Зборник радова Економског факултета, 2012, 6, стр. 31 - 43
Рад примљен: 16. март 2011.
Received: 16 March 2011.

UDC 528.2:519.564
DOI: 10.7251/ZREFIS1206031S
Изворни научни рад
Original scientific paper

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THE UPPER OPEN GEODETIC NUMBER OF A GRAPH

GORNJI OTVORENI GEODETSKI BROJ GRAFA

Summary: *For a connected graph G of order n, a set S of vertices of G is a geodetic set of G if each vertex v of G lies on a x-y geodesic for some elements x and y in S. The minimum cardinality of a geodetic set of G is defined as the geodetic number of G, denoted by g(G). A geodetic set of cardinality g(G) is called a g-set of G. A set S of vertices of a connected graph G is an open geodetic set of G if for each vertex v in G, either v is an extreme vertex of G and $v \in S$; or v is an internal vertex of an x-y geodesic for some $x,y \in S$. An open geodetic set of minimum cardinality is a minimum open geodetic set and this cardinality is the open geodetic number, og(G). An open geodetic set S in a connected graph G is called a minimal open geodetic set if no proper subset of S is an open geodetic set of G. The upper open geodetic number $og^+(G)$ of G is the maximum cardinality of a minimal open geodetic set of G. It is shown that, for a connected graph G of order n, og(G)=n, if and only if $og^+(G)=n$, and also that og(G)=3 if any only if $og^+(G)=3$. It is shown that for positive integers a and b with $4 \le a \le b$, there exists a connected graph G with og(G) =a and $og^+(G)=b$. Also, it is shown that for positive integers a, b, c with $4 \le a \le b \le c$ and $b \le 3a$, there exists a connected graph G with g(G)=a, og(G)=b and $og^+(G)= c$.*

Key words. *geodesic, geodetic number, open geodetic number, upper open geodetic number.*

JEL Classification: *C00, CO2*

MSC Classification: *05C12*

Резиме: *За повезани граф G реда n, скуп S чворова од G је геодетски скуп од G ако сваки чвор v од G лежи на x-y геодезијској линији за неке елементе x и y у S. Минимална кардиналност геодетског скупа од G дефинише се као геодетски број од G, и означава се са g(G). Геодетски скуп кардиналности g(G) се назива g-скуп од G. Скуп S чворова повезаног графа G представља отворени геодетски скуп у G ако је за сваки чвор v од G, или v екстремни чвор у G, а $v \in S$; или је v унутрашњи чвор геодезијске линије x-y при чему $x,y \in S$. Отворени геодетски скуп минималне карди-налности је минимални отворени геодетски скуп, а та кардиналност представља отворени геодетски број, og(G). Отворени геодетски скуп S у повезаном графу G назива се минимални отворени геодетски скуп ако прави подскуп у S nije отворени геодетски скуп у G. Горњи отворени геодетски број $og^+(G)$ од G пре-дстављa максималну кардиналност мини-малног отвореног геодетског скупа у G. Показано је да за повезани граф G реда n важи og(G)=n, ако и само ако је $og^+(G)=n$, и такође да је og(G)=3 ако и само ако је $og^+(G)=3$. Показано је да за позитивне цијеле бројеве a i б са особином $4 \le a \le b$, постоји повезани граф G са особинама og(G) =a, i $og^+(G)=b$. Такође је пока-зано да за позитивне цијеле бојеве a, b, c са особинама $4 \le a \le b \le c$ i $b \le 3a$, постоји пове-зани граф G са особинама (G)=a, og(G)=b, i $og^+(G)= c$.*

Кључне ријечи: *геодетски, геодетски број, отворени геодетски број, горњи отворени геодетски број.*

ЈЕЛ класификација: *C00, CO2*

MSC класификација: *05C12*

1. INTRODUCTION

By a graph $G=(V,E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by n and m respectively. For basic graph theoretic terminology we refer to Harary [6] and we refer to [1] for results on distance in graphs. The *distance* $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest u-v path in G. An u-v path of length $d(u,v)$ is called an *u-v geodesic*. It is known that this distance is a metric on the vertex set of G. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices which are adjacent with v. A vertex v is an *extreme vertex* of G if the subgraph induced by its neighbors is complete. A vertex is an *end- vertex* if its degree is 1. For a *cut- vertex* v in a connected graph G and a component H of $G–v$, the subgraph H and the vertex v together with all edges joining v and $V(H)$ is called a *branch* of G at v. A *geodetic set* of G is a set S of vertices of G such that every vertex of G is contained in a geodesic joining some pair of vertices in S. The *geodetic number* $g(G)$ of G is the cardinality of a minimum geodetic set. The geodetic number of a graph was introduced in [7] and further studied in [3,4,5,8]. A vertex x is said to *lie* on a u-v geodesic P if x is a vertex of P and x is called an internal vertex of P if $x \neq u, v$. We denote by $I[u, v]$ the set of all vertices lying on a u-v geodesic. If x is an internal vertex of an u-v geodesic, we also use the notation $x \in I(u,v)$. A set S of vertices in a connected graph G is an *open geodetic set* if for each vertex v in G, either v is an extreme vertex of G and $v \in S$; or v is an internal vertex *of* an x-y geodesic for some $x,y\in S$. An open geodetic set of minimum cardinality is a *minimum open geodetic set* and this cardinality is the *open geodetic number* $og(G)$ of G. The open geodetic number of a graph was introduced and further studied in [3, 9]. Throughout the following G denotes a connected graph with at least two vertices.

The following theorems are used in the sequel.

Theorem 1.1. [3] *Every geodetic set of a connected graph contains its extreme vertices. Also, if the set S of all extreme vertices of G is a geodetic set, then S is the unique minimum geodetic set of G.*

Theorem 1.2. [8] *Let G be a connected graph with a cut-vertex v. Then every geodetic set of G contains at least one vertex from each component of G–v.*

Theorem 1.3. [9] *Every open geodetic set of a graph G contains its extreme vertices. Also, if the set S of all extreme vertices of G is an open geodetic set, then S is the unique minimum open geodetic set of G.*

Theorem 1.4. [9] *For any tree T, the open geodetic number og(T) equals the number of end vertices of T. In fact, the set of all end vertices of T is the unique minimum open geodetic set of T.*

Theorem 1.5. [9] *Let G be a connected graph with a cut-vertex v. Then every open geodetic set of G contains at least one vertex from each component of G–v.*

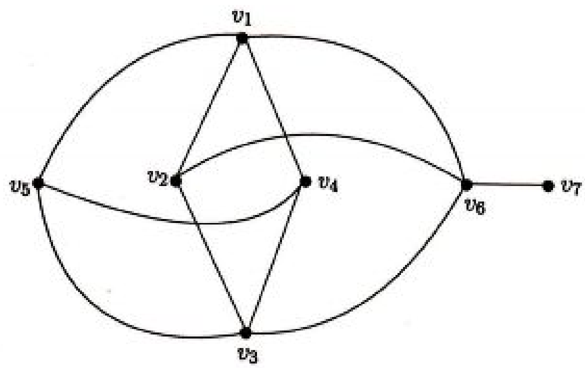
Theorem 1.6. [3] *Let G be a non-trivial connected graph that contains no extreme vertices. Then* $og(G) \geq 4$.

