligned}
\mathcal{L}_0(\mathbb{1}_Q)(x) &\geq \exp\left(S_q \tilde{\phi}(y)\right) \mathcal{L}_{\tilde{\phi}}^{q(r-1)} \mathbb{1}_Q(y) \\
&\geq (2C_l)^{-1} \exp\left(S_q \tilde{\phi}(y)\right) \\
&\geq (2C_l)^{-1} \exp\left(\inf\left(S_q \tilde{\phi}\right)\right).
\end{aligned}$$

Setting γ to be the last number of this formula finishes the proof. It is important to notice here that γ depends on q, but it is independent of r. $\square$

Definition 4.5 We say that a point $z \in B'$ has a good pullback of length $n \geq 1$ if $h^n(z) \in B'$ and $V_n(z)$, the connected component of $h^{-n}(B')$ containing z, belongs to $G_n^h(B')$. We then frequently refer to this component just as a (good) pullback.

Definition 4.6 We further say that such a good pullback $V_n(z)$ is very good if for every $0 \leq j \leq n - 1$, there exists $z_j \in h^{n-j}(V_n(z))$ such that

$$\mathrm{dist}(z_j, \partial B') > \lambda^{-j}.$$

Remark 4.7 Note that if $r \geq 1$ is chosen so large that,

$$C_\eta \theta^{rq} < \lambda^{-1} \tag{4.5}$$

(see Corollary 2.3 and also (4.1)), and V_n is a very good pullback then the sets $h^{n-j}(V_n)$ do not intersect $\partial B'$; in particular V_n is entirely contained in B'.

Having fixed q, we find γ, according to the statement of Lemma 4.4, and α, given by Lemma 4.2, applied for $Q = Q_r$. Again, it is important to note that one can find a common value γ (and thus also α), valid for all r.

The proof of Lemma 4.8 below is similar to Lemma 17 from [12]. As in Lemma 4.2 let Z_n be the set of all points $x \in B'$ for which

$$\# \left\{ 0 < i \le n : h^i(x) \in B' \text{ and } g^{ir-(r-1)}(x) \in Q \right\} > \alpha n.$$

It follows from Lemma 4.2 that $\mu_\phi(Z_n) \ge \mu(B') - \exp(-\delta n)$.

Lemma 4.8 *Let*

$$Y_n := \left\{ x \in Z_n : \#\{0 \le j \le n : \text{the pullback } V_j(x) \text{ is good}\} < (\alpha/2)n \right\}.$$

Then

$$\mu_\phi(Y_n) \le (3C_q)^n \exp(-rqn\beta\alpha/4),$$

where the constants β, C_q come from Corollary 2.6.

Thus, with $r \ge 1$ sufficiently large (depending on q and the constants above) we have that

$$\mu_\phi(Y_n) < \exp(-rqn\beta\alpha/8). \tag{4.6}$$

Proof As we have mentioned, the proof is in the spirit of the proof of Lemma 17 from [12]. Nevertheless, it requires several modifications, so we outline it below and refer to [12] for omitted details. Recall that g and h have been chosen so that $h = g^r$ and $g^{r-1}(Q) = B'$.

Let $x \in Y_n$, and consider a part of the trajectory of x:

$$x, g(x), \ldots, g^r(x) = h(x), \ldots, g^{(n-1)r}(x) = h^{n-1}(x), g^{(n-1)r+1}(x)$$

We now describe the 'configuration' M of integers, defined by the trajectory of x. First, the set of indexes $I = \{1, \ldots, n\}$ is divided into two disjoint groups:

$I_1 = \{j \in I : \text{such that } g^{rj-(r-1)}(x) \in Q \text{ (in particular, } h^j(x) = g^{rj}(x) = g^{r-1}(g^{rj-(r-1)}(x)) \in B), \text{ but the pullback } V_j(x) \text{ is not good} \}$
$I_2 = \{j \in I : \text{such that either } g^{rj-(r-1)}(x) \in Q \text{ and the pullback } V_j(x) \text{ is good or } g^{rj-(r-1)}(x) \notin Q \}$

