A Monetary Theory with Non-Degenerate Distributions*

Guido Menzio
University of Pennsylvania
(gmenzio@sas.upenn.edu)

Shouyong Shi
University of Toronto
(shouyong@chass.utoronto.ca)

Hongfei Sun
Queen's University
(hfsun@econ.queensu.ca)

May 2009

Abstract

At any given point of time in an actual economy, some individuals hold more money than other individuals do. This non-degenerate distribution of money holdings among individuals is a rationale for a range of policies designed for reallocating liquidity among individuals. However, monetary theory has often abstracted from this non-degenerate distribution for tractability reasons. In this paper, we construct a tractable search model of money with a non-degenerate distribution of money holdings. We model search as a directed process in the sense that buyers know the terms of trade before visiting particular sellers, as opposed to undirected search that has dominated the literature. In this model, the distribution of money holdings among individuals is non-degenerate. We show that this distribution affects individuals' decisions not directly, but rather indirectly only through a one-dimensional variable – the seller's future marginal value of money. This result drastically reduces the state space of individuals' decisions and makes the model tractable. We analytically characterize a monetary equilibrium, using lattice-theoretic techniques, and prove existence of a monetary steady state. In the equilibrium, buyers follow a stylized spending pattern over time, and the money distribution has a persistent wealth effect.

JEL classifications:
Keywords: Money; Distribution; Search; Lattice-Theoretic.

---

*Menzio: Department of Economics, University of Pennsylvania, 3718 Locust Walk Philadelphia, Pennsylvania 19104, USA. Shi: Department of Economics, University of Toronto, 150 St. George Street, Toronto, Ontario, Canada, M5S 3G7; Sun: Department of Economics, Queen's University, 94 University Ave., Kingston, Ontario, Canada, K7L 3N6. Shi gratefully acknowledges financial support from the Bank of Canada Fellowship and from the Social Sciences and Humanities Research Council of Canada. The opinion expressed in the paper is our own and does not reflect the view of the Bank of Canada. All errors are ours.

1. Introduction

Money is unevenly distributed among individuals in an actual economy at any given point of time. This distribution of money holdings has important implications for efficiency and welfare in the economy. In addition to the financial sector, a range of monetary and banking policies are designed to reallocate liquidity. For example, many central banks use open market operations and overnight markets to supply liquidity or channel liquidity from one set of individuals to another. Despite such importance of a non-degenerate distribution of money holdings, monetary theory has often abstracted form it, largely for tractability reasons. In this paper, we construct a tractable model with a microfoundation of money and a non-degenerate distribution of money holdings. We prove existence of a monetary steady state and analyze its properties.

A model should have at least three features in order to capture the importance of money distribution. First, the model should have a strong microfoundation for money by deriving a role for money from an explicitly specified physical environment. Only with such a microfoundation can one coherently assess the welfare effect of monetary policy. Second, money distribution should induce a wealth effect in the sense that different levels of money holdings yield different marginal values of money. This wealth effect is necessary for a redistribution of money among individuals to affect aggregate welfare. Third, the model should generate the spending pattern that an individual does not rebalance money holdings every period, even if he has the ability to do so. Instead, an individual should take time to run down the money balance. With this spending pattern, the wealth effect of money distribution is persistent.

The recent microfoundation of money (i.e., search theory of money) provides a coherent framework to explain the role of money. The theory models a market as being decentralized where each trade involves only a small group (usually two) of individuals and where the ability to keep record of individuals' trading histories is limited. In this environment, objects with no intrinsic values, such as money, can improve resource allocation. In addition to deriving a role for fiat money, search theory can be a natural framework for examining the importance of money distribution. Since exchange is decentralized, individuals experience different realizations of the matching shocks which naturally lead to a non-degenerate distribution of money holdings. Such dispersion in money holdings can persist, in principle, because an individual with a high money balance may want to spend the balance down in a sequence of trades rather than all at once.

However, this microfoundation of money has imposed assumptions to make money distribution degenerate in order to maintain tractability. Since the distribution is an aggregate state variable with large dimensionality, it is difficult to characterize an equilibrium in which the distribution directly affects individuals' decisions. To avoid this difficulty, Shi (1997) assumes that each household consists of a large number of members who share consumption and utility, and Lagos and Wright (2005) assume that individuals have quasi-linear preferences over a good which can be traded in a centralized market to immediately rebalance money holdings. With either assumption, there is no dispersion in money holdings among the buyers who enter the decentralized market.[1]

We use search theory of money as the microfoundation of money, but we do not impose the assumptions to make money distribution degenerate. The main deviation of our model from the literature is that it models search as a directed process, as opposed to undirected search. That is, buyers know the terms of trade before visiting particular sellers. In the model, there is a continuum of submarkets, each of which specifies the participants' matching probabilities and the quantities of money and goods to be traded in a match. Observing these terms, buyers choose which submarket to enter and sellers choose how many trading posts to maintain in each submarket. There is a cost of maintaining a trading post, and free entry of trading posts ensures that expected net profit is zero for every trading post. A matching function with constant returns to scale determines the total number of matches in each submarket. In equilibrium, individuals' decisions on which submarket to enter will ensure that the ratio of buyers to trading posts in each submarket is consistent with the specified matching probabilities.

In our model, an individual chooses in every period whether to be a buyer or a seller. Sellers are individuals who have spent their money down to sufficiently low levels and choose to produce. We assume that production and sales are organized by firms each of which hires a large number of producers with the same preferences. A firm pays competitive wage and distributes profit to the producers according to each producer's labor input. In addition to production, a firm chooses the number of trading posts to maintain in each submarket. In equilibrium, all sellers exit production with the same amount of money. They have the same marginal valuation of money which is equal to the reciprocal of the nominal wage rate.

In contrast, buyers are heterogeneous in money holdings. But this non-degenerate distribution does not render our analysis intractable as it does in other models cited above. Directed search is critical for tractability because it makes the equilibrium *block recursive* in the following sense. When search is directed, buyers with different money balances optimally sort themselves into different submarkets. With free entry of trading posts, sorting implies that the tightness in each submarket is independent of the distribution of buyers across submarkets. Hence, money distribution affects individuals' decisions and market tightness not directly, but rather indirectly only through a one-dimensional variable – the seller's marginal value of money. Such block

[1]Other examples of search models of money are Shi (1995) and Trejos and Wright (1995), who assume that money is indivisible. Assuming that goods are indivisible and money is in discrete units, Green and Zhou (1998) construct a search model where money distribution is non-degenerate.

recursivity drastically reduces the state space of individuals' decisions.

The pattern in which an individual in our model manages his money balance resembles that in a standard inventory model (e.g., Baumol, 1952, and Tobin, 1956).[2] An individual with the highest money balance chooses to consume frequently and consume a large quantity. He does so by visiting a submarket in which the matching probability, money spending and the quantity of goods traded in a match are all relatively high. After each purchase, the individual's money balance falls, and so he will choose next to visit a submarket in which the matching probability, money spending and the quantity of goods traded in a match are lower. This spending pattern continues until the buyer exhausts his money. Then, he will produce and sell goods to obtain the highest money balance, after which another round of purchases will start anew.

Money distribution affects resource allocation as follows. First, the distribution implies persistent dispersion in consumption. An individual with a relatively high money balance optimally chooses not to spend all of his money at once. Such an individual has a lower marginal value of money and consumes more (in probabilistic terms) than does an individual with less money. Second, by affecting the distribution of individuals across the submarkets, money distribution affects the number of trades and, hence, affects aggregate output and consumption, even though individuals' decisions do not directly depend on the distribution. Third, by affecting sellers' marginal value of money, money distribution *indirectly* affects individuals' choices.

We analytically characterize a monetary equilibrium, prove existence of a monetary steady state, and analyze money distribution. The difficulties in analyzing the equilibrium are that a buyer's objective function is not necessarily concave and that a buyer's value function is not necessarily differentiable. As a result, we cannot assume, a priori, that the first-order conditions or envelope conditions hold. To resolve these difficulties, we use lattice-theoretic techniques (see Topkis, 1998) to prove that a buyer's optimal decisions are monotonic functions of the buyer's money balance. Using this result, we prove further that *on the equilibrium path*, a buyer's value function is differentiable and a buyer's optimal choices do obey first-order conditions. The validity of these standard conditions makes the model easy to use.

Our paper is related to the literature on directed search originated from Peters (1984), most of which studies non-monetary economies.[3] Corbae et al. (2003) have employed directed search in monetary theory, but they focus on the formulation of trading coalitions and assume that money and goods are indivisible. Rocheteau and Wright (2005) have checked the robustness of

[2]In contrast to standard inventory models, our model has a strong microfoundation of money, endogenizes the staggering pattern of spending, and aggregates the pattern into an equilibrium.

[3]Other standard references on directed search are Peters (1991), Moen (1997), Acemoglu and Shimer (1999), and Burdett, Shi and Wright (2001).

a monetary model with respect to directed search. Galenianos and Kircher (2008) and Julien et al. (2008) have integrated directed search into a monetary model with auctions. Because the latter three papers build on the model by Lagos and Wright (2005), money distribution either is degenerate or does not have important and persistent wealth effects.[4]

Finally, a few recent papers have studied the labor market using block recursivity, i.e., the property that the distribution does not directly affect individuals' decisions and market tightness. In particular, Shi (2009) formalizes block recursivity, and Gonzalez and Shi (2008) use block recursivity to analyze learning from search in an equilibrium. Our paper is obviously different from these papers in the issue. Moreover, a monetary model has two elements that have been absent in a labor search model. First, an individual's gain from a monetary trade depends not only on how the match surplus is split, but also on how all individuals in the economy value money. By affecting the endogenous value of money, money distribution may still indirectly affect individuals' decisions, despite block recursivity. Second, the money balance is an individual's state variable, which the individual can accumulate by trading and carry from one match to the next. Both elements make it more challenging to study a monetary economy than a labor market.[5]

2. A Monetary Economy with Directed Search

2.1. The model environment

Consider an economy with discrete time. There are a large number of individuals who are evenly distributed over a finite number of types. The number of types is at least three. The measure of individuals within each type is normalized to one. Goods are perishable and perfectly divisible. The individuals are specialized in production and consumption in such a way that no double coincidence of wants can exist between two individuals. No record of individuals' actions can be kept between periods, and a medium of exchange is needed in every trade. This role is served by a fiat object called money. Money is perfectly divisible and can be stored without cost. The stock of money per capita is a constant $M > 0$. The utility of consuming q units of goods is $U(q)$ and the disutility of supplying l units of labor is $h(l)$. The functions U and h are twice continuously

[4]For example, Galenianos and Kircher (2008) generate a non-degenerate distribution of prices and money holdings by assuming that sellers use second-price auctions to sell goods. If sellers could commit to posted prices, as they do in our model, the distribution of prices or money holdings would be degenerate in their model.

[5]Zhu (2005) has taken an important step to construct a search model with divisible money and a non-degenerate money distribution. However, his model is different. He first characterizes a monetary equilibrium with indivisible money and then, by pushing the size of indivisibility to zero, he shows that the limit of this equilibrium is an equilibrium in an economy with divisible money. For the proof to work, Zhu imposes strong assumptions on preferences and an exogenous upper bound on money holdings. Moreover, conducting comparative statics in his model is complicated, because one needs to prove that the comparative statics in a sequence of economies with indivisible money converge to the comparative statics in an economy with divisible money.

differentiable and have the usual properties: $U' > 0$, $U'' < 0$, $U(0) = 0$, $U'(\infty) = 0$, $U'(0)$ being large; $h' > 0$, $h'' > 0$, $h(0) = 0$, and $h'(1) = \infty$.

In every period, an individual can choose to be either a buyer or a seller, where a seller is also a producer. Production and sales are carried out through firms, each of which employs a large number of producers who have the same preferences.[6] A firm maximizes expected profit by choosing how much to produce and where to sell the goods. The cost of producing q units of goods is $\psi(q)$ units of labor, and the cost of maintaining a trading post for one period is k units of labor. A seller supplies labor to the firm. The firm pays a competitive wage rate and distributes profit as dividends to the sellers in proportion to each seller's contribution. Firms enter the economy competitively so that all firms make zero profit in the equilibrium. The function ψ is twice continuously differentiable with the usual properties: $\psi' > 0$, $\psi'' \geq 0$, and $\psi(0) = 0$. Note that a firm here does not function as an insurance device for the sellers.

The goods market is decentralized and characterized by directed search. The market consists of a continuum of submarkets and each submarket can contain many trading posts whose number is determined by free entry. A submarket specifies particular terms of trade between a buyer and a seller, together with the matching probabilities. Search is directed in the sense that when choosing which submarket to enter, buyers and sellers know the terms of trade and matching probabilities in all submarkets. Matching in each submarket is still random, because buyers and trading posts there cannot coordinate. A matching technology with constant returns to scale determines the number of matches in a submarket. Once matched, individuals trade according to the terms of trade specified for that submarket. The commitment to the terms of trade precludes bargaining. In equilibrium, the matching technology and entry decisions together imply matching probabilities that are exactly equal to those specified for the submarket.

Let (x, q) denote the terms of trade in a submarket, where x is the amount of money that a seller receives from a buyer in a trade and q the amount of goods that a buyer receives from a seller. Let b denote the matching probability for a buyer in the submarket and s the matching probability for a trading post. To describe the matching probabilities, let N_b be the number of buyers in a submarket, N_s the number of trading posts, and N_s/N_b the tightness of the submarket. Let the number of matches in the submarket be given by a function, $\mathcal{M}(N_b, N_s)$. The matching probability for a buyer in the submarket is $b = \mathcal{M}(N_b, N_s)/N_b$, and the matching probability for a trading post is $s = \mathcal{M}(N_b, N_s)/N_s$. Assuming that the function, $\mathcal{M}(.,.)$, has constant returns

[6]No firm has the technology to employ producers with different preferences. This assumption is used to prevent barter from arising within a firm. If a firm could employ producers with different preferences, it would be possible for the firm to arrange multi-lateral barter trades among the producers. This role is not what a production firm performs in the market; rather, it is the role of middlemen whom we abstract from.

to scale, we can write the matching probabilities as $b = \mathcal{M}(1, N_s/N_b)$ and $s = \mathcal{M}(N_b/N_s, 1)$. As buyers and firms choose which submarket to enter, the tightness in each submarket is a function of the terms of trade in that submarket. Thus, both s and b are functions of (x, q), as we will see later in (2.3). For this reason, we refer to a submarket by (x, q) alone, although a submarket is described by (b, s) and (x, q) together.