Theorem 1.7. [3] *For every connected graph G with no extreme vertices,* $max\{g(G),4\} \leq og(G) \leq 3g(G)$.

2. THE UPPER OPEN GEODETIC NUMBER OF A GRAPH

Definition 2.1. *An open geodetic set S in a connected graph G is called a minimal open geodetic set if no proper subset of S is an open geodetic set of G. The upper open geodetic number* $og^+(G)$ *of G is the maximum cardinality of a minimal open geodetic set of G.*

Example 2.2. *For the graph G given in Figure* 2.1, *it is easily verified that no* 3-*element subset of vertices is an open geodetic set. The set* $S=\{v_1,v_3,v_5,v_7\}$ *is an open geodetic* *set of G and so* $og(G)=4$. *Also, it is easy to see that S is the unique minimum open geodetic set of G. The set* $S'=\{v_1,v_2,v_3,v_4,v_7\}$ *is an open geodetic set of G. Since S is not a subset of* S' *and no* 4- *element subset other than S is an open geodetic set of G, it follows that* S' *is a minimal open geodetic set of G. It is also easily seen that* S' *is the unique minimal open geodetic set of G. Thus* $og^+(G)=5$.



G

Figure 2.1

Remark 2.3. *Every minimum open geodetic set of a graph G is a minimal open geodetic set of G and the converse is not true. For the graph G given in Figure* 2.1, $S'=\{v_1,v_2,v_3,v_4,v_7\}$ *is a minimal open geodetic set but not a minimum open geodetic set of G.*

The following proposition is clear.

Proposition 2.4. *For the complete graph* $G=K_n$ $(n\geq 2)$, $og(G)=og^+(G)=n$.

Theorem 2.5. *If G is a connected graph of order n, then* $2\leq og(G)\leq og^+(G)\leq n$.

Proof. Any open geodetic set needs at least two vertices and so $og(G)=2$. Since every minimal open geodetic set is an open geodetic set, $og(G)\leq og^+(G)$. Also, since $V(G)$ is an open geodetic set of G, it is clear that $og^+(G)\leq n$. Thus $2\leq og(G)\leq og^+(G)\leq n$.

Remark 2.6. *The bounds in Theorem* 2.5 *are sharp.*

For any non-trivial path P, $og(P)=2$. For any non-trivial tree T, the set of all end vertices of T is the unique minimum open geodetic set of T so that $og(T)=og^+(T)$. For the complete graph K_n, $og^+(K_n)=n$ *for* $n\geq 2$. Also, all the inequalities in the Theorem 2.5 are strict. For the graph G given in Figure 2.1, $og(G)=4$, $og^+(G)=5$ and $n=7$.

Theorem2.7. *For a connected graph G of order n,* $og(G)=n$ *if and only if* $og^+(G)=n$.

Proof. Let $og^+(G)=n$. *Then* $S=V(G)$ is the unique minimal open geodetic set of G. Since no proper subset of S is an open geodetic set, it is clear that S is the unique minimum open geodetic set of G and so $og(G)=n$. The converse follows from Theorem 2.5.

Corollary 2.8. *If G is a graph of order n such that* $og(G)=n-1$, *then* $og^+(G)=n-1$.

Problem 2.9. *Characterize graphs G of order n for which* $og(G)=og^+(G)=n-1$.

Theorem 2.10. *No cut-vertex of a connected graph G belongs to any minimal open geodetic set of G.*

Proof. Let S be any minimal open geodetic set of G. Let $v\in S$. We prove that v is not a cut-vertex of G. Suppose that v is a cut-vertex of G. Let $G_1,G_2,\ldots,G_k$ $(k\geq 2)$ be the components of $G-v$. Then v is adjacent to at least one vertex of each G_i for $1\leq i\leq k$. Let $S'=S-\{v\}$. We show that S' is an open geodetic set of G. Let x be a vertex of G. If x is an extreme vertex of G, then $x\neq v$ and so by Theorem 1.3, $x\in S'$. If x is not an extreme vertex, then, since S is an open geodetic set of G, $x\in I(u,w)$ for some $u,w\in S$. If $v\neq u,w$, then $u,w\in S'$. If $v=u$, then $v\neq w$. Assume without loss of generality that $w\in G_1$. By Theorem 1.5, S contains a vertex w' from G_i $(2\leq i\leq k)$. Then $w'\neq v$. Since v is a cut-vertex of G, we have $I(w,u)\subseteq I(w,w')$. Hence $x\in I(w,w')$, where w, $w'\in S'$. Thus S' is an open geodetic set of G. This contradicts that S is a minimal open geodetic set of G.

Corollary 2.11. *For any tree T with k end- vertices,* $og(T) = og^+(T) = k$.

Proof. This follows from Theorems 1.3, 1.4 and 2.10

Lemma 2.12. *Let G be a connected graph. If G has a minimal open geodetic set S of cardinality* 3, *then all the vertices in S are extreme*.

Proof. Let $S = \{u,v,w\}$ be a minimal open geodetic set of G. Then $og(G) \leq 3$. Suppose that the vertex w is not extreme. We consider three cases.

Case1. u and v are non-extreme. Then u,v,w are all non-extreme and by Theorem 1.3, G has no extreme vertices. Hence by Theorem 1.6, we see that $og(G) \geq 4$, which is a contradiction.

Case 2. u is extreme and v is not extreme. Since S is an open geodetic set of G, we have $v \in I(u,w)$ and $w \in I(u,v)$. These in turn, give $d(u,w) > d(u,v)$ and $d(u,v) > d(u,w)$. Hence $d(u,w) > d(u,w)$, which is a contradiction.