Note that there are at most $n - \alpha n/2$ elements in I_2, since x visits Q at least αn times, but there are less than $\alpha n/2$ good pullbacks. The configuration $M = M(x)$ is a sequence $m_1 > m_1' \ge m_2 > m_2' \ge \cdots \ge m_k > m_k'$ defined as follows: First, let

$$m_1 = \max\{m \in I_1\}$$

Since the pullback $V_{m_1}(x)$ is not good, there exists $m_1' < m_1$ such that $h^{m_1'}(x) \in B_{m_1-m_1'}$. If $m_1' - m_1 = 1$, then we can say more: then $h^{m_1'}(x) \in B_r^g$, as $g(h^{m_1'}(x)) = g^{rm_1-(r-1)}(x) \in Q = G_{r-1}^g$.

Next, m_2 is chosen as the largest element ($\leq m_1'$), belonging to the group I_1. Inductively, we find m_2', m_3, m_3', etc, until m_k' such that all $1, \ldots, m_k'$ are in I_2.

In this way, the set $I = \{1, \ldots, n\}$ has been divided into 'blocks' $(m_i', m_i]$ and 'gaps'. Since every element in a gap is in I_2, the sum of lengths of blocks $\triangle m_i := m_i - m_i')$ is at least $\alpha n / 2$.

Consider the set Y_n^M of points in Y_n, sharing the same configuration M. The set Y_n^M is contained in $C := f^{-m_1'}(W_{\triangle m_1}) \cap f^{-m_2'}(W_{\triangle m_2}) \cap \ldots f^{-m_k'}(W_{\triangle m_k})$, where W_m is equal to B_m for $m \geq 2$, and $W_1 = B_r^g$. Since $\mu_\phi(C) = \int \mathcal{L}_0(\mathbb{1}_C) d\mu_\phi$, it is enough to find a bound for $\mathcal{L}_0(\mathbb{1}_C)$. Since $\mathcal{L}_0(\mathbb{1}) = \mathbb{1}$, it is easy to see that $\mathcal{L}_0(\mathbb{1}_C)$ can be bounded from above by $\prod_{j=1}^k \mathcal{L}_0^{\triangle m_j} \mathbb{1}_{W_{\triangle m_j}}$. The latter is estimated using the following.

Lemma 4.9 *For every $m \in \mathbb{N}$, $\mathcal{L}_0 \mathbb{1}_{W_m} \leq C_q \exp(-rqm\frac{\beta}{2})$.*

Proof This follows from Corollary 2.5, applied for $n = r$ and from Corollary 2.6. □

Now, since the total length of blocks, $\sum \triangle m_i$, is larger than $\frac{\alpha n}{2}$, this gives:

$$\sup \mathcal{L}_0 \mathbb{1}_C \leq C_q^n \exp\left(-rq\left(\sum \triangle m_i\right)\frac{\beta}{2}\right) \leq C_q^n \exp\left(-rq\alpha n\frac{\beta}{4}\right).$$

Since there are at most 3^n possible configurations, this gives:

$$\mu_\phi(Y_n) \leq (3C_q)^n \exp(-rqn\beta\alpha/4).$$

Thus (4.6) holds if r is large enough.

The proof of the next lemma is almost the same as the proof of Lemma 18 from [12]. The only modification is that ∂U is to be replaced by $\partial B'$.

Lemma 4.10 *Let $R_n \subset B'$ be the set of points in $x \in Z_n$ that satisfy the following two requirements.*

(1) $x \in B \setminus Y_n$, i.e. the points in R_n have at least $\frac{\alpha}{2}n$ good pullbacks, but
(2) No good pullback $V_m(x)$ with $m \leq n$, is very good.

Then

$$\mu_\phi(R_n) \leq (3C)^n \tilde{\lambda}^{-\alpha n/2},$$

where the constant C (again: independent of r) comes from Lemma 4.11 below.

We omit the proof of this lemma, referring to [12], but, since the value of the constant C in the statement of Lemma 4.10 is important, we explain it in Lemma 4.11.