It is convenient to express s as a function of b. To do so, solve the tightness, N_s/N_b, from $b = \mathcal{M}(1, N_s/N_b)$ and substitute into $s = \mathcal{M}(N_b/N_s, 1)$. Denote the result as

$$s = \mu(b). \tag{2.1}$$

Because all properties of the function $\mu(.)$ come from those of $\mathcal{M}$, we will treat $\mu(.)$ as a primitive of the model and refer to it as the *matching function*. We impose the following assumption:

Assumption 1. *For all $b \in [0, 1]$, the matching function $\mu(b)$ satisfies: (i) $\mu(b) \in [0, 1]$, with $\mu(0) = 1$ and $\mu(1) = 0$, (ii) $\mu'(b) < 0$, and (iii) $[1/\mu(b)]$ is strictly convex, i.e., $2(\mu')^2 - \mu\mu'' > 0$.*

Part (i) is a regularity condition. Part (ii) requires that if the matching probability is high for a buyer in a submarket, it must be the case that there are a relatively large number of trading posts per buyer in the submarket; in this case, the matching probability for each trading post in the submarket must be relatively low. We will show later that part (ii) implies that for any given amount of a buyer's spending, x, the quantity of goods obtained at a trading post must decrease with b in order to induce entry of trading posts into the submarket. Thus, for any given x, a seller's total cost of production is lower in submarkets with higher b. Part (iii) of the above assumption requires that this production cost be strictly concave in b for any given x. That is, producers must be compensated with increasingly larger reductions in the cost of production in order to create additional trading posts to increase buyers' matching probability. This requirement is necessary for the tradeoff between the matching probability and the terms of trade across submarkets to yield a unique optimal choice of a submarket for a buyer.[7]

The following example gives a matching function that satisfies Assumption 1:

Example 2.1. *Consider the function $\mathcal{M} = N_s N_b (N_s^\rho + N_b^\rho)^{-1/\rho}$. (The special case with $\rho = 1$ is the so-called telegraph matching function.) With this function, $s = \mu(b) = (1 - b^\rho)^{1/\rho}$. For all $\rho \in (0, \infty)$, $\mu(b)$ satisfies Assumption 1.*

Normalize all nominal variables by the stock of money per capita, M. Let m denote an individual's money balance at the beginning of a period (divided by M). Denote ω as the

[7] We do not impose the assumption $\mu'' \leq 0$, because it is neither necessary for the analysis nor reasonable for usual examples of the matching function.

marginal value of such normalized money in terms of labor so that $1/\omega$ is the nominal wage rate. We focus on equilibria where money has a positive value $\omega > 0$.

Let $V(m)$ be an individual's *ex ante value function*, measured before the individual chooses to be a seller or a buyer in a period. As is common, we will start the analysis by assuming that $V(.)$ is continuous, bounded, increasing and concave. After analyzing individuals' optimal choices, we will explore the contraction mapping theorem to verify that V indeed has these properties (see Theorem 3.3). However, we do not assume that V is differentiable, because differentiability is not preserved by contraction mapping in general.

In every period, an individual first chooses whether to be a seller or a buyer and then makes the trading decisions. A firm chooses the number of trading posts in each submarket. We will analyze these decisions recursively, i.e., a firm's decision first, an individual's trading decisions second, and finally the choice between being a seller and a buyer.

2.2. A firm's decision

A firm chooses $d\,N(x,q)$ for each (x,q), i.e., the number of trading posts in submarket (x,q). Consider an arbitrary submarket (x,q). Let $s(x,q)$ denote the matching probability for a trading post in the submarket. For a trading post in the submarket, the maintenace cost is k and the expected cost of production of goods (sales) is $\psi(q)s(x,q)$ in terms of labor.[8] The expected money receipts from selling at the post are $xs(x,q)$. Recall that $1/\omega$ is the nominal wage rate. Hence, the firm's total demand for labor, l^d, and profit, D, are given as follows:

$$l^d = \int [k + \psi(q)\, s(x,q)]\, d\,N(x,q),$$

$$D = \int \left\{ x\, s(x,q) - \frac{1}{\omega}\, [k + \psi(q) s(x,q)] \right\} d\,N(x,q).$$

The firm chooses $d\,N(x,q)$ for each (x,q) to maximize profit. The expected profit of operating a trading post in submarket (x,q) is the expression inside the integral for D. If this profit is strictly positive, the firm will choose $d\,N(x,q) = \infty$, but this case will never occur in an equilibrium under free entry of trading posts.[9] If this profit is strictly negative, the firm will choose $d\,N(x,q) = 0$. If this profit is zero, the firm is indifferent about different non-negative and finite levels of $d\,N(x,q)$.

[8]The formula of the production cost implicitly assumes that the production cost must be incurred on the spot of each sale. This assumption is not necessary for our analysis, although it has been used in most search models of money. If production is centralized in a firm, instead, the marginal cost of selling goods in a trade is independent of the quantity of the particular sale because this quantity is negligible relative to the firm's total output. This case corresponds to the special case $\psi'' = 0$.

[9]More precisely, if $d\,N(x,q) = \infty$, the matching probability for a trading post in submarket (x,q) will be zero. In this case, the expected profit of operating a trading post in submarket (x,q) is $-k < 0$, which contradicts the optimality of the choice $d\,N(x,q) = \infty$.

Thus, the optimal choice of $d\,N(x,q)$ satisfies:

$$k \geq s(x,q)\,[\omega x - \psi\,(q)] \quad \text{and} \quad d\,N(x,q) \geq 0, \tag{2.2}$$

where the two inequalities hold with complementary slackness.[10] This implies $D = 0$.

For all pairs (x,q) such that $k < \omega x - \psi\,(q)$, the submarket has $d\,N(x,q) > 0$, and (2.2) holds as equality. For all pairs (x,q) such that $k \geq \omega x - \psi\,(q)$, the submarket has $d\,N(x,q) = 0$, and so $s(x,q) = 1$ and $b(x,q) = 0$. Putting the two cases together, we obtain the following matching probability for a trading post in submarket (x,q):

$$s(x,q) = \begin{cases} \frac{k}{\omega x - \psi(q)}, & \text{if } k \leq \omega x - \psi\,(q) \\ 1, & \text{otherwise.} \end{cases} \tag{2.3}$$

A buyer's matching probability in submarket (x,q) is $b(x,q) = \mu^{-1}\,(s(x,q))$. Equation (2.3) states intuitively that the matching probability for a selling post increases in the quantity of goods traded in a match, q, and decreases in the quantity of money traded, x. Note that for any given ω, $s(x,q)$ and $b(x,q)$ are independent of the distributions of buyers and trading posts.

2.3. A seller's decision

A seller chooses labor supply, l, at the competitive wage rate $1/\omega$. The disutility of labor is $h(l)$. In addition to wages, a seller receives dividends, $l\,D/l^d$, but these are equal to zero because $D = 0$. Thus, a seller who enters production with m units of money will have a money balance, $y \equiv m + \frac{l}{\omega}$, at the end of the period whose value will be $\beta V(y)$. The value function of this seller, denoted as $W_s(m)$, obeys the following Bellman equation:

$$W_s(m) = \max_{l \in [0,1]} \left[\beta V(m + \frac{l}{\omega}) - h(l) \right]. \tag{2.4}$$

The optimal choice of l is unique. It is intuitive to expect that the higher a seller's money holdings, the less he wants to contribute to the firm. Moreover, a seller's end-of-period money balance, y, is strictly increasing in the money balance that the seller has at the beginning of the period.[11] Denote the optimal choice of l as $l(m)$. The implied end-of-period money balance is $y(m) = m + \frac{l(m)}{\omega}$. The following lemma summarizes the properties of the optimal choice and a seller's value function (see Appendix A for a proof):

[10] We impose (2.3) for all possible submarkets (x,q), not just for the submarkets that are active in the equilibrium. It will become clear that the equilibrium has only a finite number of active submarkets. Thus, this condition imposes restrictions on beliefs out of the equilibrium. As a restriction needed to complete the markets, it is common in directed search models, e.g., Moen (1997), Acemoglu and Shimer (1999), and Shi (2009).

[11] Throughout this paper, "increasing" means "non-decreasing", "concave" means "weakly concave", etc. The modifier "stictly" is added when a property is strict.

Lemma 2.2. *Assume that $V(.)$ is continuous, bounded, increasing and concave. Then, $W_s(.)$ is continuous, bounded, increasing, and concave. For each m, the optimal choice $l(m)$ is unique and decreasing in m, while $y(m)$ is unique and strictly increasing in m. For all m such that $l(m) > 0$, the function $W_s(.)$ is differentiable at m, the function $V(.)$ is differentiable at $y(m)$, and the optimal choice satisfies the first-order condition. That is,*

$$W_s'(m) = \beta V'(m + \frac{l(m)}{\omega}) = \omega h'(l(m)). \tag{2.5}$$

We focus on the case where the optimal choice $l(m)$ is interior. This focus is innocuous. Because $h'(1) = \infty$, it is never optimal for an individual to choose $l = 1$, regardless of the individual's money balance. On the other hand, the choice $l(m) = 0$ may be optimal only if the individual's money balance m is sufficiently high, but no one will hold such a high money balance in an equilibrium. Even if an individual starts with such a high money balance, the individual will spend it down. Once that happens, the individual will never accumulate money back to such a high level that will induce $l(m) = 0$.

The above lemma holds for all $m \geq 0$. Of particular interest is the case $m = 0$, because we will show later that all sellers enter production with zero money balance in the equilibrium. For a seller with $m = 0$, denote the optimal labor supply as $l^* = l(0)$ and the end-of-period money balance as $m^* = y(0) = l^*/\omega$. This seller's value function is $W_s\,(0) = \beta V(m^*) - h(\omega m^*)$. Lemma 2.2 implies that $V'(m^*) = W_s'(0)/\beta$ exists and that m^* satisfies:

$$V'(m^*) = \frac{\omega}{\beta} h'(\omega m^*). \tag{2.6}$$

2.4. A buyer's decision

A buyer chooses which submarket to enter by choosing (x, q). It is more convenient to use (x, b) as the choice variables. To do so, we express q as a function of (x, b) by substituting $s = \mu\,(b)$ into (2.3). For any submarket with $\omega x > k/\mu\,(b)$, the substitution yields:

$$q = Q(x, b) \equiv \psi^{-1} \left(\omega x - \frac{k}{\mu\,(b)} \right). \tag{2.7}$$

For any submarket with $\omega x \leq k/\mu\,(b)$, we can set $Q(x, b) = 0$. The function $Q(x, b)$ gives the quantity of goods that a seller is willing to sell for the balance x with a matching probability $\mu\,(b)$. Denote $u(x, b) = U\,(Q(x, b))$.

In submarket (x, b), if the buyer gets a match, he obtains $Q(x, b)$ units of goods which yield utility $u(x, b)$, and his money balance falls to $(m - x)$ after the purchase. In such a trade, the buyer faces the money constraint $x \leq m$. If the buyer does not get a match, he carries the balance, m, to the next period, which yields a discounted value $\beta V(m)$. The buyer's value function, $W_b(m)$, obeys the following Bellman equation:

$$W_b(m) = \max_{b\in[0,1],\ x\in[0,m]} \{\beta V(m) + b\left[u(x,b) + \beta V\left(m-x\right) - \beta V(m)\right]\}. \tag{2.8}$$

Denote a buyer's optimal choices of (x, b) as $(x^*(m), b^*(m))$. The quantity of goods purchased by the buyer is $q^*(m)$ and the buyer's residual money balance after the trade is $\phi(m)$, where

$$q^*(m) \equiv Q\left(x^*(m), b^*(m)\right), \quad \phi(m) \equiv m - x^*(m). \tag{2.9}$$

We will analyze these optimal choices in more detail later in section 3.

2.5. The choice between being a buyer and a seller

The decision on whether to be a buyer or a seller is given by

$$\tilde{V}(m) = \max\{W_s(m), W_b(m)\}. \tag{2.10}$$

We cannot take $\tilde{V}$ as the ex ante value function V, because $\tilde{V}$ is not concave for some m. It is easy to see that a buyer's matching probability is zero if $m < k/\omega$. For such low money balances, an individual's value as a buyer cannot exceed that as a seller. On the other hand, at high money balances, an individual's value as a buyer exceeds that as a seller. Hence, $\tilde{V}$ is strictly convex for low money balances. Moreover, we will see later that $W_b(m)$ may not be concave at some m, in which case $\tilde{V}(m)$ may not be concave even if $\tilde{V}(m) = W_b(m)$.

For concavity of V, we allow individuals to participate in lotteries before choosing whether to be a buyer or a seller. It is useful to clarify in advance that no lottery will be used on the equilibrium path (see section 4.1); however, it is necessary to specify the lotteries in order to define V for arbitrary levels of money holdings. Consider a two-point lottery, (L_1, L_2, π), where L_1 and L_2 are the realizations of the lottery and π is the probability that L_2 occurs. For an individual with money m, the optimal lottery and the value function solve:

$$V(m) = \max_{(L_1,L_2,\pi)} \left[(1-\pi)\,\tilde{V}\left(L_1\right) + \pi\,\tilde{V}\left(L_2\right)\right] \tag{2.11}$$

subject to:

$$\begin{aligned}&(1-\pi)\,L_1 + \pi\,L_2 = m,\\ &\pi \in [0,1] \text{ and } L_i \geq 0 \text{ for } i = 1, 2.\end{aligned}$$

Denote the solutions for L_1, L_2 and π as $L_1(m)$, $L_2(m)$ and $\pi(m)$, respectively.

Of particular interest is the lottery for low money holdings. This lottery specifies $L_1 = 0$ and $L_2 = m_0$, and is depicted in Figure 1. The tangent line connecting points A and B is the function

$V(m)$ for $m \in [0, m_0]$. Since $L_1 = 0$ in this lottery, the winning probability for an individual who puts m into the lottery is m/m_0. Thus, the winning size m_0 is determined as

$$m_0 = \arg\max_{L \geq m} \left[\frac{m}{L} \tilde{V}(L) + \left(1 - \frac{m}{L}\right) \tilde{V}(0) \right]. \tag{2.12}$$

It is clear that m_0 is independent of the individual's money holdings m, provided $m \leq m_0$. Moreover, we will show that $\tilde{V}$ is differentiable at m_0, with $\tilde{V}'(m_0) = W_b'(m_0) = V'(m_0)$ (see part (iii) of Theorem 3.3). Thus, m_0 is given by the first-order condition of the above problem.

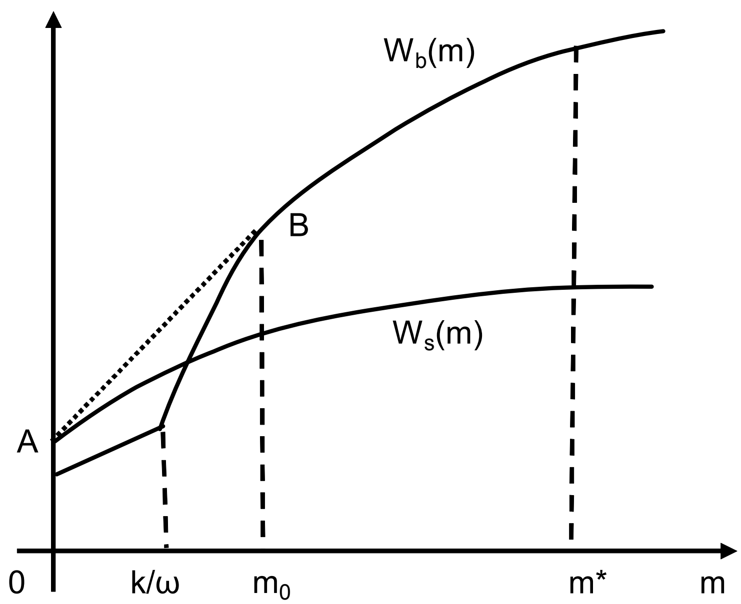

Figure 1. Lottery for low money holdings

2.6. Distributions of money holdings and trading posts

Consider a steady state. Let $G(m)$ be the measure of individuals holding a money balance less than or equal to m immediately after the outcomes of the lotteries are realized and before individuals go to the market. Let $G_a(m)$ denote the distribution of money holdings after the trade is completed in the period. Denote supp(G) as the support of G, and $dG(m)$ as the measure of individuals holding the particular level of money balance, m.

Let us calculate the change in the distribution in an arbitrary period. For any $m \in [0, m^*]$, the change in the measure of individuals in this group before and after trading is:

$$G_a(m) - G(m) = -dG(0) + \int_{m < m' \leq \phi^{-1}(m)} b^*(m')\, dG(m'). \tag{2.13}$$

The term $dG(0)$ is the measure of producers before trade. Because all producers acquire the balance m^*, they move above m after one period. The last term in (2.13) is the flow of buyers whose balances are reduced by trade from levels above m to levels less than or equal to m.