Case 3. u and v are extreme. Since S is an open geodetic set of G, we have $w \in I(u,v)$. Let $d(u,v) = k$ and let P be a u-v geodesic of length k, $d(u,w) = l_1$ and $d(w,v) = l_2$. Then $l_1 + l_2 = k$. Let P' be the u-w subpath of P and P'' the w-v subpath of P. We prove that $S' = \{u,v\}$ is an open geodetic set of G. Let x be any vertex of G such that $x \notin S'$. Since $S = \{u,v,w\}$ is a minimal open geodetic set of G with w non-extreme, u and v extreme, it follows that u and v are they only two extreme vertices of G. Hence x is not extreme. Since S is an open geodetic set of G, we have $x \in I(u,v)$ or $x \in I(u,w)$ or $x \in I(v,w)$. If $x \in I(u,v)$, there is nothing to prove. If $x \in I(u,w)$, let Q be a u-w geodesic in which x lies internally. Let R be the u-v walk obtained from Q followed by P''. Then the length of R is k and so R is a u-v geodesic containing x. Thus $x \in I(u,v)$. Similarly, if $x \in I(v,w)$, we can prove that $x \in I(u,v)$. Hence S' is an open geodetic set of G, which contradicts that S is a minimal open geodetic set of G. This completes the proof.

Theorem 2.13. *For a connected graph G,* $og(G) = 3$ *if an only if* $og^+(G) = 3$.

Proof. Let $og(G) = 3$. Let S be a minimum open geodetic set of G. Since every minimum open geodetic set is also a minimal open geodetic set, by Lemma 2.12, all the three vertices in S are extreme. Hence it follows from Theorem 1.3 that S is the unique minimal open geodetic set of G so that $og^+(G)=3$. Conversely, let $og^+(G)=3$. Let S' be a minimal open geodetic set of G of cardinality 3. By Lemma 2.12, all the vertices in S' are extreme. Hence it follows form Theorem 1.3 that S' is the unique minimum open geodetic set of G so that $og(G) = 3$.

Theorem 2.14. *For every two positive integers a and b with* $4 \leq a \leq b$, *there exists a connected graph G with* $og(G)=a$ *and* $og^+(G)=b$.

Proof. If $a = b$, let $G = K_{1,a}$. Then by Corollary 2.11, $og(G) = og^+(G) = a$. Let $4 \leq a < b$. Let $H = \overline{K}_2 + C_{b-a+3}$ with $V(K_2) = \{x,y\}$ and $V(C_{b-a+3}) = \{v_1,v_2,\ldots,v_{b-a+3}\}$. Let G be the graph in Figure 2.2 obtained from H by adding a–3 new vertices $u_1, u_2, \ldots, u_{a-3}$ and joining each u_i $(1 \leq i \leq a-3)$ with y. It is clear that $S = \{u_1, u_2, \ldots, u_{a-3}\}$ is not an open geodetic set of G. Also, it is easily seen that $S \cup \{w,z\}$, where $w,z \notin S$, is not an open geodetic set of G. Let $S' = S \cup \{x,v_i,v_j\}$, where v_i and v_j are non-adjacent.. Then it is clear that S' is an open geodetic set of G and so $og(G) = a$.

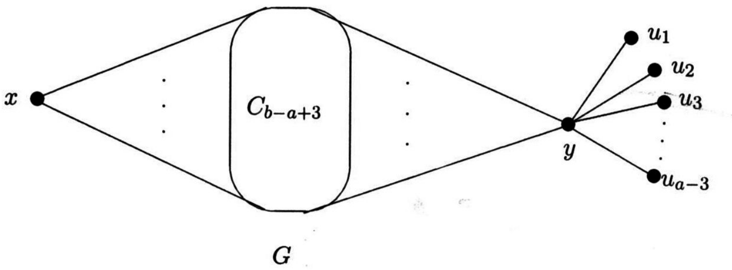


Figure 2.2

We now prove that $og^+(G) = b$. It is clear that $T=S\cup\{v_1,v_2,\ldots,v_{b-a+3}\}$ is an open geodetic set of G. We show that T is a minimal open geodetic set of G. On the contrary, assume that W is a proper subset of T such that W is an open geodetic set of G. Then there exists a vertex $v\in T$ such that $v\notin W$. By Theorem 1.3, it is clear that $v = v_j$ for some $j(1\le j \le b-a+3)$. Then v_{j+1} does not lie on a geodesic joining any pair of vertices of W and so W is not an open geodetic set of G, which is a contradiction. Hence T is a minimal open geodetic set of G so that $og^+(G) \ge b$. Now, since y is a cut-vertex of G, y does not belongs to any minimal open geodetic set of G. Suppose that $og^+(G)=b+1$. Let X be a minimal open geodetic set of cardinality $b+1$. Then $X=V(G)-\{y\}$ and S' is a proper subset of X so that X is not a minimal open geodetic set, which is a contradiction. Hence $og^+(G)= b$.

Lemma 2.15. *Let G be a connected graph with v a cut-vertex such that G–v has a component having no extreme vertices. Then every open geodetic set of G contains at least three vertices from each such component of G–v.*

Proof. Let C be a component of $G–v$ having no extreme vertices. Then C must contain at least three vertices. Let S be any open geodetic set of G. By Theorem 1.5, it follows that $S\cap V(C)\neq \phi$.

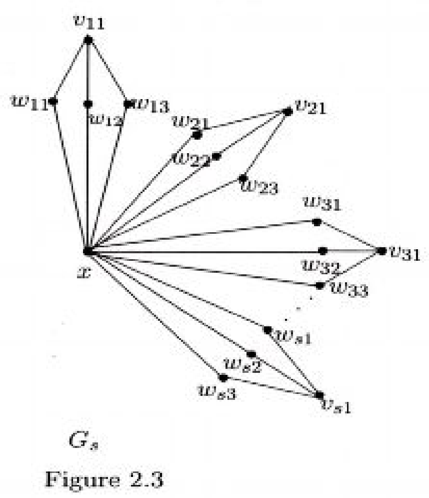
If $S\cap V(C)=\{x\}$,then $x\notin I(u,w)$ for any $u,w\in S$. Hence $|S\cap V(C)|\ge 2$. If $S\cap V(C)=\{x,y\}$, then $x\in I(y,w)$ for some $w\in S$ and $y\in I(x,z)$ for some $z\in S$. It follows that $x\in I(y,v)$ and $y\in I(x,v)$. Then

$$d(x, y) = d(y,v) - d(x, v) \quad (1)$$
$$\text{and } d(x,y) = d(x,v) - d(y,v) \quad (2)$$

From (1) and (2), we see that $d(x,v) = d(y,v)$. Hence $d(x,y) =0$. This gives $x=y$, which is a contradiction. Thus $|S\cap V(C)|\ge 3$.