Lemma 4.11 *Denote by GB_k^h the subfamily of good components for h^k, which are entirely contained in $B(\partial B', \lambda^{-k})$. Then*

$$\sup \mathcal{L}_0^k \mathbb{1}_{GB_k} \leq C\tilde{\lambda}^{-k},$$

where $C = \frac{C_l}{\mu_\phi(B')}$.

Proof We have

$$\tilde{\lambda}^{-k} \geq \mu_\phi(B(\partial B', \lambda^{-k})) = \int \mathbb{1}_{B(\partial B', \lambda^{-k})} d\mu_\phi \geq \int \mathcal{L}_0^k \mathbb{1}_{GB_k} d\mu_\phi$$

$$\geq \mu(B') \inf \mathcal{L}_0^k \mathbb{1}_{GB_k}$$

$$\geq \frac{1}{C_l} \sup \mathcal{L}_0^k \mathbb{1}_{GB_k} \cdot \mu_\phi(B').$$

$\square$

In order to make the measure $\mu_\phi(R_n)$ converge to zero exponentially fast, we thus require λ to be so large that

$$\lambda^{\frac{\alpha}{6}} = \tilde{\lambda}^{\frac{\alpha}{2}} > 3C \tag{4.7}$$

(remember that $\lambda = \tilde{\lambda}^3$). Here, again, $C = \frac{C_l}{\mu_\phi(B')}$. So, denoting

$$\lambda'' = \tilde{\lambda}^{\frac{\alpha}{2}} \cdot \frac{1}{3C} > 1$$

we have $\mu_\phi(R_n) \leq (\lambda'')^{-n}$.

The constants used in the construction. To clarify the situation, we summarize here the procedure of fixing consecutive constants (and iterates of the map f) used in the construction.

(1) The constant l is taken from Lemma 2.2, applied for $\varepsilon = 1/2$.
(2) The ball B (and, consequently, also B') is chosen so that B satisfies the assumptions of Lemma 2.2, and, moreover, so that the statement of Lemma 3.2 is satisfied, for every λ large enough.
(3) The constant q depends on B.
(4) The constant α depends on q.
(5) Since in the statement of Lemma 3.2 λ can be always replaced by its power, now, having fixed q and α, we fix λ so large that (4.7) holds.
(6) Finally, r is fixed. The iterate r depends on q, α and λ.

5 Construction of the induced system

Let

$$X = \bigcup_{n=1}^{\infty} Z_n \backslash (Y_n \cup R_n)$$

It directly follows from Lemmas 4.2, 4.4, 4.8, 4.10 that

$$\mu_\phi(X) = \mu_\phi(B'). \tag{5.1}$$

Given $x \in X$ let $n(x)$ be the smallest integer $n \geq 1$ such that the pullback $V_n(x)$ is very good.

We define

$$F(x) = h^{n(x)}(x).$$

Keep this $x \in X$ and put $n = n(x)$. Note that, if $y \in V_n(x)$ then this procedure, applied to y leads to the same component V_n. Indeed, by the definition of the induced map, we use the earliest very good pullback. Thus, if $F(y) \neq h^n(y)$ then $F(y) = h^m(y)$ for some $m < n$. Let $V_m(y)$ be the corresponding pullback. Then $V_m(y) \cap V_n(x) \neq \emptyset$ as y belongs to both of these sets, but $V_n(x) \not\subseteq V_m(y)$ since $x \in V_n(x) \setminus V_m(y)$. Let us consider $h^m(V_m(y)) = U$ and $h^m(V_n(x))$. The latter is an element of the pullback chosen for x, a component of $h^{-(n-m)}(B)$, and, since $V_n(x)$ must intersect $\partial V_m(y)$, also $h^m(V_n(x))$ intersects $\partial B'$. But this is impossible by the definition of very good pullbacks. Let $\mathcal{D}$ be the countable family of all sets $V_{n(x)}(x), x \in X$, defined in this way. We have just shown that the function $n : X \to \mathbb{N}$ is constant on each set $D \in \mathcal{D}$, and so it can and will be treated as a function from $\mathcal{D} \to \mathbb{N}$. In particular, the map

$$F : \bigcup_{D \in \mathcal{D}} \to B'$$

is well-defined and its inverse branches $F_D^{-1} : B' \to D, D \in \mathcal{D}$, form an infinite iterated function systems, which, with a slight abuse of notation, will be also referred to as F. We denote by J_F the limit set of the iterated function system F, i.e.