In a steady state, $G(m)$ is also the measure of individuals in the next period whose holdings are less than or equal to m after the lotteries are realized. Then,

$$G(m) - G_a(m) = \int_{m<m'\leq L_1^{-1}(m)} \left[1-\pi\left(m'\right)\right] dG_a\left(m'\right) - \int_{L_2^{-1}(m)<m'\leq m} \pi\left(m'\right) dG_a\left(m'\right). \quad (2.14)$$

The first term on the RHS is the group of individuals who hold $m' > m$ at the end of this period and their lotteries in the next period have the realizations $L_1(m') \leq m$. The second term is the group of individuals who hold $m' \leq m$ at the end of this period and their lotteries in the next period have the realizations $L_2(m') > m$. The two equations above determine G and G_a.

We can also calculate the distribution of trading posts across the submarkets. Because the buyers who hold a money balance m visit only the submarket $(x^*(m), q^*(m))$, the set of submarkets active in the equilibrium is:

$$\left\{(x^*(m), q^*(m)) : m \in \operatorname{supp}(G),\ m \geq m_0\right\}.$$

In submarket $(x^*(m), q^*(m))$, the measure of buyers is $dG(m)$. As an accounting identity, the number of matched buyers in every submarket must be equal to the number of matched trading posts. That is, $b\left[dG(m)\right] = s\left[dN\left(x^*(m), q^*(m)\right)\right]$. From this relationship we can compute the measure of trading posts in submarket $(x^*(m), q^*(m))$ as

$$dN\left(x^*(m), q^*(m)\right) = \frac{b^*(m)}{s\left(x^*(m), q^*(m)\right)} dG(m).$$

2.7. Definition of a monetary equilibrium and block recursivity

A *stationary monetary equilibrium* (i.e., a monetary steady state) consists of a seller's choice: l; a buyer's choice of the submarket to enter: (x, q); lotteries: $(L_1, L_2, \pi)(m)$; value functions: $(W_s, W_b, V)(m)$; the matching probability function for a post in each submarket: $s(x, q)$; future marginal value of money: ω; and money distribution: G. These components satisfy $\omega > 0$ and the following requirements: (i) Given ω and the function $s(x, q)$, a seller's choice solves (2.4) and induces the value function $W_s(m)$; (ii) Given ω and the function $b = \mu^{-1}(s(x, q))$, a buyer's optimal choices solve (2.8) and induce the value function $W_b(m)$; (iii) Given ω, each individual's choice of the lottery at the beginning of each period solves (2.11) and induces the value function $V(m)$; (iv) Free entry: the expected profit of a trading post in each active submarket is zero, and so the function $s(x, q)$ satisfies (2.3); (v) Aggregate consistency: ω is such that the average (normalized) amount of money holdings is indeed 1, i.e., $\int m d\,G(m) = 1$; (vi) The distribution of money holdings, G, satisfies (2.13) and (2.14), and it is stationary.

The equilibrium is block recursive in the following sense: Given ω, individuals' decisions and matching probabilities are independent of money distribution. The only possibility for money distribution to affect matching probabilities and individuals' decisions is to affect the one-dimensional variable, ω. In particular, for any given ω, the objects in (i) - (iv) above are determined independently of money distribution, and so the distribution can affect these objects only by affecting ω through the aggregate consistency condition (v).

Block recursivity is important for tractability because it drastically reduces the state space of individuals' decisions. Let us review how it arises in sections 2.2 – 2.5. Start with a firm's decision on how many trading posts to maintain in each submarket. Given ω, competitive entry of firms determines the matching probability function $s(x, q)$, independently of the distribution. In turn, the function $s(x, q)$, together with $b = \mu^{-1}(s)$, provides all the relevant information for the tradeoff between the terms of trade and the matching probabilities across the submarkets. Given this information, an individual chooses to search in the submarket that maximizes the expected gain from trading. Thus, given the functions $s(x, q)$ and $\mu^{-1}(s)$, an individual's optimal decision is independent of how many individuals are distributed in other submarkets.

As explained by Shi (2009), directed search and free-entry of trading posts are critical for block recursivity. When search is directed, buyers with different money balances optimally sort themselves into different submarkets. A buyer with a particular money balance only cares about the tradeoff between the matching probability and the quantities of trade in the submarket in which he will enter, and not about how many buyers are distributed in other submarkets. Similarly, a firm chooses how many trading posts to maintain in a submarket, knowing that the submarket will be only visited by the buyers with a particular money balance. With free entry of trading posts, the tightness in the submarket and, hence, the matching probability will be exactly equal to the one that delivers the optimal tradeoff for the particualr group of buyers.

3. Analyzing a Buyer's Optimal Decisions and the Value Functions

We devote this section to the analysis of a buyer's optimization problem, (2.8), and the value functions. One reason for doing so is that a buyer's optimal decisions are a main cause of a non-degenerate money distribution that arises in our model. Another reason is that the analysis is a challenge that requires significant deviations from standard approaches.

The difficulty arises because a buyer's objective function in (2.8) may not be concave jointly in (x, b, m), even if V is concave. One reason is that the objective function is the product of two functions; even if the two functions are concave, the product may not be concave. Another reason is that the objective function involves the difference between $V(m-x)$ and $V(m)$, both of which are concave in m. The lack of joint concavity of the objective function implies that the value function, $W_b(m)$, is not necessarily concave. Because the ex ante value function V is defined using $\tilde{V}$ which, in turn, is the maximum of the seller's and buyer's value function, the lack of concavity of W_b renders inapplicable most of the well-known methods for establishing differentiability of V (e.g., Benveniste and Scheinkman, 1979). Without differentiability of V, we cannot presume that usual first-order conditions and envelope conditions apply.[12]

To resolve the difficulty, we employ lattice-theoretic techniques (see Topkis, 1998) to prove that a buyer's optimal choices are monotone in the money balance, as stated in Lemma 3.1. Such monotonicity enables us to establish limited forms of differentiability of W_b and V, as stated in Lemma 3.2, which are useful for establishing first-order conditions and envelope conditions. Finally, we will prove that V is differentiable in Theorem 3.3.[13]

To use lattice-theoretic techniques, let us use Assumption 1 on the matching function, $\mu(b)$, and the assumptions on the cost function, $\psi(q)$, to derive the following properties:

$$Q_1(x,b) > 0,\ Q_2(x,b) < 0,\ Q(x,b) \text{ is concave, and } Q_{12} \geq 0. \tag{3.1}$$

It is easy to explain the monotonicity of Q. A higher money payment by a buyer induces a seller to sell a larger quantity of goods. On the other hand, for any given expenditure, a buyer must accept a relatively low quantity of goods in order to have a relatively high matching probability. This is because the cost of production must be relatively low in order to induce firms to set up a large number of trading posts needed to deliver the high matching probability for a buyer.

Concavity of Q in x means intuitively that the marginal benefit to a buyer from a higher money balance is diminishing. Similarly, concavity of Q in b means that the marginal cost to a buyer of having a high matching probability is increasing. Note that part (iii) of Assumption 1 is used to ensure that Q is concave in b. Moreover, Q is concave in (x,b) jointly because $\psi(q) = \omega x - [\mu(b)]^{-1} k$, which is separable in x and b, and because ψ^{-1} is a concave function. These features of ψ also explain why $Q_{12} \geq 0$, with strict inequality if ψ is strictly convex.

The property $Q_{12} \geq 0$ implies that Q is supermodular in (x,b), and strictly so if ψ is strictly convex. Since this supermodularity will play an important role in our analysis, let us explain

[12] The well-known result that a concave function is differentiable almost everywhere does not help our analysis much. Because the support of the money distribution has only a finite number of points, as we will see later, it might be possible in principle that the ex ante value function is non-differentiable at those points.

[13] Gonzalez and Shi (2008) have employed the lattice-theoretic approach to analyze optimal learning from directed search. Although there are similarities between the analyses there and here, there are also many differences because the optimization problems are different in the two models.

it further. Consider two pairs, (x_1, b_1) and (x_2, b_2), with $x_2 > x_1$ and $b_2 > b_1$. In the current context, supermodularity of $Q(x,b)$ requires that $Q(x_2, b_2) - Q(x_1, b_2) \geq Q(x_2, b_1) - Q(x_1, b_1)$. That is, the additional benefit to a buyer from having a higher money balance increases with b. Our model meets this requirement for the following reason. In submarkets where a buyer has a relatively high matching probability, the amount of goods that a seller is willing to sell for any given amount of money balance is relatively low. At such a low quantity of goods, a seller's marginal cost of production is low, and so a seller is willing to increase the quantity of goods by more for any given increase in the amount of money spent by a buyer.

The composite function, $u(x,b) = U(Q(x,b))$, has similar properties:

$$u_1(x,b) > 0,\ u_2(x,b) < 0,\ u(x,b) \text{ is strictly concave, and } u_{12} > 0. \tag{3.2}$$

Note that regardless of whether ψ is strictly or weakly convex, concavity and supermodularity of $u(x,b)$ are strict because the utility function $U(q)$ is strictly concave.

Despite strict supermodularity of $u(x,b)$, the objective function in (2.8) may not be supermodular in (x,b). To get around this problem, we decompose the maximization problem into two steps. In the first step, we fix b and characterize the optimal choice of x. For any given (b,m), the optimal choice of x maximizes the buyer's surplus from trade defined as follows:

$$t(x,b,m) = u(x,b) + \beta V(m-x) - \beta V(m). \tag{3.3}$$

Denote this optimal choice and the maximized function as

$$\tilde{x}(b,m) = \arg\max_{x\in[0,m]} t(x,b,m), \quad t^*(b,m) = t(\tilde{x}(b,m), b, m). \tag{3.4}$$

With the properties of u in (3.2), we prove that $t(x,b,m)$ is supermodular in (x,b,m). Topkis' (1998) theorems then imply that the optimal choice, $\tilde{x}(b,m)$, and the maximized function, $t^*(b,m)$, are both increasing functions of (b,m) (see Appendix B). In the second step, we characterize the optimal choice of b, which maximizes the function $b\,t^*(b,m)$. We prove that this function is supermodular in (b,m). Thus, the optimal choice, $b^*(m)$, and the maximized function, $b^*(m)t^*(b^*(m),m)$, are increasing functions of m.

By changing the choice variables from (x,b) to (x,q) and $(m-x,b)$, in turn, we use a similar procedure to prove that the quantity of goods purchased, $q^*(m)$, and the residual money balance, $\phi(m)$, are increasing functions of the money balance that the buyer brings into the trade. Moreover, $W_b(m)$ is a strictly increasing function whenever $b^*(m) > 0$. The following lemma states the properties of a buyer's optimal choices and value function (see Appendix B for a proof):

Lemma 3.1. *Assume that $V(.)$ is continuous, bounded, increasing and concave. Then, $W_b(.)$ is continuous, bounded, and increasing. If $b^* = 0$, the choice of x is irrelevant. If $b^* > 0$, the following results hold: (i) $x^*(m)$ and $b^*(m)$ exist and are unique for each m; (ii) $x^*(m)$, $b^*(m)$, $q^*(m)$ and $\phi(m)$ are increasing and continuous functions; (iii) $W_b(m)$ is strictly increasing.*

Optimal choices of (x, b) are unique for each level of money balance because a buyer's objective function is strictly quasi-concave in the choices. Monotonicity of optimal choices is also intuitive. Relative to a buyer who holds a lower quantity of money, a buyer with more money will choose to enter a submarket where he will have a higher matching probability; moreover, once he is matched in the submarket, he will spend a larger amount of money, buy a larger quantity of goods, and leave the trade with a higher money balance. As a result, a buyer with more money will obtain a higher present value.

Uniqueness and monotonicity deserve further remarks. First, a buyer is *not indifferent* between different combinations of (x, q, b) across the submarkets. Instead, a unique combination is optimal for a buyer, given the buyer's money balance. Second, buyers optimally sort themselves into different submarkets according to money holdings. Buyers with a relatively higher money balance choose to enter a submarket where x, q and b are all relatively high. Such sorting is important for the equilibrium to be block recursive, as explained at the end of last section.

The above features enable us to characterize optimal choices in more detail. For any given (x, m), the function $t(x, b, m)$ defined in (3.3) is differentiable with respect to b. Since optimal choices are unique for each m, then for any given (x, m), the optimal choice of b satisfies the following first-order condition:

$$u(x,b) + b\,u_2(x,b) \leq \beta\left[V(m) - V(m-x)\right] \quad \text{and} \quad b \geq 0, \tag{3.5}$$

where the two inequalities hold with complementary slackness. Note that this condition does not require differentiability of V. In contrast, the first-order condition for x^* and the envelope condition do require at least a limited version of differentiability of V. We establish such differentiability in the following lemma (see Appendix C for a proof):

Lemma 3.2. *Assume that $V(.)$ is continuous, bounded, increasing and concave. In addition, assume that $b^*(m) > 0$. Then, the following results hold. (i) $W_b(.)$ is differentiable at m if and only if $V(.)$ is so. (ii) For any m such that $\phi(m) > 0$, the derivative $V'(\phi(m))$ exists and satisfies the first-order condition:*[14]

$$V'(\phi(m)) = \frac{1}{\beta} u_1\left(x^*(m), b^*(m)\right). \tag{3.6}$$

[14]If $\phi(m) = 0$, then (3.6) is replaced with $V'(0) \leq \frac{1}{\beta} u_1\left(m, b^*(m)\right)$.

(iii) For any $m > 0$ such that $W_b(m) = V(m)$, the derivatives $W_b'(m)$ and $V'(m)$ exist and are given by the envelope condition:

$$W_b'(m) = V'(m) = \frac{b^*(m)}{1 - \beta\left[1 - b^*(m)\right]} u_1\left(x^*(m), b^*(m)\right). \tag{3.7}$$

(iv) For any m such that $W_b(m) = V(m)$ and $\phi(m) > 0$, the functions $b^*(.)$ and $\phi(.)$ are strictly increasing at m; the function $V(.)$ is strictly concave at $\phi(m)$; and $V(\phi(m)) = W_b(\phi(m))$.

Part (i) of the above lemma is not surprising, given that optimal choices are unique and monotone. Part (ii) states that for any given m such that the money constraint in a trade does not bind, a buyer's optimal expenditure, $x^*(m)$, is characterized by the first-order condition. In this case, even if $V(m)$ is not differentiable at m, V is differentiable at $\phi(m)$, i.e., at the buyer's residual money balance induced by the optimal choice. Put in another way, the buyer will choose the spending optimally so that the residual money balance will not be at a level where V is not differentiable. This result can be explained as follows. Suppose that a spending level, x, results in $V(m - x)$ being non-differentiable, i.e., $V'(m - x^+) > V'(m - x^-)$. By reducing x marginally, the opportunity cost of spending decreases by a discrete amount. Since the marginal benefit, given by $u_1(x, b)$, falls continuously with the decrease in x, the net marginal gain must be positive. Thus, spending marginally less than x is strictly better than spending x.

Part (iii) states that if the ex ante value function $V(.)$ is equal to the buyer's value function $W_b(.)$ at m, then both value functions are differentiable at m, and the common derivative is given by the envelope condition. To obtain this result, we use the fact that a concave function has both left-hand and right-hand derivatives (see Royden, 1988, pp113-114). When $W_b(m) = V(m)$ at m, we are able to use the Bellman equation, (2.8), to prove that the left-hand derivative of $W_b(m)$ is equal to the left-hand derivative of $V(m)$, that the right-hand derivative of $W_b(m)$ is equal to the right-hand derivative of $V(m)$, and that the left-hand and right-hand derivatives are equal to the right-hand side of (3.7).

Part (iv) provides mild conditions under which the policy functions $b^*(.)$ and $\phi(.)$ are strictly increasing at m. Under the same conditions, the ex ante value function V is strictly concave at the residual money balance, $\phi(m)$, and is equal to a buyer's value function at such a balance. Also, part (iii) of the lemma holds at this residual balance.