Next, we show that every three positive integers a,b,c with $4\le a\le b\le c$ and $b\le 3a$ is realizable as the geodetic number, open geodetic number and upper open geodetic number of some connected graph. For this purpose we introduce the following special graphs G_s, H_l and H given in Figures 2.3, 2.4 and 2.5 respectively.

For integers i and s with $1\le i\le s$, let each F_i be a copy of $K_{2,3}$ with partite sets $V_{i1}=\{v_{i1}, v_{i2}\}$ and $V_{i2}=\{w_{i1},w_{i2},w_{i3}\}$. Let G_s be the graph in Figure 2.3 obtained from the F_i by identifying the s vertices $v_{i2}\,(1\le i\le s)$. Let x be the common vertex representing the identified vertices. It is clear that $S=\{v_{11},v_{21},\ldots,v_{s1}\}$ is the unique minimum geodetic set of G_s so that $g(G_s) = s$. Since each vertex $v_{i1}\,(1\le i\le s)$ lies only on a geodesic joining any two of the three vertices w_{i1},w_{i2},w_{i3}, it follows that $S\cup\{w_{i1}, w_{i2}: 1\le i\le s\}$ is a minimum open geodetic set of G_s and so $og(G_s) =3s$.



G_s

Figure 2.3

For $l \geq 3$, let $H_l = C_l + \bar{K}_2$ be the graph given in figure 2.4, where $V(\bar{K}_2)=\{x, y\}$ and C_l is the cycle with $V(C_l) =\{z_1,z_2\ldots,z_l\}$. It is clear that $S = \{x,y\}$ is the unique minimum geodetic set of H_l so that $g(H_l) =2$. Moreover, $S \cup \{z_i, z_j\}$, where z_i and z_j $(1 \leq i, j \leq l)$ are non-adjacent vertices of C_l, is a minimum open geodetic set of H_l so that $og(H_l) = 4$.

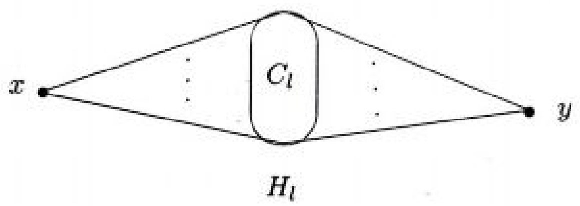


Figure 2.4

Let H be the graph given in Figure 2.5. It is clear that $S = \{u_1,u_3\}$ is a geodetic set of H and so $g(H)=2$. Since H is a graph without extreme vertices, by Theorem 1.6, $og(H) \geq 4$. It is easily verified that no 4-element subset of $V(H)$ is an open geodetic set of H. Now, $S_1 = \{u_1, u_2, u_3, w_1, w_4\}$ is an open geodetic set of H and so $og(H) = 5$.

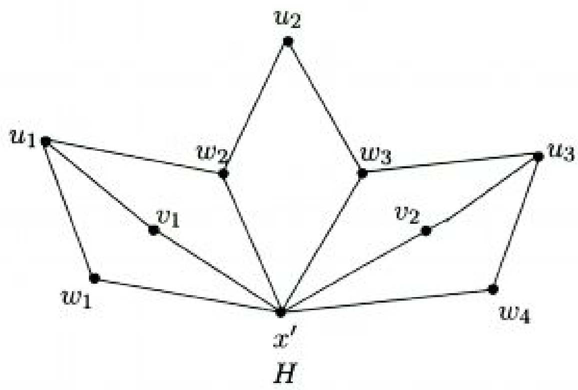


Figure 2.5

The arguments used in determining the geodetic number and open geodetic number of the graphs G_s, H_l and H will also be used in the proof of the following theorem.

Theorem 2.16. *For any three positive integers a,b,c with* $4 \leq a \leq b \leq c$ *and* $b \leq 3a$, *there exists a connected graph G such that* $g(G) = a$, $og(G) = b$ *and* $og^+(G) = c$.

Proof. For $a=b=c$, the star $K_{1,\, a}$ has the desired properties. Let $b = a+p$, where $1 \leq p \leq 2a$. We consider three cases.

Case 1. $a < b < c$.

First suppose that $p \neq 2a–1$. We consider two subcases.

Subcase 1a. p is even. First, let $p = 2$. Let G be the graph in Figure 2.6 obtained form the graph H_{c-b+3} by adding $a–1$ new vertices $u_1,u_2,\ldots,u_{a-1}$ and joining the edges $u_i\, x$ $(1 \leq i \leq a-1)$. It is clear that $S = \{u_1, u_2, ..., u_{a-1}\}$ is not a geodetic set of G and $S_1 = S \cup \{y\}$ is a geodetic set of G and so by Theorem 1.1, $g(G)=a$. Now, let $S_2=S_1 \cup \{z_i, z_j\}$, where z_i and z_j $(1 \leq i, j \leq c-b+3)$ are non-adjacent vertices on C_{c-b+3}. By Theorem 1.3, every open geodetic set contains S and it is easily verified that S_2 is a minimum open geodetic set of G and so $og(G)=a+2=b$. Let $S_3=S \cup \{z_1, z_2, ..., z_{c-b+3}\}$. Then it is clear that S_3 is an open geodetic set of G. We show that S_3 is a minimal open geodetic set of G. Otherwise, there exists a proper subset W of S_3 such that W is an open geodetic set of G. Then there exists a vertex say $v \in S_3$ such that $v \notin W$. By Theorem 1.3, it is clear that $v = z_j$ for some $j (1 \leq j \leq c-b+3)$. Let z_l be a vertex on the cycle C_{c-b+3} such that it is adjacent to z_j. Then z_l is not an internal vertex of a geodesic joining any pair of vertices of W and so W is not an open geodetic set of G, which is a contradiction. Hence S_3 is a minimal open geodetic set of G so that $og^+(G) \geq c$. Now, suppose that $og^+(G) \geq c+1$. Let X be a minimal open geodetic set with $|X| \geq c+1$. Then $X = V(G) - \{x\}$. Since S_2 is an open geodetic set properly contained in X, we see that X is not a minimal open geodetic set of G, which is a contradiction. Hence $og^+(G) = c$.