$$J_F = \bigcap_{n=0}^{\infty} F^{-n}(X). \tag{5.2}$$

The argument leading to (5.1) gives in fact more. Namely:

$$\mu_\phi(J_F) = \mu_\phi(B'). \tag{5.3}$$

It immediately follows from the construction of the system F and Lemma 4.10, that

$$m_\phi\left(\bigcup\{D : n(D) = n\}\right) \leq (\lambda'')^{-n} \tag{5.4}$$

for some $\lambda'' > 1$ and all $n \geq 1$. Let us record the following, proved in the same way as Lemma 19 in [12], essential property of this induced system.

Lemma 5.1 *If D_1, D_2 are two domains in $\mathcal{D}$, $F_{|D_1} = h^n$, $F_{|D_2} = h^m$ then for $0 \leq s < n, 0 \leq t < m$ either $h^s(D_1) \cap h^t(D_2) = \emptyset$ or the closure of one of these sets is contained in the other set.*

For the sake of Proposition 5.4, we need to extend the potential ϕ beyond the Julia sets $J(f)$.

Lemma 5.2 *The function ϕ can be extended in a Hölder continuous manner, with the same Hölder exponent, to the whole projective space $\mathbb{P}^k$.*

This lemma is well-known. From now on, we assume that the potential ϕ is defined and Hölder continuous in the whole projective space $\mathbb{P}^k$.

As we have passed to induced system, we shall modify the potential ϕ accordingly to this inducing process. First, if $D \in \mathcal{D}$ is one of the domains on which F is defined, then we put, for all $x \in D$, that

$$\hat{\phi}(x) = \sum_{k=0}^{n(D)-1} S_{qr}\phi(h^k(x)).$$

Then, for all Borel sets $A \subset D_e$ we have that,

$$m_\phi(F(A)) = m_\phi(h^{n(D)}(A)) = \int_A \exp\left(-\sum_{k=0}^{n(D)-1} S_{qr}\phi \circ h^k\right) dm_\phi$$

$$= \int_A \exp\left(-\hat{\phi}(x)\right) dm_\phi(x).$$

Along with (5.3) this entails the following.

Lemma 5.3 *The probability measure m_ϕ is $\exp(-\hat{\phi})$-conformal for the map $F : J_F \to J_F$.*

Having Lemmas 5.3 and 5.2, the general theory of infinite iterated function systems, as developed in [8,9], see also [10], gives the following.

Proposition 5.4 *There exists a unique probability F-invariant measure $\mu_{\hat{\phi}}$ which is equivalent to m_ϕ. Moreover the Radon–Nikodym derivative $\hat{\rho} := \frac{d\mu_{\hat{\phi}}}{dm_\phi}$ is bounded above and separated below from zero. This Radon–Nikodym derivative $\hat{\rho}$ has a continuous extension $\hat{\rho} : B' \to (0; +\infty)$ to the whole ball B' and this extension is a fixed point of the following transfer operator.*

$$\mathcal{L}_{\hat{\phi}}(v)(x) = \sum_{y \in F^{-1}(x)} \exp \hat{\phi}(y) v(y).$$

This is a bounded linear operator acting on the Banach space $C_b(B)$ of all bounded real-valued continuous functions defined on B, and it is easy to see that this operator is almost periodic.

6 Real analyticity of topological pressure

For every Hölder continuous potential $\phi : J(f) \to \mathbb{R}$, let

$$\Delta_\phi := \left\{ t \in \mathbb{R} : \sup(t\phi) - \inf(t\phi) < \kappa_f \right\},$$

where κ_f was defined in Theorem 1.1. Obviously, Δ_ψ is an open subset of $\mathbb{R}$. Having all the material of the previous sections i.e. Sects. 4 and 5, particularly formula (5.4), we can repeat verbatim Sect. 6, Real Analyticity of the Pressure Function, from [12] to get the following.