Together, the results in Lemma 3.2 significantly simplify the characterization of optimal choices. In section 4.1, we will verify that the hypotheses in the above lemma are satisfied. In this case, a buyer's optimal choices in the equilibrium satisfy the first-order conditions, (3.5) and (3.6), and the value functions satisfy the envelope condition, (3.7).

Now we turn to the ex ante value function V, defined by (2.11). The latter equation defines a mapping for the ex ante value function, V. To see this, note that (2.4) defines a seller's value function W_s as a mapping of V, (2.8) defines a buyer's value function W_b as a mapping of V, and (2.10) defines $\tilde{V}$ as a mapping of V. As a result, the right-hand side of (2.11) is a mapping of V. Let us denote this mapping as $\mathcal{F}$ and express (2.11) as $V(m) = \mathcal{F}V(m)$. That is, V is a fixed point of $\mathcal{F}$. The following theorem states existence, uniqueness, and other properties of the fixed point; in particular, the theorem confirms that the fixed point, V, has the properties that we have assumed so far (see Appendix D for a proof).

Theorem 3.3. *$\mathcal{F}$ has a unique fixed point, V, which is continuous, bounded, increasing and concave. Moreover, the following results hold: (i) $V'(m)$ exists for all $m \geq 0$ and $W_b'(m)$ exists for all such m that $b^*(m) > 0$; (ii) $V(m) > W_s(m) > 0$ for all $m > 0$, $V(0) = W_s(0) > 0$, and $W_s(m) \geq W_b(m)$ for all $m \in [0, k/\omega]$; (iii) There exists $m_0 > 0$ such that $V(m_0) = W_b(m_0)$ and $V'(m_0) = W_b'(m_0) > 0$; (iv) $b^*(m_0) > 0$ and $\phi(m_0) = 0$; (v) $V'(m_0) > V'(m^*)$ and $m_0 < m^*$.*

Result (i) states that the ex ante value function is differentiable for all money holdings and that a buyer's value function is differentiable wherever the buyer has a positive trading probability. Recall that a seller's value function is differentiable whenever $l(m) > 0$. Thus, standard first-order conditions and envelope conditions hold for all individuals' optimization problems. Result (ii) gives the relative level of various value functions. In particular, an individual will choose to be a buyer only when his money balance is relatively high and that, when his money balance is relatively low, he will participate in the lottery whose winning prize m_0 is defined by (2.12). Result (iii) states the properties of the value functions at m_0. Result (iv) states that after winning the lottery with prize m_0, an individual strictly prefers buying goods to not buying, and he spends all of his money in such a trade. Result (v) compares the relative position of m_0 with m^*.

4. Monetary Equilibrium

4.1. Equilibrium pattern of spending

Let us define a sequence of functions $\{\phi^i\}_{i\geq 0}$ recursively by $\phi^0(m) = m$ and $\phi^{i+1}(m) = \phi^i(m)$ for $i = 0, 1, 2, ...$ Denote T as the positive integer such that $\phi^{T-1}(m^*) \geq m_0 > \phi^T(m^*)$. That is, T is the number of purchases that a buyer with m^* can make according to the policy function ϕ before the money balance falls below m_0. We prove the following lemma in Appendix E:

Lemma 4.1. *The following results hold in a monetary equilibrium: (i) $b^*(m^*) > 0$ and $\phi^T(m^*) = 0$; (ii) $V(m^*) = W_b(m^*)$; (iii) For all $i = 1, 2, ..., T-1$, V is strictly concave at $\phi^i(m^*)$ and $V(\phi^i(m^*)) = W_b(\phi^i(m^*))$.*

Part (i) of the above lemma states that a buyer with a balance m^* strictly prefers trading to not trading and that a buyer will eventually run out of money after repeated purchases. Part (ii) says that the ex ante value function is equal to the buyer's value function at m^*. Part (iii) describes similar properties of the ex ante value function at subsequent levels of equilibrium money holdings $\phi^i(m^*)$.

Lemma 4.1 implies the following dynamics of an individual's money balance. When the individual is a seller, he acquires the balance, m^*. A buyer with a balance m^* spends $x^*(m^*)$ units of money and retains $\phi(m^*)$ units. If $\phi(m^*) \geq m_0$, the individual will continue to be a buyer in the next period, where his spending will be $x^*(\phi(m^*))$ and his residual money balance will be $\phi^2(m^*)$. This process continues until the Tth time in which the individual has purchased goods. The individual will enter the Tth trade with a money balance $\phi^{T-1}(m^*)$, and will exit the trade with a balance $\phi^T(m^*) = 0$. After the trade, the individual will become a seller. In this round of purchases, a buyer spends less and less as each purchase reduces his money balance.

Note that no lottery is used on the equilibrium path. Starting from the money balance m^*, an individual's ex ante value function is equal to the value function as a buyer (see part (ii) of Lemma 4.1), which implies that the individual with m^* does not participate in lotteries. After purchasing goods, the individual will have a residual money balance, $\phi(m^*)$. If $\phi(m^*) = 0$, then the individual will have no money to buy the lottery and, hence, will directly become a producer next period. If $\phi(m^*) > 0$, the individual's ex ante value function will be strictly concave at $\phi(m^*)$ and equal to $W_b(\phi(m^*))$ (see part (iii) of Lemma 4.1). Strict concavity of V at $\phi(m^*)$ implies that $\phi(m^*) \geq m_0$. Again, the buyer will not use a lottery in the next period. This process continues until the individual runs out of money.

Another implication of Lemma 4.1 is that on the equilibrium path, a buyer's optimal choices are characterized by the first-order conditions, (3.5) and (3.6), and that the value functions satisfy the envelope condition, (3.7). We have explained earlier that the optimal choice of b always satisfies the first-order condition (3.5). For the optimal choice of x and the value functions, start with a buyer whose money balance is m^*. Because $V(m^*) = W_b(m^*)$ by part (ii) of Lemma 4.1, V is differentiable at m^*, and the derivative $V'(m^*)$ satisfies the envelope condition (3.7) (and is also equal to ω/β). Because $\phi(m^*) > 0$, the buyer's optimal choice of spending, $x^*(m^*)$, satisfies the first-order condition (3.6) with $m = m^*$. If $\phi(m^*) \geq m_0$, then $V(\phi(m^*)) = W_b(\phi(m^*))$ (see part (iii) of Lemma 4.1). In this case, the derivative $V'(\phi(m^*))$ exists and satisfies the envelope condition (3.7) with $m = \phi(m^*)$. This process continues until the Tth time in which the buyer has a match. When entering the Tth match, the buyer has a balance $\phi^{T-1}(m^*) \geq m_0$. In this match, the derivative $V'(\phi^{T-1}(m^*))$ exists and satisfies the envelope condition (3.7) with $m = \phi^{T-1}(m^*)$.

Moreover, the buyer's optimal choice of money spending is given by the binding money constraint, $x^*(\phi^{T-1}(m^*)) = \phi^{T-1}(m^*)$.

4.2. Equilibrium distribution of money holdings

The equilibrium path of money holdings described in the previous subsection significantly simplifies the characterization of the distribution of money holdings. The equilibrium path implies that the support of the distribution of money holdings, G, is

$$\text{supp}\,(G) = \{\phi^i(m^*)\}_{i=0}^{T-1} \cup \{0\}. \tag{4.1}$$

Since the support of the distribution contains only a finite number of values, let us denote $g(m) = d\,G(m)$ as the measure of individuals who hold the balance m immediately before entering the goods market. Similarly, let $g_a(m) = d\,G_a(m)$ be the measure of individuals who hold the balance m after trading is completed in the period. The support of G_a is the same as supp(G).

Because no lotteries are used on the equilibrium path, (2.13) and (2.14) yield the following equations for the stationary distribution:

$$0 = g\,(0) - b^*(m^*)g(m^*); \tag{4.2}$$

$$0 = b^*(\phi^{i-1}(m^*))g(\phi^{i-1}(m^*)) - b^*(\phi^i(m^*))g(\phi^i(m^*)) \text{ for } 1 \leq i \leq T-1; \tag{4.3}$$

$$g\,(0) = b^*(\phi^{T-1}(m^*))g(\phi^{T-1}(m^*)). \tag{4.4}$$

The first two equations calculate the net change between the current and the next period in the measure of individuals who hold $\phi^i(m^*)$, where $i = 0, 1, ..., T-1$. The outflow is the measure of individuals with $\phi^i(m^*)$ who successfully trade in the current period. For $i = 0$, the inflow is the measure of individuals who are producers in the current period; for $i \geq 1$, the inflow is the measure of individuals with $\phi^{i-1}(m^*)$ who successfully trade in the current period. Finally, (4.4) calculates the measure of producers in the next period, as the measure of individuals with $\phi^{T-1}(m^*)$ who successfully trade in the current period.

Equations (4.2) – (4.4) solve for the steady state distribution:

$$\left.\begin{array}{l} g(\phi^i(m^*)) = \frac{g(0)}{b^*(\phi^i(m^*))} \text{ for } 0 \leq i \leq T-1; \\ \\ g\,(0) = \left[1 + \sum_{i=0}^{T-1} \frac{1}{b^*(\phi^i(m^*))}\right]^{-1}. \end{array}\right\} \tag{4.5}$$

The results in (4.5) are simple to explain. Because a buyer moves only to the next lower level of money holdings after each purchase, the outflow must be the same for all equilibrium levels $\phi^i(m^*)$ in order to maintain a steady state. That is, $b^*(\phi^i(m^*))g(\phi^i(m^*))$ must be the same in a steady state for all $i = 0, 1, ..., T-1$. Moreover, this outflow must be equal to the inflow from producers, as in the first line of (4.5). The last line comes from substituting $g(\phi^i(m^*))$ into the requirement that the total measure of individuals in the economy must be equal to one.

4.3. Existence of a monetary steady state

In previous subsections, we have characterized the equilibrium patterns of spending for any given money distribution and any given marginal value of money of the sellers, ω. We have also characterized the equilibrium distribution of money holdings for any given ω. For existence of a stationary monetary equilibrium, it suffices to find $\omega > 0$ that satisfies requirement (v) in the definition of an equilibrium. That is, a seller's marginal value of money must be such that all money is held by the individuals. Since the support of money distribution is given by (4.1), we can express requirement (v) in the equilibrium definition explicitly as

$$\sum_{i=0}^{T-1} \phi^i(m^*) g(\phi^i(m^*)) = 1. \tag{4.6}$$

Because ω appears in $Q(x, b)$ defined by (2.7) and, hence, in the function $u(x, b)$, optimal choices $(m^*, \phi^i(m^*))$ depend on ω. So does the distribution of money holdings. Thus, the above equation is indeed an equation of ω. In general, such dependence on ω can be complicated.

Fortunately, the equilibrium possesses a property that enables us to simplify the task of determining ω. To describe this property, let us recognize the appearance of ω in a buyer's decision problem through $Q(x, b)$ by modifying the notation for a buyer's value function as $W_b(m, \omega)$. Similarly, modify the ex ante value function as $V(m, \omega)$, a seller's value function as $W_s(m, \omega)$, and define $\tilde{V}(m, \omega) = \max\{W_s(m, \omega), W_b(m, \omega)\}$. Multiply all nominal variables by ω to obtain the "real" values of these variables. Denote such real values with a caret. For example, the real value of money is $\hat{m} = \omega m$, the real value of spending is $\hat{x} = \omega x$, the real value of a buyer's residual balance after a purchase is $\hat{\phi} = \omega \phi$, and the real value of the realization of a lottery is $\hat{L} = \omega L$. The following lemma states the properties of such real values:

Lemma 4.2. *In a steady state, the value functions and real values of optimal choices depend on ω only through $\hat{m} = \omega m$ and not through ω separately. That is, (i) there exist functions w_s, w_b, $\tilde{v}$, and v such that $W_s(m, \omega) = w_s(\hat{m})$, $W_b(m, \omega) = w_b(\hat{m})$, $\tilde{V}(m, \omega) = \tilde{v}(\hat{m})$, and $V(m, \omega) = v(\hat{m})$; (ii) Real values of optimal choices in a steady state, $(\hat{m}^*, \hat{m}_0, \hat{x}, \hat{\phi}, \hat{L}_1, \hat{L}_2)$, are independent of ω.*

Proof. We have proven that the function $V(m, \omega)$ is unique. If there exists v such that $V(m, \omega) = v(\hat{m})$ for all m, then $v(\hat{m})$ is the unique ex ante value function. In the process of verifying this result, we also show that other value functions depend on ω only through $\hat{m}$ and that optimal choices of $(\hat{m}^*, \hat{m}_0, \hat{x}, \hat{\phi}, \hat{L}_1, \hat{L}_2)$ are independent of ω.

Suppose that $V(m,\omega) = v(\hat{m})$. First, we transform a seller's problem in (2.4). Note that V on the right-hand side of (2.4) is the ex ante value function in the next period. The money balance y that an agent brings to the next period will have a real value $\omega_{+1}y = \hat{y}\omega_{+1}/\omega$, where ω_{+1} is a seller's marginal value of money in the next period. Thus, we can rewrite (2.4) as

$$w_s(\hat{m}) = \max_{\hat{y}\geq\hat{m}} \left[\beta v(\hat{y}\omega_{+1}/\omega) - h(\hat{y}-\hat{m})\right].$$

Clearly, the optimal choice of $\hat{y}$ depends on ω_{+1}/ω, but not on ω or ω_{+1} alone. In particular, a seller with $\hat{m}=0$ will acquire a balance $\hat{m}^*$ that satisfies $\beta v'(\hat{m}^*\omega_{+1}/\omega)\omega_{+1}/\omega = h'(\hat{m}^*)$. Clearly, $\hat{m}^*$ is independent of ω in a steady state.

Second, we transform a buyer's problem. Since ω appears in $s(x,q)$ defined by (2.3) only through the term ωx, we can slightly abuse the notation to write a seller's matching probability as $s(\hat{x},q)$. Similarly, write the function Q defined by (2.7) as $Q(\hat{x},b)$ and the function u as $u(\hat{x},b)$. With the conjecture $V(m,\omega) = v(\hat{m})$, a buyer's problem can be transformed as

$$w_b(\hat{m}) = \max_{b\in[0,1],\ \hat{x}\in[0,\hat{m}]} \left\{\beta v(\hat{m}\omega_{+1}/\omega) + b\left[u(\hat{x},b) + \beta v((\hat{m}-\hat{x})\omega_{+1}/\omega) - \beta v(\hat{m}\omega_{+1}/\omega)\right]\right\}.$$

Again, the ratio ω_{+1}/ω appears on the right-hand side of the equation because the function v is the value function in the next period, which is defined as a function of future real money balance $\hat{m}_{+1} = \hat{m}\omega_{+1}/\omega$. Optimal choices $(\hat{x}^*, b^*)$ depend on ω_{+1}/ω, but not on ω and ω_{+1} alone, and they are independent of ω in the steady state. Similarly, the residual money balance, $\hat{\phi} = \hat{m} - \hat{x}^*$, is independent of ω in the steady state.