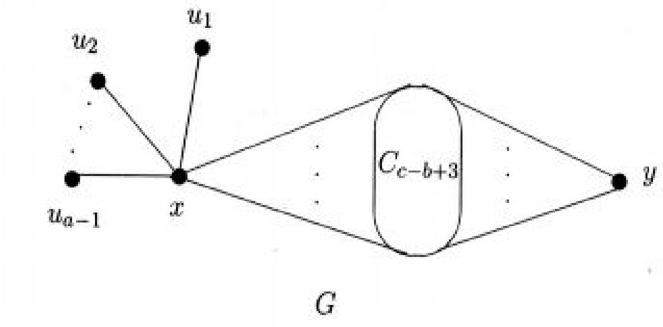


Figure 2.6

Now, let $p \geq 4$. Let G be the graph in Figure 2.7 obtained from $G_{\frac{p}{2}-1}$ and H_{c-b+3} by first identifying at the vertex x and then adding the new vertices $u_1, u_2, \ldots, u_{a-\frac{p}{2}}$ and then joining the edges $u_i x (1 \leq i \leq a - \frac{p}{2})$.

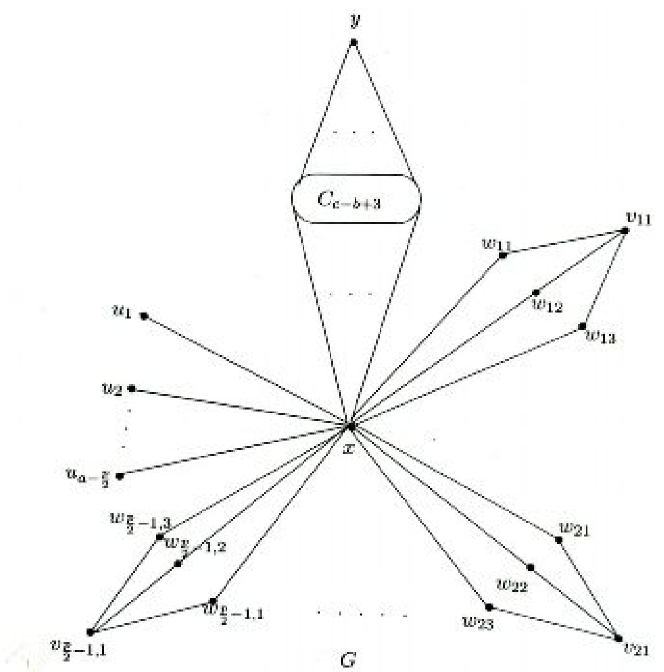


Figure 2.7

Let $S = \{ u_1, u_2, \ldots, u_{a-\frac{p}{2}} \}$. Then it follows form Theorem 1.1 and Theorem 1.2 that $S_1 = S \cup \{y\} \cup \{v_{11}, v_{21}, \ldots, v_{\frac{p}{2}-1,1}\}$ is a minimum geodetic set (in fact, S_1 is unique) and so $g(G) = a$. Let $S_2 = S_1 \cup \{ w_{i1}, w_{i2} : 1 \leq i \leq \frac{p}{2} - 1\} \cup \{z_i, z_j\}$, where z_i and z_j $(1 < i, j \leq c - b + 3)$ are non-adjacent vertices of C_{c-b+3}. By Theorem 1.3 and Lemma 2.15, S_2 is a minimum open geodetic set of G and so $og(G) = a + p = b$. Let $S_3 = S \cup \{w_{i1}, w_{i2} : 1 \leq i \leq \frac{p}{2} - 1\} \cup \{v_{11}, v_{21}, \ldots, v_{\frac{p}{2}-1,1}\} \cup \{z_1, z_2, \ldots, z_{c-b+3}\}$.

We show that S_3 is a minimal open geodetic set of G. Otherwise, there exists a proper subset W of S_3 such that W is an open geodetic set of G. Then there exists a vertex say $v \in S_3$ such that $v \notin W$. By Theorem 1.3 and Lemma 2.15, it is clear that $v = z_j$ for some $j\,(1 \le j \le c-b+3)$. Let z_l be the vertex on the cycle C_{c-b+3} such that it is adjacent to z_j. Then z_l is not an internal vertex of a geodesic joining any pair of vertices of W and so W is not an open geodetic set of G, which is a contradiction. Hence S_3 is a minimal open geodetic set of G so that $og^+(G) \ge c$. Now, suppose that $og^+(G) \ge c+1$. Let X be a minimal open geodetic set of cardinality $\ge c+1$.

First suppose that $y \in X$. By Lemma 2.15, it is clear that any two non-adjacent vertices $z_i, z_j\,(1 \le i, j \le c-b+3)$ of C_{c-b+3} must belong to X. Also, by Theorem 1.3, $u_i \in X\ (1 \le i \le a - \frac{p}{2})$. Since v_{i1} are the end vertices of any geodesic in which w_{i1} lies internally, we have $v_{i1} \in X\ (1 \le i \le \frac{p}{2} - 1)$. Also, it is clear that $w_{i1}, w_{i2} \in X\ (1 \le i \le \frac{p}{2} - 1)$. Then it is clear that S_2 is a proper subset of X such that S_2 is an open geodetic set of G with $|S_2| = a+p = b$, which is a contradiction to X a minimal open geodetic set of G.

Now, suppose that $y \notin X$. Then it is clear that $z_i \in X$ for each $i\,(1 \le i \le c-b+3)$. Then, just as above $u_i \in X\ (1 \le i \le a - \frac{p}{2})$ and $w_{11}, v_{11}, w_{12};\ w_{21}, v_{21}, w_{22}; \ldots;\ w_{a-\frac{p}{2},1}, v_{a-\frac{p}{2},1}, w_{a-\frac{p}{2},2} \in X$. Then it is clear that S_3 is a proper subset of X such that S_3 is an open geodetic set of G with $|S_3| = c$, which is a contradiction to X a minimal open geodetic set of G. Hence $og^+(G) = c$.

Subcase1b. p is odd. Then $1 \le p \le 2a-3$. For $p=1$, let G be the graph in Figure 2.8 obtained from H_{c-b+3} by adding $a-1$ new vertices $u_1, u_2, \ldots, u_{a-1}$ and joining the edges $u_1 z_1, u_i x\,(2 \le i \le a-1)$.