Theorem 6.1 *The topological pressure function*

$$\Delta_\phi \ni t \mapsto P(t\phi) \in \mathbb{R}$$

is real-analytic.

Remark 6.2 With only slight formal modification we could prove a little bit more; namely that for any Hölder continuous functions $\psi, \phi : J(f) \to \mathbb{R}$, the function

$$\Delta_\phi(\psi) \ni t \mapsto P(\psi + t\phi) \in \mathbb{R}$$

is real-analytic if $\Delta_\phi(\psi) := \big\{ t \in \mathbb{R} : \sup(\psi + t\phi) - \inf(\psi + t\phi) < \kappa_f \big\}$.

7 Stochastic properties of the equilibrium measure μ_ϕ

In this section we obtain strong transparent stochastic properties of the dynamical system (f, μ_ϕ). We deduce them from the corresponding properties of the induced system $(F, \mu_{\hat\phi})$. We follow the scheme worked out in [13] in the way it was presented in [12]. We recall it briefly now. We do this in an abstract context. Let $(\Delta_0, \mathcal{B}_0, m_0)$ be a measure space with a finite measure m_0, let $\mathcal{P}_0$ be a countable measurable partition of Δ_0 and let $T_0 : \Delta_0 \to \Delta_0$ be a measurable map such that, for every $\Delta' \in \mathcal{P}_0$ the map $T_0 : \Delta' \to \Delta_0$ is a bijection onto Δ_0. Moreover, we assume that the partition $\mathcal{P}_0$ is generating, i.e. for every $x, y \in \Delta_0$ there exists $s \geq 0$ such that $T_0^s(x), T_0^s(y)$ are in different elements of the partition $\mathcal{P}_0$. We denote by $s = s(x, y)$ the smallest integer with this property and we call it a separation time for the pair x, y. We assume also that for each $\Delta' \in \mathcal{P}_0$ the map $(T_{0|\Delta'})^{-1}$ is measurable and that the Jacobian $Jac_{m_0}(T_0)$ with respect to the measure m_0 is well-defined and positive a.e. in Δ'. The following distortion property is assumed to be satisfied.

$$\left| \frac{\mathrm{Jac}_{m_0} T_0(x)}{\mathrm{Jac}_{m_0} T_0(y)} - 1 \right| \leq C \kappa^{s(x,y)}. \tag{7.1}$$

for all elements $\Delta' \in \mathcal{P}_0$, all $x, y \in \Delta'$ and some $\kappa \in (0, 1)$. We have also a function $R : \Delta_0 \to \mathbb{N}$ ("return time") which is constant on each element of the partition $\mathcal{P}_0$. We assume that the greatest common divisor of the values of R is equal to 1. Finally, let

$$\Delta = \{(z, n) \in \Delta_0 \times \mathbb{N} \cup \{0\} : 0 \leq n < R(z)\}$$

and each point $z \in \Delta_0$ is identified with $(z, 0)$. The map T acts on Δ as

$$T(z, n) = \begin{cases} (z, n+1) & \text{if } n+1 < R(z) \\ (T_0(z), 0) & \text{if } n+1 = R(z) \end{cases}$$

The measure m_0 is spread over the whole space Δ by putting

$$\tilde{m}_{|\Delta_0} = m_0 \quad \text{and} \quad \tilde{m}_{|\Delta' \times \{j\}} = m_{0|\Delta'} \circ \pi_j^{-1}, \quad \Delta' \in \mathcal{P}_0,$$

where $\pi_j(z, 0) = (z, j)$. Thus, the measure $\tilde{m}$ is finite iff $\int_{\Delta_0} R \, dm_0 < \infty$. The separation time $s((x, n), (y, m))$ is defined to be equal to $s(x, y)$ if $n = m$ and x, y are in the same set of the partition $\mathcal{P}$. Otherwise we set $s(x, y) = 0$. Given any $\beta \in (0, 1)$ we define the space

$$C_\beta(\Delta) = \{\varphi : \Delta \to \mathbb{R} : \exists \, C_\varphi \quad \text{such that} \quad |\varphi(x) - \varphi(y)| < C_\varphi \beta^{s(x,y)} \quad \forall x, y \in \Delta\}.$$