Third, $\tilde{v}(\hat{m}) = \max\{w_s(\hat{m}), w_b(\hat{m})\}$, and the ex ante value function is

$$v(\hat{m}) = \max_{(\hat{L}_1,\hat{L}_2,\pi)} \left[(1-\pi)\,\tilde{v}(\hat{L}_1) + \pi\,\tilde{v}(\hat{L}_2)\right]$$

subject to: $(1-\pi)\,\hat{L}_1 + \pi\hat{L}_2 = \hat{m}$, $\pi\in[0,1]$ and $\hat{L}_i \geq 0$ for $i=1,2$. Because the functions $w_s(\hat{m})$ and $w_b(\hat{m})$ depend on the ratio ω_{+1}/ω, so do $\tilde{v}(\hat{m})$ and $v(\hat{m})$. However, in a steady state, these functions are independent of ω or ω_{+1}. So are the steady state values of optimal choices, $(\hat{L}_1^*, \hat{L}_2^*)$. In particular, the lottery for low money holdings yields $\hat{L}_2^* = \hat{m}_0$. As before, the above process defines a mapping of v, which is a continuous and contraction mapping. Thus, a unique function $v(\hat{m})$ exists which is independent of ω in the steady state. **QED**

Let us remark on Lemma 4.2. First, only in a steady state are the value functions and real values of optimal choices independent of ω and ω_{+1}. Outside a steady state, these objects depend on the ratio ω_{+1}/ω. Second, the independence of these steady state objects on ω implies that money is neutral at an individual's level in the long run. Moreover, we can modify (4.5) to characterize the steady state distribution of the real money balance and show that this distribution is independent of ω. Thus, money is also neutral at the aggregate level in a steady state.

Now, it is straightforward to solve ω from the requirement (4.6). Substituting the steady state distribution from (4.5), and using real values of the variables, we rewrite (4.6) as follows:

$$\sum_{i=0}^{T-1} \frac{\hat{\phi}^i(\hat{m}^*) - \omega}{b^*(\hat{\phi}^i(\hat{m}^*))} = \omega. \tag{4.7}$$

By Lemma 4.2, $\hat{m}^*$, $\hat{m}_0$, and $\hat{\phi}^i(\hat{m}^*)$ are all independent of ω in a steady state. Thus, for any given $\hat{m}^*$, the steady state value of ω is uniquely solved as

$$\omega = \left[\sum_{i=0}^{T-1} \frac{\hat{\phi}^i(\hat{m}^*)}{b^*(\hat{\phi}^i(\hat{m}^*))}\right] \Big/ \left[1 + \sum_{i=0}^{T-1} \frac{1}{b^*(\hat{\phi}^i(\hat{m}^*))}\right].$$

This solution satisfies $\omega \in (0, \infty)$, and so a monetary steady state exists. Uniqueness follows from the fact that $\hat{m}^*$ is unique (see part (ii) of Lemma 4.1).

The following theorem summarizes the above results and states other properties of the steady state (see Appendix F for a proof):

Theorem 4.3. *A unique monetary steady state exists. Money is neutral in the steady state. The steady state distribution of money holdings satisfies: $g(m^*) > g(0)$ and $g(\phi^i(m^*)) > g(\phi^{i-1}(m^*))$ for all $i = 1, 2, ..., T-1$. Moreover, $\phi(m^*) > 0$ and, hence, $T \geq 2$ if $\beta_0 \leq \beta < 1$, where β_0 is defined in Appendix F.*

Since lotteries are never used in the equilibrium, the frequency function of money holdings among buyers is a strictly decreasing function. That is, the higher the money balance, the fewer the number of buyers who hold that balance. This result arises because a relatively high money balance allows a buyer to trade with a relatively high probability, which reduces the measure of individuals staying at that level of money holdings in the steady state.[15] Similarly, the measure of sellers is smaller than the measure of buyers at any equilibrium level of money holdings, because a seller always becomes a buyer after one period of production.

A sufficient condition for money distribution among buyers to be non-degenerate in the steady state is that individuals are sufficiently patient in the sense that $\beta \geq \beta_0$. To see why this condition is sufficient, consider a buyer with the balance m^*. The buyer can spend all this balance in one

[15] This decreasing feature of the distribution of money holdings resembles that in Green and Zhou (1998), but the feature is obtained here without the restriction in Green and Zhou that goods and money are indivisible.

trade or in a sequence of trades. If the buyer spends all the money once, he consumes a large amount of goods in the period. Because the marginal utility of consumption is diminishing, there is a gain from spreading the purchases over several periods. The downside from doing so is time discounting. If the buyer is sufficiently patient, the gain outweighs the cost, in which case the buyer will gradually spend the money balance over time.

Recall that the ex ante value function is strictly concave at all levels of equilibrium money holdings. Thus, buyers who hold different amounts of money have different marginal values of money. As a buyer's money balance decreases with each purchase, his marginal value of money increases. In this sense, money distribution has a wealth effect. Moreover, because the frequency function of money holdings is strictly decreasing, more buyers have high marginal values of money than low marginal values.

5. Conclusion

In this paper, we construct a tractable search model of money with a non-degenerate distribution of money holdings. We model search as a directed process in the sense that buyers know the terms of trade before visiting particular sellers, as opposed to undirected search that has dominated the literature. In this model, the distribution of money holdings among individuals is non-degenerate. We show that this distribution affects individuals' decisions not directly, but rather indirectly only through a one-dimensional variable – the seller's marginal value of money. This result drastically reduces the state space of individuals' decisions and makes the model tractable. We analytically characterize a monetary equilibrium, using lattice-theoretic techniques, and prove existence of a monetary steady state. In the equilibrium, buyers follow a stylized spending pattern over time, and money distribution has a persistent wealth effect.

Appendix

A. Proof of Lemma 2.2

From (2.4) it is easy to verify that $W_s(.)$ is a continuous, bounded and increasing function. Because the objective function $\left[\beta V(m+\frac{l}{\omega})-h(l)\right]$ is strictly concave in (l,m) jointly, its maximized value, $W_s(m)$, is concave in m, and the optimal choice, $l(m)$, is unique. With uniqueness, the theorem of the maximum implies that $l(m)$ is a continuous function (see Stokey and Lucas with Prescott, 1989, p62).

The choice $l=1$ can never be optimal under the assumption $h'(1)=\infty$. In contrast, it may be possible that the optimal choice has $l(m)=0$ when m is sufficiently high. In this case, $l(m)=0$ is increasing (i.e., non-decreasing) in m and $y(m)=m$ is strictly increasing in m. We will see below that these monotonicity properties of $l(m)$ and $y(m)$ also hold when $l(m)>0$.

The remainder of this proof focuses on the case where $l(m)>0$. In this case, $y(m)=m+\frac{l(m)}{\omega}>m$. We rewrite (2.4) as

$$W_s(m)=\max_{y\geq m}\left[\beta V\left(y\right)-h(\omega\left(y-m\right))\right]. \tag{A.1}$$

Note that the objective function in (A.1) is strictly concave in (y,m) jointly and that the function $h(\omega(y-m))$ is continuously differentiable in (y,m). These properties and an interior $y(m)$ imply that $W_s(m)$ is differentiable (see Stokey and Lucas with Prescott, 1989, p85) and that the derivative is given as

$$W_s'(m)=\omega h'(\omega(y(m)-m))=\omega h'(l(m)). \tag{A.2}$$

From the equation $W_s'(m)=\omega h'(l(m))$, we can use concavity of $W_s(.)$ and strict convexity of $h(.)$ to deduce that $l(m)$ is decreasing in m.

Return to the original maximization problem, (2.4). Consider any $m\in[0,\bar{m}]$ such that $l(m)>0$. Because $l(m)$ is a continuous function, there exists $\varepsilon_0>0$ such that $l(m\pm\varepsilon)>0$ for all $\varepsilon\in[0,\varepsilon_0]$. For any $\varepsilon\in[0,\varepsilon_0]$, optimality of the choice l implies:

$$\begin{aligned}&W_s(m)=F(l(m),m)\geq F(l(m-\varepsilon),m),\\&W_s(m-\varepsilon)=F(l(m-\varepsilon),m-\varepsilon)\geq F(l(m),m-\varepsilon).\end{aligned}$$

where $F(l,m)$ is the objective function in (2.4). Hence,

$$\frac{F(l(m-\varepsilon),m)-F(l(m-\varepsilon),m-\varepsilon)}{\varepsilon}\leq\frac{W_s(m)-W_s(m-\varepsilon)}{\varepsilon}\leq\frac{F(l(m),m)-F(l(m),m-\varepsilon)}{\varepsilon}.$$

Since the derivative $W_s'(m)$ exists (see (A.2)), taking the limit $\varepsilon\nearrow 0$ on the above inequalities yields $W_s'(m)=\beta V'(y^-(m))$, where $y^-(m)=m^-+\frac{l(m)}{\omega}$. Note that the left-hand derivative

$V'(y^-)$ exists because a concave function has both left-hand and right-hand derivatives (see Royden, 1988, pp113-114). Similarly, we can prove that $W_s'(m) = \beta V'(y^+(m))$, where $y^+(m) = m^+ + \frac{l(m)}{\omega}$. Therefore, V is differentiable at $y(m)$ and the derivative satisfies $\beta V'(y(m)) = W_s'(m)$. This result and (A.2) establish (2.5).

Finally, rewrite the second part of (2.5) as $\beta V'(y(m)) = \omega h'(\omega(y(m) - m))$. Concavity of V and strict convexity of h imply that $y(m)$ is strictly increasing in m when $y(m) > m$. **QED**

B. Proof of Lemma 3.1

The theorem of the maximum implies that $W_b(m)$ is continuous and bounded and that a solution exists (see Stokey and Lucas with Prescott, 1989, p62). It is also straightforward to show that $W_b(.)$ is increasing. For the remainder of the proof, temporarily denote $F(x, b, m) = b\, t(x, b, m)$, where the function t is defined in (3.3). Optimal choices (x^*, b^*) maximize $F(x, b, m)$. If $b^* = 0$, the choice of x is irrelevant for the buyer because a trade does not take place. Since the choice $b = 0$ yields $F(x, b, m) = 0$, then $b^* > 0$ is optimal only if $t(x, b, m) \geq 0$. In the remainder of the proof, we only examine the case where $b > 0$ and $t(x, b, m) > 0$.

With $b > 0$ and $t(x, b, m) > 0$, we can transform the maximization problem as

$$\max_{b,x} F(x, b, m) = \exp\left\{\max_{b,m} [\ln b + \ln t(x, b, m)]\right\}.$$

The function $(\ln b)$ is concave in (x, b). Recall that $u(x, b)$ is strictly concave in (x, b) jointly. Since V is concave, then $V(m - x)$ is concave in x. Thus, $t(x, b, m)$ defined in (3.3) is strictly concave in (x, b) jointly. Since the logarithmic transformation is an increasing and concave transformation, the function $[\ln b + \ln t(x, b, m)]$ is strictly concave in (x, b) jointly. Therefore, optimal choices of (x, b) are unique.

To show that optimal choices have monotonicity as described in the lemma, we decompose the maximization problem into two: In the first step, we fix b and find the optimal choice of x; in the second step, we find the optimal choice of b.

Take the first step. For any given (b, m), the optimal choice of x is denoted $\tilde{x}(b, m)$ as in (3.4). Because $u(x, b)$ is strictly concave in x and V is concave, $\tilde{x}$ exists and is unique. Note that t is separable in b and m, and so t has increasing differences in (b, m) (but not strictly so). Take arbitrary m_1, m_2, x_1, x_2, b_1 and b_2, with $m_2 > m_1$, $x_2 > x_1$, and $b_2 > b_1$. Compute:

$$t(x_2, b, m) - t(x_1, b, m) = [u(x_2, b) - u(x_1, b)] + \beta [V(m - x_2) - V(m - x_1)].$$

Since $u(x, b)$ is strictly supermodular in (x, b), we have:

$$\begin{aligned} & [t(x_2, b_2, m) - t(x_1, b_2, m)] - [t(x_2, b_1, m) - t(x_1, b_1, m)] \\ = \ & [u(x_2, b_2) - u(x_1, b_2)] - [u(x_2, b_1) - u(x_1, b_1)] > 0. \end{aligned}$$

That is, $t(x,b,m)$ has strictly increasing differences in (x,b). Moreover,

$$\begin{aligned} & [t(x_2,b,m_2)-t(x_1,b,m_2)]-[t(x_2,b,m_1)-t(x_1,b,m_1)] \\ = \;& \beta[V(m_1-x_1)-V(m_1-x_2)]-\beta[V(m_2-x_1)-V(m_2-x_2)] \geq 0. \end{aligned}$$

The inequality follows from concavity of V (see Royden, 1988, p113) and the facts that $m_1-x_1 < m_2-x_1$, $m_1-x_2 < m_2-x_2$, and $(m_1-x_1)-(m_1-x_2) = (m_2-x_1)-(m_2-x_2) = x_2-x_1 > 0$. The inequality in the above result is strict if V is strictly concave. Thus, $t(x,b,m)$ has increasing differences in (x,m), and strictly so if V is strictly concave. Because $t(x,b,m)$ has increasing differences in (b,m), (x,b) and (x,m), it is supermodular in (x,b,m). Since the choice set, $[0,m]$, is also increasing in m, then $\tilde{x}(b,m)$ is increasing in (b,m) (see Topkis, 1998, p76), and the maximized value of t is supermodular in (b,m) (see Topkis, 1998, p70).

We can establish stronger properties of $\tilde{x}(b,m)$. If the constraint $x \leq m$ binds, then $\tilde{x}(b,m) = m$, in which case $\tilde{x}(b,m)$ is strictly increasing in m and independent of b. If the constraint $x \leq m$ does not bind, the derivative of $t(x,b,m)$ with respect to b is strictly increasing in x, in which case $\tilde{x}(b,m)$ is strictly increasing in b (see Edlin and Shannon, 1998).

Denote $t^*(b,m) = t(\tilde{x}(b,m),b,m)$ as in (3.4). From the above proof, $t^*(b,m)$ is supermodular in (b,m). Because $t(x,b,m)$ strictly decreases in b for any given (x,m), its maximized value with respect to x, $t^*(b,m)$, is strictly decreasing in b. To examine the dependence of $t^*(b,m)$ on m, take arbitrary m_1 and m_2, with $m_2 \geq m_1$. We have:

$$t(x,b,m_2)-t(x,b,m_1) = \beta[V(m_1)-V(m_1-x)]-\beta[V(m_2)-V(m_2-x)] \geq 0,$$

where the inequality follows from concavity of V. Since the above result holds for all (x,b), then

$$t^*(b,m_1) = t(\tilde{x}(b,m_1),b,m_1) \leq t(\tilde{x}(b,m_1),b,m_2) \leq t(\tilde{x}(b,m_2),b,m_2) = t^*(b,m_2).$$

Note that for the second inequality we have used the fact that $\tilde{x}(b,m_1)$ is feasible in the problem $\max_{x\leq m_2} t(x,b,m_2)$. Thus, $t^*(b,m)$ increases in m.

Now consider the optimal choice of b, denoted as $b^*(m) = \arg\max_{b\in[0,1]} f(b,m)$, where

$$f(b,m) = F(\tilde{x}(b,m),b,m) = b\,t^*(b,m).$$

We show that f is supermodular in (b,m), and so the optimal choice of b is increasing in m. Take arbitrary $b_1, b_2 \in [0,1]$, with $b_2 > b_1$, and arbitrary $m_1, m_2 \in [0,\bar{m}]$, with $m_2 > m_1$. Compute:

$$\begin{aligned} & [f(b_2,m_2)-f(b_1,m_2)]-[f(b_2,m_1)-f(b_1,m_1)] \\ = \;& b_2[t^*(b_2,m_2)-t^*(b_1,m_2)+t^*(b_1,m_1)-t^*(b_2,m_1)] \\ & +(b_2-b_1)[t^*(b_1,m_2)-t^*(b_1,m_1)]. \end{aligned}$$

Because $t^*(b,m)$ is supermodular in (b,m), the first difference is positive; because $t^*(b,m)$ is increasing in m, the second difference is also positive. Thus, $f(b,m)$ is supermodular in (b,m)

on $[0,1]\times[0,\bar{m}]$. As a result, $b^*(m)$ is increasing in m. Since $\tilde{x}(b,m)$ is increasing in (b,m), the optimal choice of x, given by $x^*(m)=\tilde{x}(b^*(m),m)$, is increasing in m.