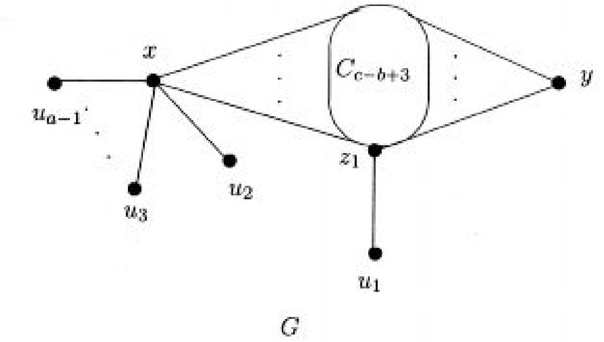


Figure 2.8

Let $S = \{u_1, u_2, \ldots, u_{a-1}\}$. By Theorem 1.1, $S_1 = S \cup \{y\}$ is a minimum geodetic set of G and so $g(G) = a$. Let z_j be a vertex on C_{c-b+3} such that z_1 and z_j are non-adjacent. Then $S_2 = S \cup \{z_j\}$ is a minimum open geodetic set of G and so $og(G) = a+1 = b$. Let $S_3 = S \cup (V(C_{c-b+3}) - \{z_1\})$. Then S_3 is an open geodetic set of G. we show that S_3 is a minimal open geodetic set of G. Otherwise, there exists a proper subset W of S_3 such that W is an open geodetic set of G. Then there exists a vertex say $v \in S_3$ such that $v \notin W$. By Theorem 1.3, it is clear that $v = z_k$ for some k such that $k \ne 1$. Let z_l and z_l' be the vertices adjacent to z_k. Suppose that $z_k \ne z_2, z_{c-b+3}$. Then it is clear that both z_l and z_l' do not lie as internal vertices of a geodesic joining a pair of vertices of W. Suppose that $z_k = z_2$ or z_{c-b+3}. Then one of the vertices z_l or z_l' does not lie as an internal vertex of any geodesic joining a pair of vertices of W. Thus W is not an open geodetic set of G, which is a contradiction. Hence S_3 is a minimal open geodetic set of G so that $og^{+}(G) \geq c$. Now, suppose that $og^{+}(G) \geq c+1$. Let X be a minimal open geodetic set of cardinality $\geq c+1$.

First, Suppose that $y \in X$. By Theorem 1.3, $u_i \in X$ $(1 \leq i \leq a-1)$. Since X is an open geodetic set there exists $z_i \in V(C_{c-b+3})$ with $z \neq z_i$ $(i = 1, 2, c-b+3)$ such that $z \in X$. Now, let $T = S \cup \{y, z\}$. Then it is clear that T is an open geodetic set contained in X such that $|T| = a+1 = b$, which is contradiction to X a minimal open geodetic set of G. Next, suppose that $y \notin X$. Then just as above $u_i \in X(1 \leq i \leq a-1)$. Also, since $y \notin X$, it is clear that, $z_i \in X$ for each $i(2 \leq i \leq c-b+3)$. Then clearly S_3 is a proper subset of X such that S_3 is an open geodetic set of G with $|S_3| = c$, which is a contradiction to X a minimal open geodetic set of G. Thus $og^{+}(G)=c$.

For $p \geq 3$, let G be the graph in Figure 2.9 obtained from the graph $G_{\frac{p+1}{2}-1}$ by first adding $a-(\frac{p+1}{2})$ new vertices $u_1, u_2, \ldots, u_{a-(\frac{p+1}{2})}$ and $a-(\frac{p+1}{2})$ new edges u_1w_{11} and $u_i x(2 \leq i \leq a-(\frac{p+1}{2}))$ and then identifying this with H_{c-b+3} at the vertex x.

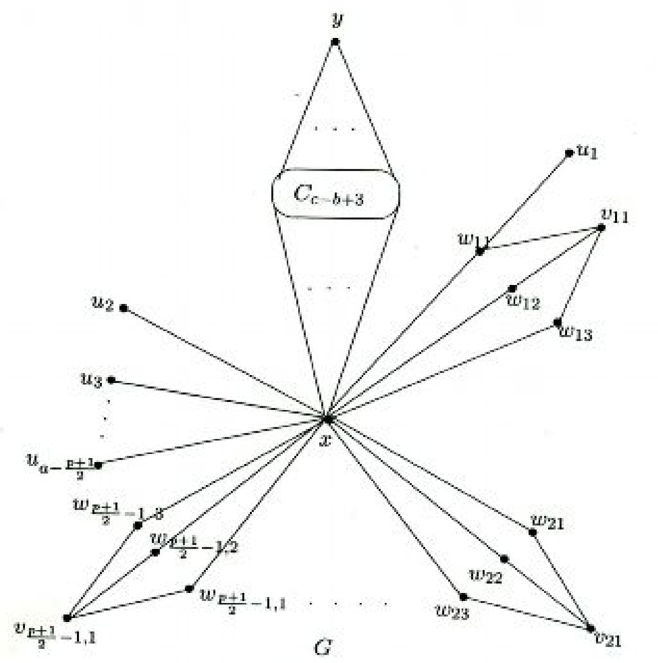


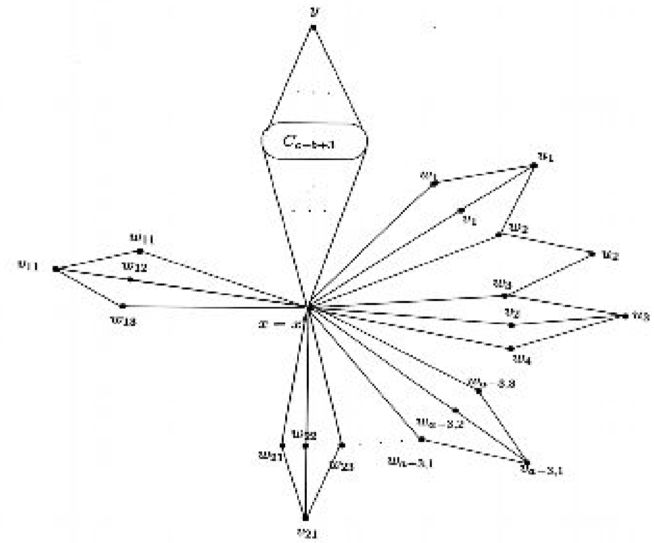
Figure 2.9

Let $S_1 = \{u_1, u_2, \ldots, u_{a-(\frac{p+1}{2})}, v_{11}, v_{21}, \ldots, v_{\frac{p+1}{2}-1,1}, y\}$ and $S_2 = S_1 \cup \{w_{12}\} \cup \{w_{i1}, w_{i2}$: $2 \leq i \leq (\frac{p+1}{2})-1\} \cup \{z_i, z_j\}$ where z_i and z_j $(1 \leq i, j \leq c-b+3)$ are non-adjacent vertices on C_{c-b+3}. Then as earlier, it can be seen that S_1 is a minimum geodetic set of G and S_2 is a minimum open geodetic set of G and so $g(G) = a$ and $og(G) = a+p = b$. Let $S_3 = \{u_1, u_2, \ldots, u_{a-(\frac{p+1}{2})}, v_{11}, v_{21}, \ldots, v_{\frac{p+1}{2}-1,1}\} \cup \{w_{12}\} \cup \{w_{i1}, w_{i2}$: $2 \leq i \leq (\frac{p+1}{2})-1\} \cup \{z_1, z_2, \ldots, z_{c-b+3}\}$. Then as in subcase 1a of case 1, it can be proved that $og^{+}(G) \geq c$. Now, let X be a minimal open geodetic set of G such that $|X| \geq c+1$.