We refer to the pentapol $\mathcal{Y} = (\Delta_0, T_0, \mathcal{P}_0, R, m_0)$ as a Young tower. The first three items of the following basic result have been proved in [13], see also [7] for the fourth item.

Theorem 7.1 *If $\mathcal{Y} = (\Delta_0, T_0, \mathcal{P}_0, R, m_0)$ is a Young tower and $\int R \, dm_0 < \infty$ then the following hold.*

(1) There exists a unique probability T-invariant measure ν, absolutely continuous with respect to $\tilde{m}$. The Radon–Nikodym derivative $d\nu/d\tilde{m}$ is bounded from below by a positive constant. The dynamical system (T, ν) is exact, thus ergodic.
(2) If $m_0(R > n) = O(\theta^n)$ for some $0 < \theta < 1$, then for every $\beta > 0$ there exists $0 < \tilde{\theta} < 1$, depending on β, such that for all functions $\psi \in L^\infty$ and all functions $g \in C_\beta$ we have,

$$Cov(\psi \circ T^n, g) = \left| \int (\psi \circ T^n) g \, d\nu - \int \psi \, d\nu \int g \, d\nu \right| = O(\tilde{\theta}^n) \qquad (7.2)$$

(3) If $m_0(R > n) = O(n^{-\alpha})$ with some $\alpha > 1$ (in particular, if $m_0(R > n) = O(\theta^n)$), then the Central Limit Theorem is satisfied for all functions $g \in C_\beta$, that are not cohomologous to a constant in $L^2(\nu)$.
(4) If $m_0(R > n) = O(n^{-\alpha})$ with some $\alpha > 4$ (in particular, if $m_0(R > n) = O(\theta^n)$), then the Law of Iterated Logarithm holds for all $\beta > 0$ and all functions $g \in C_\beta$, that are not cohomologous to a constant in $L^2(\nu)$. This means that there exists a real positive number A_g such that such that ν almost everywhere

$$\limsup_{n \to \infty} \frac{S_n g - n \int g \, d\nu}{\sqrt{n \log \log n}} = A_g.$$

Passing to our holomorphic dynamical system (f, μ_ϕ) we shall check that the assumptions of this theorem are satisfied for our induced system (F, m_ϕ). The space Δ_0 is now J_F, the limit set of the iterated function system F. The partition $\mathcal{P}_0$ consists of the sets $D \cap J_F$, $D \in \mathcal{D}$. The measure m_0 is the conformal measure m_ϕ, restricted to J_F. The map T_0 is, in our setting, the map F. The function R, the return time, is, naturally, defined as $R(D) = n(D)$. We shall check that the pentapol $\mathcal{Y}_\phi = (J_F, F, \mathcal{P}, n, m_\phi)$ is a Young Tower, i.e. it satisfies the hypotheses of Theorem 7.1. We start with the following.

Lemma 7.2 *There exists a constant $C > 0$ such that if $D \in \mathcal{D}$ and $x, y \in D$, then*

$$dist(x, y) \leq C4^{-s(x,y)}.$$

Proof The assertion follows immediately from Corollary 2.3, formula (4.1), and the definition of the separation time s. $\qquad\square$

As a fairly straightforward consequence of this lemma, we get the following.

Lemma 7.3 *There exists a constant $C' > 0$ such that if $D \in \mathcal{D}$ and $x, y \in D$, then*

$$dist(h^j(x), h^j(y)) \leq C'(1/4)^{n(D)-j+s(x,y)}$$

for all $0 \leq j \leq n(D) - 1$.