Because the solutions, $b^*(m)$ and $x^*(m)$, are unique for each m, the theorem of the maximum implies that they are continuous in m.

Because $t^*(b,m)$ is increasing in m for any given b, as proven above, $f(b,m)$ is increasing in m for any given b. This feature implies that $f(b^*(m),m)$ is increasing in m and, hence, that the function $W_b(m)$, given by $W_b(m)=f(b^*(m),m)+\beta V(m)$, is increasing in m. In Lemma B.1 below, we will prove that $W_b(.)$ is strictly increasing (in the case $b^*(m)>0$).

To prove that $q^*(m)$ is increasing in m, denote $z=(m-x)\omega+\psi(q)$ and use (q,z) as a buyer's choice variables. From the definition of z and (2.3), we can express

$$m-x=\frac{z-\psi(q)}{\omega},\quad b=\mu^{-1}\left(\frac{k}{\omega m-z}\right).$$

Because $b\geq 0$, the relevant domain of z is $[0,\omega m-k]$. The relevant domain of q is $[0,\psi^{-1}(z)]$. A buyer chooses $(q,z)\in[0,\psi^{-1}(z)]\times[0,\omega m-k]$ to solve:

$$\max_{(q,z)}\mu^{-1}\left(\frac{k}{\omega m-z}\right)\left[U(q)+\beta V\left(\frac{z-\psi(q)}{\omega}\right)-\beta V(m)\right].$$

We can divide this problem into two steps: first solve q for any given (z,m) and then solve z.

For any given (z,m), the optimal choice of q, denoted as $\tilde{q}(z)$, solves:

$$J(z)=\max_{0\leq q\leq\psi^{-1}(z)}\left[U(q)+\beta V\left(\frac{z-\psi(q)}{\omega}\right)\right].$$

Note that q and J do not depend on m for any given z. It is easy to see that the objective function above is supermodular in (q,z), and strictly so if V is strictly concave. Since the choice set, $\left[0,\psi^{-1}(z)\right]$, is also increasing in z, $\tilde{q}(z)$ and $J(z)$ increase in z.

The optimal choice of z is $z^*(m)=\arg\max_{0\leq z\leq\omega m-k}B(z,m)$, where

$$B(z,m)=\mu^{-1}\left(\frac{k}{\omega m-z}\right)[J(z)-\beta V(m)].$$

Note that if $J(z)<\beta V(m)$, the buyer can choose $z=\omega m-k$ to obtain $B=0$. Thus, focus on the case where $J(z)\geq\beta V(m)$. Under Assumption 1, it can be verified that the function $\mu^{-1}(\frac{k}{\omega m-z})$ strictly increases in m, strictly decreases in z, and is strictly supermodular in (z,m). Thus, for arbitrary $z_2>z_1$ and $m_2>m_1$, we have:

$$\begin{aligned}
& B(z_2,m_2)-B(z_1,m_2)-B(z_2,m_1)+B(z_1,m_1)\\
= & \left[\mu^{-1}\left(\tfrac{k}{\omega m_2-z_2}\right)-\mu^{-1}\left(\tfrac{k}{\omega m_1-z_2}\right)\right][J(z_2)-J(z_1)]\\
& +\left[\mu^{-1}\left(\tfrac{k}{\omega m_2-z_1}\right)-\mu^{-1}\left(\tfrac{k}{\omega m_2-z_2}\right)\right][\beta V(m_2)-\beta V(m_1)]\\
& +\left[\mu^{-1}\left(\tfrac{k}{\omega m_2-z_2}\right)-\mu^{-1}\left(\tfrac{k}{\omega m_2-z_1}\right)-\mu^{-1}\left(\tfrac{k}{\omega m_1-z_2}\right)+\mu^{-1}\left(\tfrac{k}{\omega m_1-z_1}\right)\right][J(z_1)-\beta V(m_1)].
\end{aligned}$$

The first term on the RHS is positive because $J(z)$ increases in z and $\mu^{-1}(\frac{k}{\omega m-z})$ increases in m. The second term is positive because $\mu^{-1}(\frac{k}{\omega m-z})$ decreases in z and $V(m)$ increases in m. The third term is strictly positive because $\mu^{-1}(\frac{k}{\omega m-z})$ is strictly supermodular in (z,m). Therefore, $B(z,m)$ is strictly supermodular. Since the choice set $[0,\omega m-k]$ is also increasing in m, the solution $z^*(m)$ increases in m. Since $\tilde{q}(z)$ increases in z, then $q^*(m)=\tilde{q}(z^*(m))$ increases in m.

To show that ϕ is increasing, we formulate a buyer's problem by letting the choices be (ϕ,z), where z is defined as $z=\phi\omega+\psi(q)$. From the definition of z and (2.3), we can express

$$q=\psi^{-1}(z-\phi\omega), \quad b=\mu^{-1}\left(\frac{k}{\omega m-z}\right).$$

The relevant domain of ϕ is $[0,\min\{m,z/\omega\}]$, and of z is $[0,\omega m-k]$. A buyer solves:

$$\max_{(\phi,z)}\mu^{-1}\left(\frac{k}{\omega m-z}\right)\left[U\left(\psi^{-1}(z-\phi\omega)\right)+\beta V(\phi)-\beta V(m)\right]. \tag{B.1}$$

As in the above formulation where the choices are (q,z), we can divide the maximization problem into two steps. First, for any given z, the optimal choice of ϕ solves:

$$J(z)=\max_{\phi\geq 0}\left[U\left(\psi^{-1}(z-\phi\omega)\right)+\beta V(\phi)\right]. \tag{B.2}$$

Note that we have written the constraint on ϕ as $\phi\geq 0$, instead of $\phi\in[0,\min\{m,z/\omega\}]$. The optimal choice satisfies $\phi<m$, because $\phi=m$ implies $x=0$ which is not optimal (in the case with $b>0$). Also, $\phi<z/\omega$ under the assumptions $\psi(0)=0$ and $U'(0)=\infty$. Denote the solution for ϕ as $\tilde{\phi}(z)$. Second, the optimal choice of z solves

$$W_b(m)-\beta V(m)=\max_{0\leq z\leq\omega m-k}\mu^{-1}\left(\frac{k}{\omega m-z}\right)[J(z)-\beta V(m)]. \tag{B.3}$$

Similar to the procedure used in the above formulation of the problem where the choices are (q,z), we can show that $\phi^*(m)$ and $z^*(m)$ increase in m. **QED**

Lemma B.1. *Consider the formulation of a buyer's problem, (B.1), where the choices are ϕ and $z=\phi\omega+\psi(q)$. Assume $b^*(m)>0$. The following results hold: (i) $J'(z)$ exists and is given by*

$$J'(z)=\frac{U'(\tilde{q}(z))}{\psi'(\tilde{q}(z))}>0, \tag{B.4}$$

where $\tilde{\phi}$ is defined above and $\tilde{q}(z)\equiv\psi^{-1}(z-\tilde{\phi}(z)\omega)$; (ii) The solution for z in (B.3) is unique and given by

$$0\geq J(z^*)-\beta V(m)+\frac{U'(\tilde{q}(z^*))}{\psi'(\tilde{q}(z^*))}\frac{k\mu' b^*(m)}{\mu^2} \quad \text{and} \quad z^*\leq\omega m-k, \tag{B.5}$$

where the two inequalities hold with complementary slackness; (iii) $W_b(m)$ is differentiable at m if and only if $V(m)$ is so; (iv) $W_b(m)$ is strictly increasing (if $b^*(m)>0$).

Proof. To prove part (i), let z and z' be arbitrary levels in $[0, \omega m - k]$. Note that the constraint on the choice ϕ is $\phi \geq 0$, which does not depend on z. Thus, the choice $\tilde{\phi}(z)$ is feasible in the maximization problem with z' and the choice $\tilde{\phi}(z')$ is feasible in the maximization problem with z. Using a similar proof to the one in Appendix A that established differentiability of V at y, we can prove that $J'(z^-)$ and $J'(z^+)$ both exist and are equal to the expression in (B.4).

For part (ii), we prove first that the objective function in (B.3) is strictly concave in z. For this result, recall that $\tilde{q}(z)$ is an increasing function, as shown in the proof of Lemma 3.1 using (q, z) as the choices. This result and (B.4) imply that $J'(z)$ is decreasing, i.e., that $J(z)$ is concave. Because $J(z)$ is increasing and concave, and $\mu^{-1}(\frac{k}{\omega m - z})$ is strictly decreasing and strictly concave in z, the objective function in (B.3) is strictly concave in z.

Strict concavity of the objective function implies that the solution for z is unique. Also, because the objective function is differentiable in z, the optimal choice of z is given by the first-order condition. Deriving the first-order condition, substituting $J'(z)$ from (B.4), and substituting $\mu^{-1}(\frac{k}{\omega m - z^*}) = b^*(m)$, we obtain (B.5).

For part (iii), consider an arbitrary m that satisfies the maintained hypothesis $b^*(m) > 0$. Note that $b^*(m) > 0$ implies $z^*(m) < \omega m - k$. Because $z^*(m) < \omega m - k$ and $z^*(m)$ is a continuous function, there exists $\varepsilon > 0$ such that $z^*(m + \varepsilon) < \omega m - k$ and $z^*(m) < \omega(m - \varepsilon) - k$. Consider the neighborhood $O(m) = (m - \varepsilon, m + \varepsilon)$. For any $m' \in O(m)$, the choice $z^*(m')$ is feasible in the problem where the money balance is m, and the choice $z^*(m)$ is feasible in the problem where the balance is m'. Applying to (B.3) the proof in Appendix A that established differentiability of V at y, we can derive the formulas of $W_b'(m^+)$ and $W_b'(m^-)$ for any m such that $b^*(m) > 0$. Substituting the first-order condition of z^*, these formulas yield the following generalized version of the envelope theorem:

$$W_b'(m^+) = b^*(m)\left[\omega J'(z^*) - \beta V'(m^+)\right] + \beta V'(m^+), \tag{B.6}$$

$$W_b'(m^-) = b^*(m)\left[\omega J'(z^*) - \beta V'(m^+)\right] + \beta V'(m^-). \tag{B.7}$$

Again, we have used the fact that left-hand and right-hand derivatives exist for a concave function. The above equations show that $W_b(m)$ is differentiable at m if and only if $V(m)$ is so.

Finally, we prove part (iv). Since V is concave and increasing, $V'(m^-) \geq V'(m^+) \geq 0$. Since $b^* \leq 1$ and $J'(z^*(m)) > 0$, (B.6) and (B.7) imply that $W_b'(m^-) \geq W_b'(m^+) \geq b^*(m)\omega J'(z^*) > 0$, where the last inequality has used the maintained assumptions $b^*(m) > 0$ and $\omega > 0$. Therefore, $W_b(.)$ is strictly increasing under these assumptions. **QED**

C. Proof of Lemma 3.2

Maintain the assumption $b^*(m) > 0$ in this proof. Lemma B.1 has already proven part (i) of the current lemma. For future use, compute $Q_1(x, b) = \omega/\psi'(q)$ and rewrite (B.4) as

$$J'(z^*) = \frac{U'(q^*)}{\psi'(q^*)} = \frac{u_1(x^*, b^*)}{\omega}.$$

Substituting this result for $J'(z^*)$, we rewrite (B.6) and (B.7) as

$$W_b'(m^+) = b^*(m)u_1(x^*(m), b^*(m)) + \beta(1 - b^*(m))V'(m^+) \tag{C.1}$$

$$W_b'(m^-) = b^*(m)u_1(x^*(m), b^*(m)) + \beta(1 - b^*(m))V'(m^-). \tag{C.2}$$

To prove part (ii), consider any arbitrary m such that $\phi(m) > 0$ (i.e., $x^*(m) < m$) and consider the original formulation of a buyer's maximization problem, (2.8), where the choices are (x, b). Since $x^*(m) < m$, a procedure similar to the derivation of $J'(z)$ in the previous proof but applied to (2.8) yields:

$$W_b'(m^+) = \beta\left[b^*(m)V'(\phi^+(m)) + (1 - b^*(m))V'(m^+)\right]$$

$$W_b'(m^-) = \beta\left[b^*(m)V'(\phi^-(m)) + (1 - b^*(m))V'(m^-)\right],$$

where $\phi^+(m) = m^+ - x^*(m)$ and $\phi^-(m) = m^- - x^*(m)$. Comparing these equations with (C.1) and (C.2) yields $V'(\phi^+(m)) = V'(\phi^-(m))$. The common derivative is given by (3.6).

To prove part (iii), take any $m > 0$ such that $W_b(m) = V(m)$. The lottery in (2.11) implies that $W_b(m') \le V(m')$ for all m'. Because W_b and V are continuous functions, it is straightforward to verify that $W_b'(m^+) \le V'(m^+)$ and $W_b'(m^-) \ge V'(m^-)$ at the particular level m. These inequalities, together with (C.1) and (C.2), imply:

$$V'(m^-) \le \frac{b^*(m)}{1 - \beta[1 - b^*(m)]}u_1(x^*(m), b^*(m)) \le V'(m^+).$$

On the other hand, concavity of V implies $V'(m^-) \ge V'(m^+)$. Thus, $V'(m^-) = V'(m^+)$, and the common derivative is given by the right-hand side of (3.7). By (C.1) and (C.2), $W_b'(m)$ exists and is also given by the right-hand side of (3.7).

For part (iv), let m_1 be an arbitrary balance that satisfies the hypotheses that $W_b(m_1) = V(m_1)$, $\phi(m_1) > 0$ and $b^*(m_1) > 0$. We first prove that $b^*(.)$ is strictly increasing at m_1. To this aim, let m_2 be sufficiently close to m_1 so that $\phi(m_2) > 0$ and $b^*(m_2) > 0$ (which is possible because $\phi(m)$ and $b(m)$ are continuous functions). Shorten the notation $(x^*(m_i), b^*(m_i), \phi(m_i))$

to (x_i^*, b_i^*, ϕ_i), where $x_i^* = m_i - \phi_i$ and $i = 1, 2$. Because $b_i^* > 0$, the first-order condition for b, i.e., the condition (3.5), yields:

$$u(x_i^*, b_i^*) + \beta[V(\phi_i) - V(m_i)] + b_i^* u_2(x_i^*, b_i^*) = 0.$$

Moreover, because m_1 satisfies the hypotheses for both (3.7) and (3.6), subtracting the two equations yields:

$$V'(\phi_1) - V'(m_1) = \frac{1-\beta}{\beta[1-\beta(1-b_1^*)]} u_1(x_1^*, b_1^*) > 0.$$

This result shows that V must be strictly concave in some sections of $[\phi_1, m_1]$.