If $y \in X$, then by using arguments similar to the one in Subcase 1a of Case 1, it is clear that S_2 is a proper subset of X such that S_2 is an open geodetic set of G with $|S_2| = a+p = b$, which is a contradiction to X a minimal open geodetic set.

If $y \notin X$, then just as above, it can be seen that S_3 is a proper subset of X such that S_3 is an open geodetic set of G with $|S_3| = c,$ which is a contradiction to X a minimal open geodetic set. Thus $og^+(G)=c$.

Now, Suppose that $p = 2a - 1$. Let G be the graph in Figure 2.10 obtained from $G_{a\text{-}3,}$ $H_{c\text{-}b+3}$ and H by identifying the vertices x and x'.



G

Figure 2.10

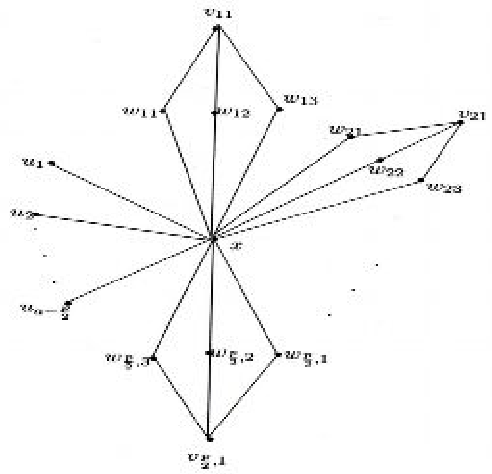
It follows from Theorem 1.2 that $S = \{y, v_{11}, v_{21}, ..., v_{a-3,1}, u_1, u_3\}$ is a minimum geodetic set of G so that $g(G)=a$. Let $S_1 = \{y, z_i, z_j,; w_{11}, v_{11}, w_{12}; w_{21}, v_{21}, w_{22}; ..., w_{a\text{-}3,1}, v_{a\text{-}3,1}, w_{a\text{-}3,2};$ $w_1, u_1, u_2, u_3, w_4\}$, where z_i *and* z_j $(1 \leq i, j \leq c - b + 3)$ are non-adjacent vertices on $C_{c\text{-}b+3.}$ By Lemma 2.15, it is straight forward to verify that S_1 is a minimum open geodetic set of G so that og $(G)=a+p=b$. Let $S_2=\{z_1, z_2, \ldots, z_{c\text{-}b+3}, w_{11}, v_{11}, w_{12}; w_{21}, v_{21}, w_{22}; ..., w_{a\text{-}3,1}, v_{a\text{-}3,1}, w_{a\text{-}3,2};$ $w_1, u_1, u_2, u_3, w_4\}$. We show that S_2 is a minimal open geodetic set of G. Otherwise, there exists a proper subset W of S_3 such that W is an open geodetic set of G. Then there exists a vertex, say $v \in S_3$ such that $v \notin W$. It is easily seen that $v = z_j$ for some $j\,(1 \leq j \leq c - b + 3)$. Let z_l be the vertex on the cycle $C_{c\text{-}b+3}$ such that it is adjacent to z_j. Then z_l is not an internal vertex of a geodesic joining any pair of vertices of W, and so W is not an open geodetic set of G, which is a contradiction. Hence S_2 is a minimal open geodetic set of G so that $og^+(G) \geq c$. Now, let X be a minimal open geodetic set of cardinality $\geq c + 1$.

Suppose that $y \in X$. By Lemma 2.15, $z_i, z_j \in X\,(1 \leq i, j \leq c - b + 3)$, where z_i and z_j are non-adjacent vertices on $C_{c\text{-}b+3}$. Also, it is easy to see that X must contain the vertices $w_{11}, v_{11}, w_{12}; w_{21}, v_{21}, w_{22};..., w_{a-3,1}, v_{a-3,1}, w_{a-3,2}; w_{1,} u_1, u_2, u_3, w_4$. Then clearly S_1 is a proper subset of X such that S_1 is an open geodetic set of G, which is a contradiction to X a minimal open geodetic set.

Suppose that $y \notin X$. Then X must contain all the vertices on $C_{c\text{-}b+3}$. Also, it is easy to see that X must contain the vertices $w_{11}, v_{11}, w_{12}; w_{21}, v_{21}, w_{22};..., w_{a-3.1}, v_{a-3.1},$ $w_{a-3.2}; w_{1,} u_1, u_2, u_3, w_4$. Then clearly S_2 is a proper subset of X such that S_2 is an open geodetic set of G with $|S_2| = c,$ which is a contradiction to X a minimal open geodetic set. Thus $og^+(G)$ $=c$.

Case 2. $a < b=c$. First suppose that $p \neq 2a - 1$. We consider two subcases.

Subcase 2a. p is even. Let G be the graph in Figure 2.11 obtained from the graph $G_{\frac{p}{2}}$ by adding $a-\frac{p}{2}$ new vertices $u_1,u_2,...,u_{a-\frac{p}{2}}$ and the $a-\frac{p}{2}$ new edges $u_ix(1\le i\le a-\frac{p}{2})$.