Proof Recall that for each $D \in \mathcal{D}$ we have $F|_D = h^{n(D)}$ and h expands by a factor equal at least 4 on each set $h^i(D)$ for all $i = 0, \ldots, n - 1$. Therefore we can very generously estimate,

$$dist(h^j(x), h^j(y) \leq (1/4)^{n(D)-j} dist(F(x), F(y))$$

By virtue of Lemma 7.2, this yields

$$dist(h^j(x), h^j(y) \leq (1/4)^{n(D)-j} C4^{-s(F(x),F(y))}.$$

Now, if $s(x, y) > 1$, then $s(F(x), F(y)) = s(x, y) - 1$, and if $s(x, y) = 1$, then $s(F(x), F(y)) \geq s(x, y) - 1$. In any case inserting this to the last display, gives

$$dist(h^j(x), h^j(y) \leq 4C(1/4)^{n(D)-j+s(x,y)}.$$

The proof is complete. $\qquad\square$

Lemma 7.4 *The pentapol $\mathcal{Y}_\phi = (J_F, F, \mathcal{P}, n, m_\phi)$ is a Young Tower, i.e. it satisfies the hypotheses of Theorem 7.1. In addition, $\tilde{m}_\phi(\Delta) < +\infty$.*

Proof First, we need to show that the formula (7.1) holds. To do this fix an arbitrary domain $D \in \mathcal{D}$ and arbitrary two points $x, y \in J_F \cap D$. Recalling that the function $S_{qr}\phi : J(f) \to \mathbb{R}$ is Hölder continuous with some exponent $\alpha > 0$, and using Lemma 7.3, we can write as follows.

$$\left| \log \frac{\mathrm{Jac}_{m_\phi} F(y)}{\mathrm{Jac}_{m_\phi} F(x)} \right| = \left| \sum_{j=0}^{n(D)-1} S_{qr}\phi(h^j(y)) - \sum_{j=0}^{n(D)-1} S_{qr}\phi(h^j(x)) \right|$$

$$\leq \sum_{j=0}^{n(D)-1} \left| S_{qr}\phi(h^j(y)) - S_{qr}\phi(h^j(x)) \right|$$

$$\leq \sum_{j=0}^{n(D)-1} C_1 \mathrm{dist}^\alpha(h^j(y), h^j(x))$$

$$\leq C_1 C' \sum_{j=0}^{n(D)-1} (1/4)^{n(D)-j+s(x,y)}$$

$$= C_1 C'(1/4)^{s(x,y)} \sum_{j=0}^{n(D)-1} (1/4)^{n(D)-j}$$

$$\leq C_1 C'(1/4)^{s(x,y)} \sum_{j=0}^{\infty} (1/4)^j$$

$$= (4/3)C_1 C'(1/4)^{s(x,y)}.$$

and formula (7.1) is established.

We also need to take care of the last assumption in Theorem 7.1 requiring that the greatest common divisor of all the values of $n(D)$, $D \in \mathcal{D}$, is equal to 1. If for our induced system this value is equal to some integer $s > 1$, then we replace the map h by its iterate h^s. The return times are now equal to $n(D)/s$, $D \in \mathcal{D}$, and their greatest common divisor equals 1.

The finiteness of $\tilde{m}_\phi(\Delta)$ follows immediately from (5.4) and the definition of the return time R. □

Now consider $\pi : \Delta \to \mathbb{P}^k$, the natural projection from the abstract tower Δ to the projective space $\mathbb{P}^k$ given by the formula

$$\pi(z, n) = h^n(z).$$

Then

$$\pi \circ T = h \circ \pi,$$
$$\tilde{m}_\phi\big|_{J_F} \circ \pi^{-1} = m_0 = m_\phi, \tag{7.3}$$

and

$$\tilde{m}_{\phi|D \times \{n\}} \circ \pi^{-1} = m_{\phi|D \times \{0\}} \circ h^{-n} = m_{0|D} \circ h^{-n}$$