Since the proofs for strict monotonicity of $b^*(m_1)$ at m_1 are similar in the cases $m_2 > m_1$ and $m_2 < m_1$, let us consider only the case where $m_2 > m_1$. In this case, we want to prove that $b_2^* > b_1^*$. By Lemma 3.1, $x_2^* \geq x_1^*$, $b_2^* \geq b_1^*$ and $\phi_2 \geq \phi_1$. Subtract the first-order conditions for b_1^* and b_2^* (see above), and re-organize:

$$\begin{aligned} 0 = \; & \{u(x_2^*, b_2^*) + \beta[V(m_2 - x_2^*) - V(m_2)]\} - \{u(x_1^*, b_2^*) + \beta[V(m_2 - x_1^*) - V(m_2)]\} \\ & + \beta\{V(m_1) - V(m_1 - x_1^*) + V(m_2 - x_1^*) - V(m_2)\} \\ & + u(x_1^*, b_2^*) - u(x_1^*, b_1^*) + b_2^* u_2(x_2^*, b_2^*) - b_1^* u_2(x_1^*, b_1^*). \end{aligned}$$

The first difference on the right-hand side of the above equation is equal to $[t(x_2^*, b_2^*, m_2) - t(x_1^*, b_2^*, m_2)]$, where t is defined by (3.3). This difference is greater than or equal to zero, because x_2^* maximizes $t(x, b_2^*, m_2)$ and because x_1^* is a feasible choice of x in such a maximization problem (as $x_1^* \leq x_2^*$). The second difference is strictly positive because V is strictly concave in some sections of $[\phi_1, m_1] \subset [\phi_1, m_2]$. For the third difference on the right-hand side of the above equation, note that $u_2(x_2^*, b_2^*) \geq u_2(x_1^*, b_2^*)$, because $u_{12} > 0$ and $x_2^* \geq x_1^*$. Thus,

$$0 > u(x_1^*, b_2^*) - u(x_1^*, b_1^*) + b_2^* u_2(x_1^*, b_2^*) - b_1^* u_2(x_1^*, b_1^*).$$

The right-hand side of the above inequality is a strictly decreasing function of b_2^*, and it is equal to 0 when $b_2^* = b_1^*$. Thus, $b_2^* > b_1^*$. Therefore, $b^*(.)$ is strictly increasing at m_1.

Let us continue to prove the rest of part (iv). Since (3.7) and (3.6) hold for $m = m_1$, we can combine the two equations to obtain:

$$V'(\phi(m_1)) = V'(m_1)\left[\frac{1-\beta}{\beta b^*(m_1)} + 1\right].$$

Because $b^*(m_1)$ is strictly increasing at m_1 and V is concave, the right-hand side above is strictly decreasing in m_1. In this case, the above equation shows that V must be strictly concave at $\phi(m_1)$ and that $\phi(.)$ must be strictly increasing at m_1. Strictly concavity of V at $\phi(m_1)$ implies that $W_b(\phi(m_1)) = V(\phi(m_1))$. To see why, suppose that the inequality does not hold. Because $W_b(m') \leq V(m')$ for all m', the supposition yields $W_b(\phi(m_1)) < V(\phi(m_1))$. In this case, V around $\phi(m_1)$ must be a linear segment generated by the lottery in (2.11), which contradicts strict concavity of V at $\phi(m_1)$. **QED**

D. Proof of Theorem 3.3

It is easy to verify that the mapping $\mathcal{F}$, defined by the right-hand side of (2.11), satisfies Blackwell's sufficient conditions for contraction mapping; thus, $\mathcal{F}$ has a unique fixed point V that is continuous and bounded (see Stokey and Lucas with Prescott, 1989). With Lemmas 2.2 and 3.1, it is clear that $\mathcal{F}$ maps increasing functions into increasing functions. Thus, the fixed point of $\mathcal{F}$ is an increasing function. Moreover, since $\tilde{V}$ is a continuous function defined on a closed interval, two-point lotteries make it a concave function (see Appendix F in Menzio and Shi, 2009, for the proof). Thus, $\mathcal{F}$ maps concave functions into concave functions, and the fixed point V is concave.

To establish other properties of V stated in the theorem, we start with part (ii). For a seller with any money balance m, the choice of not contributing to a firm yields the value $\beta V(m)$. Because this choice is always feasible, $W_s(m) \geq \beta V(m)$ for all m. For a buyer who holds $m \leq k/\omega$, all feasible choices of $x \leq m$ yield $Q(x,b) = 0$. Thus, for such a buyer, the optimal choice of b is $b^*(m) = 0$, and the value is $W_b(m) = \beta V(m) \leq W_s(m)$. It is clear that $V(0) = \tilde{V}(0) = W_s(0)$. Also, $V(0) \geq 0$, because an individual without money can always choose not to trade. To prove $V(0) > 0$, suppose $V(0) = 0$, to the contrary. In this case, $0 = V(0) \geq W_s(0) \geq \beta V(0) = 0$, and so $W_s(0) = V(0) = 0$. Using the definition of $W_s(0)$, we have $\beta V(m^*) - h(\omega m^*) = 0$. Since $m^* = 0$ satisfies this equation and since m^* is unique, then $m^* = 0$. This result contradicts the existence of a monetary equilibrium, because it implies that the value of money is zero. Therefore, $V(m) \geq W_s(m) \geq W_s(0) = V(0) > 0$ for all m.

To complete the proof of part (ii), we prove that $V(m) > W_s(m)$ for all $m > 0$. For all $m > 0$ such that the constraint $y \geq m$ is binding, (A.1) yields $W_s(m) = \beta V(m) < V(m)$. Now consider $m > 0$ such that the constraint $y \geq m$ is not binding. Contrary to part (ii), suppose $V(\tilde{m}) = W_s(\tilde{m})$ for some $\tilde{m} > 0$ such that $y(\tilde{m}) > \tilde{m}$. In this case, (2.5) in Lemma 2.2 implies that $V'(y(\tilde{m})) > 0$, and so $V'(\tilde{m}^-) > 0$. Then,

$$V'(\tilde{m}^-) \leq W_s'(\tilde{m}) = \beta V'(y(\tilde{m})) \leq \beta V'(\tilde{m}^-) < V'(\tilde{m}^-).$$

The first inequality follows from the fact that $V(m) \geq W_s(m)$ for all m and that $W_s'(\tilde{m})$ exists. The second inequality follows from concavity of V and $V'(\tilde{m}^-) > 0$. The above result implies the contradiction that $V'(\tilde{m}^-) < 0$. Therefore, $V(m) > W_s(m)$ for all $m > 0$.

For part (iii), we prove first that there is some $m' \in (0, \infty)$ such that $W_b(m') > W_s(m')$. Suppose, to the contrary, that $W_b(m) \leq W_s(m)$ for all $m \in (0, \infty)$. Then, $\tilde{V}(m) = W_s(m)$ for all m. Since $W_s(m)$ is concave (see Lemma 2.2), $\tilde{V}(.)$ is concave in this case, and so $V(m) =$

$\tilde{V}(m) = W_s(m)$ for all m. In this case, (A.1) yields

$$V(m) = \max_{y \geq m} \left[\beta V(y) - h(\omega(y-m))\right], \quad \text{all } m > 0.$$

If $y(m) = m$, the above equation yields $V(m) = 0$, which contradicts part (ii) of the theorem. If $y(m) > m$, Lemma 2.2 implies that $W_s(.)$ is differentiable at m and that $W_s'(m) = \beta V'(y(m)) > 0$ (see (2.5)). Since $W_s(m) = V(m)$ for all $m > 0$ in this case, $V'(m) = W_s'(m) = \beta V'(y(m)) \leq \beta V'(m)$. This implies $V'(m) = 0 = V'(y(m))$, which contradicts $V'(y(m)) > 0$.

Continue the proof of part (iii). For an individual with a money balance $m \in (0, k/\omega)$, the lottery with $L_1 = 0$ and $L_2 = m'$ yields a value higher than $\tilde{V}(m)$, where m' is described above. Thus, these individuals will participate in lotteries. However, m' may not be necessarily be the optimal winning prize of the lottery for these individuals. Let m_0 be such an optimal winning prize (defined by (2.12)). Clearly, $m_0 > k/\omega > 0$, $V(m_0) = \tilde{V}(m_0) = W_b(m_0)$, and $V(m) \geq \tilde{V}(m)$ for all $m \in [0, m_0]$. Since $V(m_0) = W_b(m_0)$, part (iii) of Lemma 3.2 implies $V'(m_0) = W_b'(m_0) > 0$.

For part (iv), suppose $b^*(m_0) = 0$, contrary to the stated result. In this case, $W_b(m_0) = \beta V(m_0)$. Since $V(m_0) = W_b(m_0)$ by part (iii) above, then $V(m_0) = 0$, which contradicts part (ii) that $V(m) > 0$ for all $m \geq 0$. Thus, it must be true that $b^*(m_0) > 0$. Since $V(m_0) = W_b(m_0)$, (3.7) holds for $m = m_0$. With $b^*(m_0) > 0$, (3.7) implies $V'(m_0) < u_1(x^*(m_0), b^*(m_0))/\beta$. Since $V(m)$ is linear for $m \in [0, m_0]$, then $V'(\phi(m_0)) = V'(m_0) < u_1(x^*(m_0), b^*(m_0))/\beta$. If $\phi(m_0) > 0$, then (3.6) holds for $m = m_0$, which yields the contradiction that $V'(\phi(m_0)) = u_1(x^*(m_0), b^*(m_0))/\beta$. Thus, it must be true that $\phi(m_0) = 0$.

For part (v), note that $V'(m_0)$ exists by part (iii) above and that $V'(m^*)$ exists by Lemma 2.2. We prove that $V'(m_0) > V'(m^*)$ which, by concavity of V, implies $m_0 < m^*$. Recall that the lottery with the winning prize m_0 implies that $V(m) = V(0) + V'(m_0)m$ for all $m \in [0, m_0]$ and, hence, that $V'(m) = V'(m_0)$ for all such m. Thus, it is always true that $V'(m^*) \leq V'(m_0)$: If $m^* \leq m_0$, then $V'(m^*) = V'(m_0)$; If $m^* > m_0$, concavity of V implies $V'(m^*) \leq V'(m_0)$. To prove the result $V'(m_0) > V'(m^*)$, suppose that $V'(m^*) = V'(m_0)$, to the contrary. In this case, both $V(m_0)$ and $V(m^*)$ lie on the same tangent line (or its extension) that connects $V(0)$ and $V(m_0)$. So, $V'(m^*) = V'(0) = W_s'(0) = \beta V'(m^*)$. This implies $0 = V'(m^*) = V'(m_0)$, which contradicts $V'(m_0) > 0$. Thus, $V'(m_0) > V'(m^*)$ and $m_0 < m^*$.

Finally, we prove part (i). Note first that the availability of the lottery with the winning prize m_0 implies that $V'(m)$ exists for all $m \in [0, m_0]$. To show that $V'(m)$ exists for all m, suppose to the contrary that $V'(\tilde{m}^-) > V'(\tilde{m}^+)$ for some $\tilde{m} > m_0$. Then V must be strictly concave at $\tilde{m}$. Given that $V(m) > W_s(m)$ for all $m > 0$, we must have $V(\tilde{m}) = W_b(\tilde{m})$. Note that $b^*(\tilde{m}) \geq b^*(m_0) > 0$. Part (iii) of Lemma 3.2 applies, which leads to the contradiction that

$V'(\tilde{m})$ exists. Therefore, $V'(m)$ must exist for all $m \geq 0$. Thus, part (i) of Lemma 3.2 implies that $W_b'(m)$ exists for all such m that $b^*(m) > 0$. **QED**

E. Proof of Lemma 4.1

For part (i), note that the choice $b = 0$ is always feasible to a buyer with a money balance m^*, and it yields the value $\beta V(m^*)$ for a buyer. Thus, $W_b(m^*) \geq \beta V(m^*) = W_s(m^*)$, and the inequality is strict if the optimal choice satisfies $b^*(m^*) > 0$. Suppose $b^*(m^*) = 0$, contrary to the desired result. Since $b^*(.)$ is an increasing function (see Lemma 3.1), then $b^*(m) = 0$ for all $m \in [0, m^*]$, and so $W_b(m) = \beta V(m)$ for all $m \in [0, m^*]$. Because $W_s(m) \geq \beta V(m)$ for all m (as the choice $l = 0$ is always feasible to a seller), $\tilde{V}(m) = W_s(m)$ in this case for all $m \in [0, m^*]$. Since $W_s(.)$ is a concave function (see Lemma 2.2), then $V(m) = W_s(m)$ for all $m \in [0, m^*]$. In particular, $V(m^*) = W_s(m^*) = \beta V(m^*)$. Since this result implies $0 = V(m^*) = \omega m^*$, which contradicts $\omega > 0$, it must be the case that $b^*(m^*) > 0$.

To prove $\phi^T(m^*) = 0$, consider first the case where $W_b(m^*) = V(m^*)$. If $\phi(m^*) = 0$, then $T = 1$ and $\phi^T(m^*) = 0$. If $\phi(m^*) > 0$, part (iv) of Lemma 3.2 implies that V is strictly concave at $\phi(m^*)$ and that $W_b(\phi(m^*)) = V(\phi(m^*))$. This argument can be repeated to show that $\phi^T(m^*) = 0$. Now consider the case where $W_b(m^*) < V(m^*)$. In this case, the individual with m^* participates in a lottery. Let m_a and m_b be the two realizations of the lotteries. Then, $W_b(m_i) = V(m_i)$ for $i = a, b$. Applying the previous argument to m_a and m_b separately, we can show that $\phi^{T_a}(m_a) = \phi^{T_b}(m_b) = 0$, where T_i is defined by $\phi^{T_i - 1}(m_i) > m_0 \geq \phi^{T_i}(m_i)$ for $i = a, b$.

The proof of part (ii) of Lemma 4.1 is to be completed.

For part (iii), note that part (ii) implies that the conditions in part (ii) of Lemma 3.2 are satisfied at $m = m^*$. Thus, $V(\phi(m^*)) = W_b(\phi(m^*))$, and V is strictly concave at $\phi(m^*)$. These features imply that if $\phi^2(m^*) > 0$, then the conditions in part (ii) of Lemma 3.2 are satisfied at $m = \phi(m^*)$, in which case $V\left(\phi^2(m^*)\right) = W_b\left(\phi^2(m^*)\right)$ and V is strictly concave at $\phi^2(m^*)$. Continuing this process, we conclude that if $\phi^i(m^*) > 0$, then $V(\phi^i(m^*)) = W_b(\phi^i(m^*))$ and V is strictly concave at $\phi^i(m^*)$. **QED**

F. Proof of Theorem 4.3

The text preceding the theorem has established that a unique monetary steady state exists and that money is neutral in the steady state.

To prove that the frequency function of money holding among buyers is decreasing, note that $\phi^i(m^*) = \phi^{i-1}(m^*) - x^*(\phi^{i-1}(m^*)) < \phi^{i-1}(m^*)$ for all $1 \leq i \leq T$. By part (iv) of Theorem 3.3, $b^*(m_0) > 0$. Because $\phi^i(m^*) \geq m_0$ for all $1 \leq i \leq T - 1$ and $b^*(.)$ is an increasing function,

$b^*(\phi^i(m^*)) > 0$ for all $1 \le i \le T-1$. Thus, for all $1 \le i \le T-1$, $\phi^i(m^*)$ satisfies part (iv) of Lemma 3.2, which implies that $b^*(.)$ is strictly increasing at $\phi^i(m^*)$. The first line of (4.5) implies that $g(m^*) > g(0)$ and $g(\phi^i(m^*)) > g(\phi^{i-1}(m^*))$ for all $i = 1, 2, ..., T-1$.

Define $(\bar{q}, \underline{q}, \beta_0)$ by the following equations:

$$\frac{U'(\bar{q})}{\psi'(\bar{q})} = h'(\psi(\bar{q}) + k); \qquad U(\underline{q})\frac{\psi'(\underline{q})}{U'(\underline{q})} - \psi(\underline{q}) = k; \tag{F.1}$$

$$\beta_0 = \frac{U(\bar{q})}{h(\psi(\bar{q}) + k) + \frac{U'(\bar{q})}{\psi'(\bar{q})}[\psi(\underline{q}) + k]}. \tag{F.2}$$

We prove that if $\beta \ge \beta_0$, then $\phi(m^*) > 0$, in which case $T \ge 2$. To this end, suppose $\phi(m^*) = 0$. Shorten the notation $b^*(m^*)$ as b^* and $q^*(m^*)$ as q^*. Because $\phi(m^*) = 0$, then

$$\frac{1}{\beta}u_1(m^*, b^*) \ge V'(0) > V'(m^*) = \frac{\omega}{\beta}h'(\omega m^*) \ge \frac{\omega}{\beta}h'(\psi(q^*) + k).$$

The first inequality comes from the optimality condition for $\phi(m^*) = 0$, the second inequality from the fact that V has a strictly concave section in $[0, m^*]$, and the equality from (2.6). The last inequality holds because h is strictly convex and $\omega m^* = \psi(q^*) + \frac{k}{\mu(b^*)} \ge \psi(q^*) + k$. Substituting $u_1(m^*, b^*) = \omega U'(q^*)/\psi'(q^*)$, we obtain the following necessary condition for $\phi(m^*) = 0$:

$$\frac{U'(q^*)}{\psi'(q^*)} > h'(\psi(q^*) + k).$$

This condition is equivalent to $q^* < \bar{q}$, where $\bar{q}$ is defined in (F.1).