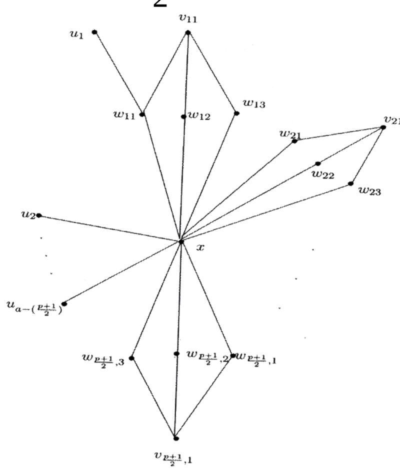


G

Figure 2.11

Let $S_1=\{u_1,u_2,...,u_{a-\frac{p}{2}},v_{11},v_{21},...,v_{\frac{p}{2},1}\}$ and $S_2 = S_1\cup\{w_{i1},w_{i2}:1\le i\le\frac{p}{2}\}$. Then, as earlier it can be seen that S_1 is a minimum geodetic set of G and S_2 is a minimum open geodetic set of G so that $g(G)=a$ and $og(G)=a+p=b$. Now, it also follows that $og^+(G)\ge b$. suppose that $og^+(G)\ge b+1$. Let X be a minimal open geodetic set of cardinality $\ge b+1$. Then it can be seen as earlier that S_2 is a proper subset of X such that S_2 is an open geodetic set of G with $|S_2|=a+p=b$, which is contradiction to X a minimal open geodetic set. Hence $og^+(G)=b=c$.

Subcase 2b. p is odd. Then $1\le p\le 2a-3$. Let G be the graph in Figure 2.12 obtained from the graph $G_{\frac{P+1}{2}}$ by adding $a-(\frac{P+1}{2})$ new vertices, $u_1,u_2,..,u_{a-(\frac{p+1}{2})}$ and $a-(\frac{p+1}{2})$ new edges u_ix ($2\le$ i $\le a-(\frac{P+1}{2})$) and the edge $u_1\, w_{11}$.

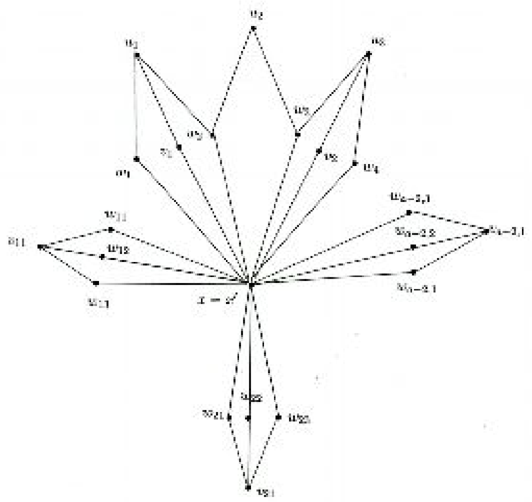


G

Figure 2.12

Let S_1 = $\{u_1,u_2,..,u_{a-(\frac{p+1}{2})},v_{11},v_{21},\ldots,v_{\frac{P+1}{2},1}\}$ and $S_2=S_1\cup\{w_{12}\}\cup\{w_{i1},w_{i2}$: $2\le$ i $\le\frac{P+1}{2}\}$. Then, as earlier it can be seen that S_1 is a minimum geodetic set of G and S_2 is a minimum open geodetic set of G so that $g(G)=a$ and $og(G)=a+p=b$. Now, it also follows that $og^{+}(G)\geq b$. Suppose that $og^{+}(G)\geq b+1$. Let X be a minimal open geodetic set of G with $|X| \geq b+1$. Then, it can be seen as earlier that S_2 is a proper subset of X such that S_2 is an open geodetic set G with $|S_2| = a + p = b,$ which is a contradiction to X a minimal open geodetic set. Hence $og^{+}(G) = b = c$.

Now, suppose that $p = 2a-1$. Let G be the graph in Figure 2.13 obtained from the graph $G_{a\text{-}2}$ and H by identifying the vertices x and x'.



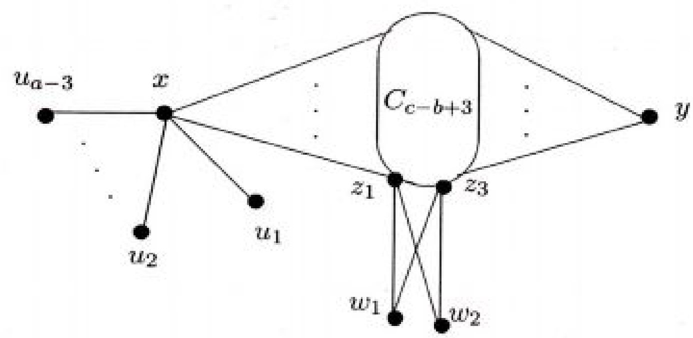
G

Figure 2.13

It follows from Theorem1.2 that $S = \{v_{11}, v_{12}, \ldots, v_{a\text{-}2,1}, u_1, u_3\}$ is a minimum geodetic set of G so that $g(G)=a$. Let $S_1 = \{w_1, u_1, u_2, u_3, w_4;\ w_{11}, v_{11}, w_{12};\ w_{21}, v_{21}, w_{22}; \ldots, w_{a\text{-}2,1}, v_{a\text{-}2,1}, w_{a\text{-}2,2}\}$. It is straight forward to verify that S_1 is a minimum open geodetic set of G so that $og(G)=a+p=b$. Now, it also follows that $og^{+}(G)\geq b$. Suppose that $og^{+}(G)\geq b+1$. Let X be a minimal open geodetic set of G with $|X| \geq b+1$. Then it can be seen that S_1 is an open geodetic set contained in X with $|S_1| = a + p = b$, which is a contradiction to X a minimal open geodetic set of G. Hence $og^{+}(G)=b=c$.

Case 3. $a = b < c$.

Let G_1 be the graph obtained from $H_{c\text{-}b\text{+}3}$ by adding $a-3$ new vertices $u_1, u_2, \ldots, u_{a\text{-}3}$ and $a-3$ new edges $u_i x$ $(1 \leq i \leq a-3)$. Let G be the graph in Figure 2.14 obtained from G_1 by adding two new vertices w_1 and w_2 and joining both w_1 and w_2 to z_1 and z_3.



G

Figure 2.14

Let $S_1 = \{u_1,u_2,\ldots,u_{a\text{-}3}, z_1,z_3,y\}$. Then it can be seen that S_1 is both a minimum geodetic set and a minimum open geodetic set of G so that $g(G)=og(G)=a$. Let $S_2=\{u_1,u_2,\ldots, u_{a\text{-}3},z_1,z_2,\ldots,z_{c\text{-}b+3}\}$. Then as earlier, it can be seen that S_2 is a minimal open geodetic set of G, so that $og^+(G)\geq |S_2| = c.$ Let X be a minimal open geodetic set with $|X| \geq c+1$. Then it is clear that S_1 is a proper subset of X and so X is not a minimal open geodetic set, which is a contradiction. Hence $og^+(G)=c$. Thus the proof of the theorem is complete.

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