for all $D \in \mathcal{D}$ and all $0 \leq n \leq n(D)$. Now, the measure $\tilde{m}_{\phi|D \times \{n\}} \circ \pi^{-1}$ is absolutely continuous with respect to m_ϕ with the Radon–Nikodym derivative equal to $J_{D,n} := \mathrm{Jac}_{m_\phi}(h^{-n})$ in $h^n(D)$ and zero elsewhere. Therefore,

$$\int_{\mathbb{P}^k} \sum_{D \in \mathcal{D}} \sum_{n=0}^{n(D)-1} J_{D,n} \, dm_\phi = \sum_{D \in \mathcal{D}} \sum_{n=0}^{n(D)-1} \int_{\mathbb{P}^k} J_{D,n} \, dm_\phi = \sum_{D \in \mathcal{D}} \sum_{n=0}^{n(D)-1} \int_{h^n(D)} J_{D,n} \, dm_\phi$$

$$= \sum_{D \in \mathcal{D}} \sum_{n=0}^{n(D)-1} \tilde{m}_{\phi|D \times \{n\}} \circ \pi^{-1}(h^n(D))$$

$$= \sum_{D \in \mathcal{D}} \sum_{n=0}^{n(D)-1} \tilde{m}_{\phi|D \times \{n\}} \circ \pi^{-1}(\mathrm{P}^k)$$

$$= \tilde{m}_\phi \circ \pi^{-1}(\mathbb{P}^k) = \tilde{m}_\phi(\Delta)$$

$$< +\infty,$$

where in writing the inequality sign we used the last assertion of Lemma 7.4. Thus, the function $\sum_{D \in \mathcal{D}} \sum_{0 \le n < n(D)} J_{D,n}$ is integrable with respect to the measure m_ϕ. This implies immediately that the measure $\tilde{m}_\phi \circ \pi^{-1}$ is absolutely continuous with respect to the measure m_ϕ with the Radon–Nikodym derivative equal to $\sum_{D \in \mathcal{D}} \sum_{0 \le n < n(D)} J_{D_i,n}$. Hence, the measure $\nu \circ \pi^{-1}$ is also absolutely continuous with respect to m_ϕ. Since ν is F-invariant and $\pi \circ T = h \circ \pi$, the measure $\nu \circ \pi^{-1}$ is h-invariant. But the measure μ_ϕ is h-invariant ergodic and equivalent with the conformal measure m_ϕ. Hence, $\nu \circ \pi^{-1}$ is absolutely continuous with respect to the ergodic measure μ_ϕ. Invariance and ergodicity of $\nu \circ \pi^{-1}$ yield this.

Lemma 7.5

$$\nu \circ \pi^{-1} = \mu_\phi.$$

Having this, we can prove in exactly the same way as in [12], the following.

Theorem 7.6 *For the dynamical system (f, μ_ϕ) the following hold.*

(1) *For every $\alpha \le 1$, every α-Hölder continuous function $g : J(f) \to \mathbb{R}$ and every bounded measurable function $\psi : J(f) \to \mathbb{R}$, we have that*

$$\left| \int \psi \circ f^n \cdot g \, d\mu_\phi - \int g \, d\mu_\phi \int \psi \, d\mu_\phi \right| = O(\theta^n)$$

for some $0 < \theta < 1$ depending on α.

(2) *The Central Limit Theorem holds for every Hölder continuous function $g : J(f) \to \mathbb{R}$ that is not cohomologous to a constant in $L^2(\mu_\phi)$, i.e. for which there is no square integrable function η for which $g = \mathrm{const} + \eta \circ f - \eta$. Precisely this means that there exists $\sigma > 0$ such that*

$$\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} g \circ f^j \to \mathcal{N}(0, \sigma)$$

in distribution.

(3) *The Law of Iterated Logarithm holds for every Hölder continuous function g : $J(f) \to \mathbb{R}$ that is not cohomologous to a constant in $L^2(\mu_\phi)$. This means that there exists a real positive constant A_g such that such that μ_ϕ almost everywhere*

$$\limsup_{n \to \infty} \frac{S_n g - n \int g \, d\mu}{\sqrt{n \log \log n}} = A_g.$$

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