We prove that if $\beta \ge \beta_0$, then the supposition $\phi(m^*) = 0$ leads to the contradiction $q^* \ge \bar{q}$. With $\phi(m^*) = 0$, the fact $W_b(m^*) \le V(m^*)$ yields:

$$W_b(m^*) = \beta V(m^*) + b^*[u(m^*, b^*) + \beta V(0) - \beta V(m^*)] \le V(m^*).$$

Recall that $V(0) = W_s(0) = \beta V(m^*) - h(\omega m^*)$. Using this equation and the first-order condition for $b^*(m^*)$, we solve for $V(0)$ and $V(m^*)$. Substituting the results into the above inequality yields:

$$u(m^*, b^*) - \beta h(\omega m^*) + b^*(1 + \beta b^*) u_2(m^*, b^*) \ge 0.$$

Substituting $u(m^*, b^*) = U(q^*)$ and $\omega m^* \ge \psi(q^*) + k$, and computing $u_2(m^*, b^*)$, we get the following necessary condition for the above inequality:

$$U(q^*) - \beta h(\psi(q^*) + k) - \beta\frac{U'(q^*)}{\psi'(q^*)}\left\{b^*\left(\frac{1}{\beta} + b^*\right)\frac{[-k\mu'(b^*)]}{[\mu(b^*)]^2}\right\} \ge 0. \tag{F.3}$$

Since $b^* \le 1$, it can be verified that $(1-\beta)^2 + \beta b^*(1-2\beta) > 0$. (This is clearly true if $2\beta \ge 1$. If $2\beta < 1$, then $(1-\beta)^2 + \beta b^*(1-2\beta) \ge (1-\beta)^2 + \beta(1-2\beta) = 1 - \beta - \beta^2 > 0$.) This result implies $\frac{1}{\beta} + b^* > 1 + \frac{\beta b^*}{1-\beta}$, and so the expression inside $\{.\}$ in (F.3) is strictly greater than $a(b^*)$ where

$$a(b^*) \equiv b^*\left[1 + \frac{\beta b^*}{1-\beta}\right]\frac{[-k\mu'(b^*)]}{[\mu(b^*)]^2}. \tag{F.4}$$

We will prove below that $a(b^*) > \psi(\underline{q}) + k$. Thus, a necessary condition for (F.3) is

$$U(q^*) - \beta h(\psi(q^*) + k) - \beta \frac{U'(q^*)}{\psi'(q^*)}[\psi(\underline{q}) + k] > 0.$$

The left-hand side above is strictly increasing in q^* for all $q^* \leq \bar{q}$. If $\beta \geq \beta_0$ (where β_0 is defined by (F.2)), the left-hand side above is non-positive at $q^* = \bar{q}$ and, hence, strictly negative for all $q^* < \bar{q}$. Thus, if $\beta \geq \beta_0$, the supposition $\phi(m^*) = 0$ leads to the contradiction $q^* \geq \bar{q}$.

Finally, we prove $a(b^*) > \psi(\underline{q}) + k$. Under Assumption 1, $a(b^*)$ strictly increases in b^*. Because $b^* \geq b^*(m_0)$, it suffices to show that $a(b^*(m_0)) > \psi(\underline{q}) + k$. Let us characterize optimal choices at m_0. Shorten the notation $b^*(m_0)$ as b_0. The lottery in (2.12) implies that $V(m) = V(0) + mV'(0)$ for all $m \in [0, m_0]$. By Theorem 3.3, $b_0 > 0$, $\phi(m_0) = 0$, and $W_b(m_0) = V(m_0)$. With these results, the Bellman equation, (2.8), yields:

$$(1-\beta)\left[V(0) + m_0 V'(0)\right] = \max_{b \in [0,1]} b\left[u(m_0, b) - \beta m_0 V'(0)\right].$$

Since $W_b(m_0) = V(m_0)$, Lemma 3.2 implies $W_b'(m_0) = V'(m_0) = V'(0)$, which yields:

$$V'(0) = \frac{b_0 u_1(m_0, b_0)}{1 - \beta + \beta b_0}.$$

Substituting $V'(0)$ and using the fact $V(0) > 0$ in the Bellman equation, we get:

$$u(m_0, b_0) - m_0 u_1(m_0, b_0) > 0. \tag{F.5}$$

Since $b_0 > 0$, b_0 satisfies the first-order condition. Substituting $V'(0)$ from the above into this first-order condition yields:

$$u(m_0, b_0) - \frac{\beta b_0 m_0}{1 - \beta + \beta b_0} u_1(m_0, b_0) + b_0 u_2(m_0, b_0) = 0.$$

Substituting $u(m_0, b_0)$ from (F.5), the above equation yields:

$$-\left(1 + \frac{\beta b_0}{1-\beta}\right) b_0 u_2(m_0, b_0) > m_0 u_1(m_0, b_0). \tag{F.6}$$

Let us derive necessary conditions for (F.5) and (F.6). Shorten the notation $q^*(m_0)$ as q_0. Substituting $u(m_0, b_0) = U(q_0)$, $u_1(m_0, b_0) = \omega U'(q_0)/\psi'(q_0)$ and $\omega m_0 = \psi(q_0) + \frac{k}{\mu(b_0)} \geq \psi(q_0) + k$, we have the following necessary condition for (F.5):

$$U(q_0)\frac{\psi'(q_0)}{U'(q_0)} - \psi(q_0) > k.$$

This condition is equivalent to $q_0 > \underline{q}$, where $\underline{q}$ is defined in (F.1). Similarly, substituting u_1, u_2 and $\omega m_0 \geq \psi(q_0) + k$, we have the following necessary condition for (F.6): $a(b_0) > \psi(q_0) + k$. Because $q_0 > \underline{q}$, then $a(b_0) > \psi(\underline{q}) + k$, as desired. **QED**

References

[1] Acemoglu, D., and R. Shimer (1999): Efficient Unemployment Insurance, Journal of Political Economy, 107, 893-928.

[2] Baumol, W.J. (1952): The Transactions Demand for Cash: An Inventory Theoretic Approach, Quarterly Journal of Economics, 66, 545-556.

[3] Benveniste, L.M., and J.A. Scheinkman (1979): On the Differentiability of the Value Function in Dynamic Models of Economics, Econometrica, 47, 727-732.

[4] Burdett, K., S. Shi, and R. Wright (2001): Pricing and Matching with Frictions, Journal of Political Economy, 109, 1060-1085.

[5] Corbae, D., Temzelides, T., and R. Wright (2003): Directed Matching and Monetary Exchange, Econometrica, 71, 731-756.

[6] Edlin, A.S., and C. Shannon (1998): Strict Monotonicity in Comparative Statics, Journal of Economic Theory, 81, 201-219.

[7] Galenianos, M., and P. Kircher (2008): A Model of Money with Multilateral Matching, Journal of Monetary Economics, 55, 1054-1066.

[8] Gonzalez, F.M., and S. Shi (2008): An Equilibrium Theory of Learning, Search and Wages, Working Paper 328, University of Toronto.

[9] Green, E., and R. Zhou (1998): A Rudimentary Model of Search with Divisible Money and Prices, Journal of Economic Theory, 81, 252-271.

[10] Julien, B., Kennes, J. and I. King (2008): Bidding for Money, Journal of Economic Theory, 142, 196-217.

[11] Lagos, R., and R. Wright (2005): A Unified Framework for Monetary Theory and Policy Analysis, Journal of Political Economy, 113, 463-484.

[12] Menzio, G., and S. Shi (2009): Block Recursive Equilibria for Stochastic Models of Search on the Job, manuscript, University of Toronto.

[13] Moen, E. (1997): Competitive Search Equilibrium, Journal of Political Economy, 105, 694–723.

[14] Peters, M. (1984): Bertrand Equilibrium with Capacity Constraints and Restricted Mobility, Econometrica, 52, 1117-1129.

[15] Peters, M. (1991): Ex Ante Price Offers in Matching Games: Non-Steady State, Econometrica, 59, 1425-1454.

[16] Rocheteau, G., and R. Wright (2005): Money in Competitive Equilibrium, in Search Equilibrium, and in Competitive Search Equilibrium, Econometrica, 73, 175-202.

[17] Royden, H.L. (1988): Real Analysis. New York: Macmillan.

[18] Shi, S. (1995): Money and Prices: A Model of Search and Bargaining, Journal of Economic Theory, 67, 467-496.

[19] Shi, S. (1997): A Divisible Search Model of Fiat Money, Econometrica, 65, 75-102.

[20] Shi, S. (2009): Directed Search for Equilibrium Wage-Tenure Contracts, Econometrica, 77, 561-584..

[21] Stokey, N., and R.E. Lucas, Jr., with E.C. Prescott (1989): Recursive Methods in Economic Dynamics. Cambridge, Massachusetts: Harvard University Press.

[22] Tobin, J. (1956): The Interest-Elasticity of Transactions Demand for Cash, Review of Economics and Statistics, 38, 784-793.

[23] Topkis, D.M. (1998): Supermodularity and Complementary. New Jersey: Princeton University Press.

[24] Trejos, A., and R. Wright (1995): Search, Bargaining, Money and Prices, Journal of Political Economy, 103, 118-141.

[25] Zhu, T. (2005): Existence of a Monetary Steady State in a Matching Model: Divisible Money, Journal of Economic Theory, 123, 135-160.

A List of the Notation

$U(q)$: utility of consuming q units of goods;

$\psi(q)$: labor input of producing q units of goods;

$h(l)$: disutility of supplying l units of labor;

l: labor supply of a seller;

l^d: demand for labor by a firm;

$y = m + l/\omega$: a seller's end-of-period money balance after production;

M: aggregate stock of money per capita;

m: an individual's holdings of money, normalized by M;

(x, q): terms of trade in a submarket; x is the quantity of money (normalized by M) given by the buyer to the seller for q units of goods;

k: cost of maintaining a trading post for a period (in terms of labor);

N_s, N_b: temporary notation for the number of trading posts and buyers, respectively, in a submarket;

$\mathcal{M}(N_b, N_s)$: aggregate number of matches in a submarket with N_b buyers and N_s trading posts;

$s = \mathcal{M}(N_b, N_s)/N_s$: matching probability of a selling post in a submarket;

$b = \mathcal{M}(N_b, N_s)/N_s$: matching probability of a buyer in a submarket;

$s = \mu(b)$: expressing a post's matching probability as a function of a buyer's matching probability, obtained after combining the expressions for s and b to eliminate N_s/N_b;

$s(x, q)$: expressing a selling post's matching probability as a function of the terms of trade, (x, q), in that submarket, which is an equilibrium object;

$b(x, q) = \mu^{-1}(s(x, q))$: expressing a buyer's matching probability as a function of the terms of trade, (x, q), in that submarket;

$Q(x, b)$: defined as q that satisfies $b(x, q) = b$, i.e., the quantity of goods offered at a selling posting that asks for x units of money from a buyer and offers a matching probability b to the buyer;

ρ: a parameter in an example of the matching function;

(x, b): an alternative way to express the terms of trade in a submarket;

$u(x, b) = U(Q(x, b))$: utility of consumption obtained from a trade in submarket (x, b);

D: a firm's profit (in terms of money);

ω: value of money in terms of labor; $1/\omega$ is the nominal wage;

$W_s(m)$: value function of a seller with money holdings m;

$W_b(m)$: value function of a buyer with money holdings m;

$\tilde{V}(m) = \max\{W_s(m), W_b(m)\}$;

$V(m)$: value function of an individual with money holdings m;

L_1, L_2: the two realizations in a lottery;

π: the probability with which L_2 is realized in the lottery;

$\mathcal{F}$: the mapping on the value function defined by the Bellman equation;

β: discount factor;

$z = (m - x)\,\omega + \psi\,(q)$: used in the appendix;

m^*: maximizer of $[\beta V(m) - \omega m]$;

$\bar{m}$: upper bound on m, used in the appendix;

$x^*(m)$, $b^*(m)$: a buyer's optimal choice of x and b, respectively, given money holdings m;

x^*_m and b^*_m: shortened notation for $x^*(m)$ and $b^*(m)$, and used only in the appendix;

$\tilde{x}\,(b, m)$: the optimal choice of x given (b, m), with $x^*(m) = \tilde{x}\,(b^*(m), m)$;

$m^+ \equiv \lim_{\varepsilon \downarrow 0}\,(m + \varepsilon)$;

$m^- \equiv \lim_{\varepsilon \downarrow 0}\,(m - \varepsilon)$;

m': an arbitrary level of money holdings, used as opposed to m;

m_0: winning size of money in a lottery participated by the individuals who hold low quantities of money;

m_a: average money holdings (an infrequent notation);

m_b: the highest level of money holdings at which the money constraint binds in trade (an infrequent notation);

$G(m)$: measure of individuals whose money holdings are equal to or less than m after the outcomes of the lotteries are realized and before the goods market opens;

$G_a(m)$: measure of individuals whose holdings are equal to or less than m after production and trading in a period;

$d\,G(m)$ and $d\,G_a(m)$: density functions associated with G and G_a, respectively;

$g(m) = d\,G(m)$ and $g_a(m) = d\,G_a(m)$: used when G is discrete;

G_{+1}: distribution function in the next period;

$\phi(m) = m - x^*(m)$: a buyer's money holdings after purchase;

$d\,N(x, q)$: number of trading posts in submarket (x, q);

$t\,(x, b, m) = u(x, b) + \beta V\,(m - x) - \beta V(m)$: used in the appendix;

$F\,(x, b, m) = b\,t\,(x, b, m)$: used in the appendix;

$\tilde{x}\,(b, m)$: a buyer's optimal choice of x given (b, m), a notation used in the appendix;

$t^*\,(b, m) = t\,(\tilde{x}\,(b, m)\,, b, m)$: a notation used in the appendix;

$f\,(b, m) = F\,(\tilde{x}\,(b, m)\,, b, m) = b\,t^*\,(b, m)$: a notation used in the appendix;

$J\,(z) = \arg\max_q \left[U\,(q) + \beta V\left(\frac{z - \psi(q)}{\omega}\right)\right]$, where $z = (m - x)\,\omega + \psi\,(q)$; used in the appendix;

$\tilde{q}\,(z)$: optimal choice of q in the above problem (for any given (z, m)); used in the appendix;

$\tilde{\phi}\,(z) = [z - \psi\,(\tilde{q}\,(z))]\,/\omega$: left-over money balance by a buyer in the above problem; used in the appendix;

$B\,(z, m) = \mu^{-1}\left(\frac{k}{\omega m - z}\right)\left[U\,(q) + \beta V\left(\frac{z - \psi(q)}{\omega}\right) - \beta V(m)\right]$, used in the appendix;

$F^*(m) = F\,(x^*_m, b^*_m, m)$: a notation used in the appendix.

β_0: critical level of β in the condition $\beta \geq \beta_0$ to guarantee $\phi(m^*) = 0$;

$\bar{q}, \underline{q}$: critical levels of q used to define β_0 in the appendix;

$a(b)$: a function defined in the appendix.