Fang *Journal of Inequalities and Applications* (2015) 2015:41
DOI 10.1186/s13660-015-0561-3

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Some relationships among the constraint qualifications for Lagrangian dualities in DC infinite optimization problems

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Abstract

In this paper, we establish some relationships among several constraint qualifications, which characterize strong Lagrangian dualities and total Lagrangian dualities for DC infinite optimization problems.

MSC: 90C26; 90C46

Keywords: basic constraint qualification; conical epigraph hull property; DC programming

1 Introduction

Consider the following DC infinite optimization problem:

$$(\mathrm{P})\quad \begin{array}{ll} \mathrm{Min} & f(x)-g(x), \\ \mathrm{s.\,t.} & f_t(x)-g_t(x)\le 0, \quad t\in T, \\ & x\in C, \end{array} \tag{1.1}$$

where T is an arbitrary (possibly infinite) index set, C is a nonempty convex subset of a locally convex Hausdorff topological vector space X and $f, g, f_t, g_t : X \to \overline{\mathbb{R}} := \mathbb{R}\cup\{+\infty\}$, $t\in T$, are proper convex functions. This problem has been studied extensively by many researchers. For example, the authors in [1–11] studied Lagrange dualities, Farkas lemmas, and optimality condition in the case when $g = g_t = 0$, $t \in T$ and the authors in [12] established the Fenchel-Lagrange duality in the case when $X = \mathbb{R}^n$ and T is finite, and Sun *et al.* gave some dualities and Farkas-type results in [13, 14]. In particular, the authors in [15] defined the dual problem of (1.1) by

$$(\mathrm{D})\quad \sup_{\lambda\in\mathbb{R}_+^{(T)}}\ \inf_{w^*\in H^*} L(w^*,\lambda), \tag{1.2}$$

where $H^* = \operatorname{dom} g^* \times \prod_{t\in T}\operatorname{dom} g_t^*$, and the Lagrange function $L : H^*\times\mathbb{R}_+^{(T)} \to \overline{\mathbb{R}}$ for (1.1) is defined by

$$L(w^*,\lambda) := g^*(u^*) + \sum_{t\in T}\lambda_t g_t^*(v_t^*) - \Bigl(f+\delta_C+\sum_{t\in T}\lambda_t f_t\Bigr)^*\Bigl(u^*+\sum_{t\in T}\lambda_t v_t^*\Bigr) \tag{1.3}$$

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for any $(w^*,\lambda) \in H^* \times \mathbb{R}_+^{(T)}$ with $w^* = (u^*,(v_t^*)) \in H^*$ and $\lambda = (\lambda_t) \in \mathbb{R}_+^{(T)}$, and they established some Lagrangian dualities between (P) and (D).

Usually, the main interest for the above optimization problems is focused on two aspects: one is about strong Lagrangian duality and the other is about total Lagrangian duality. For the strong Lagrangian duality for problem (1.1), one seeks conditions ensuring

$$\inf_{x\in A}\{f(x)-g(x)\} = \max_{\lambda\in\mathbb{R}_+^{(T)}} \inf_{w^*\in H^*} L(w^*,\lambda); \tag{1.4}$$

and, for the problem of total Lagrangian duality, one seeks conditions ensuring the following equality holds:

$$\min_{x\in A}\{f(x)-g(x)\} = \max_{\lambda\in\mathbb{R}_+^{(T)}} \inf_{w^*\in H^*} L(w^*,\lambda), \tag{1.5}$$

where $A := \{x \in C : f_t(x) - g_t(x) \le 0, \text{for each } t \in T\}$. To establish the strong Lagrangian duality, the authors in [15] introduced the following constraint qualification (the conical $(WEHP)$):

$$\begin{aligned}\operatorname{epi}(f-g+\delta_A)^* = \bigcup_{\lambda\in\mathbb{R}_+^{(T)}} \Biggl(\bigcap_{(u^*,v^*)\in H^*} \Biggl(\operatorname{epi}\Bigl(f+\delta_C+\sum_{t\in T}\lambda_t f_t\Bigr)^* - \bigl(u^*,g^*(u^*)\bigr) \\ - \sum_{t\in T}\lambda_t\bigl(v_t^*,g_t^*(v_t^*)\bigr)\Biggr)\Biggr),\end{aligned}$$

and to consider the total Lagrangian duality, the authors in [16] introduced two constraint qualifications: the quasi-$(WBCQ)$

$$\partial(f-g+\delta_A)(x) \subseteq \bigcup_{\lambda\in\mathbb{R}_+^{(T)}} \Biggl(\bigcap_{(u^*,v^*)\in\partial H(x)} \Biggl(\partial\Bigl(f+\delta_C+\sum_{t\in T(x)}\lambda_t f_t\Bigr)(x) - u^* - \sum_{t\in T(x)}\lambda_t v_t^*\Biggr)\Biggr),$$

and the $(WBCQ)$

$$\partial(f-g+\delta_A)(x) \subseteq \bigcup_{\lambda\in\mathbb{R}_+^{(T)}} \Biggl(\bigcap_{(u^*,v^*)\in H^*} \Biggl(\partial\Bigl(f+\delta_C+\sum_{t\in T(x)}\lambda_t f_t\Bigr)(x) - u^* - \sum_{t\in T(x)}\lambda_t v_t^*\Biggr)\Biggr),$$

where $\partial H(x) := \partial g(x) \times \prod_{t\in T}\partial g_t(x)$, for each $x \in X$ and $T(x) := \{t \in T : f_t(x) - g_t(x) = 0\}$.

In this paper, we continuous to study the general case, that is, C is not necessarily closed and $f, g, f_t, g_t, t \in T$, are not necessarily lsc. Our main aim in the present paper is focused on the relationships among the conical $(WEHP)$, the quasi-$(WBCQ)$, and the $(WBCQ)$. The paper is organized as follows. The next section contains some necessary notations and preliminary results. In Section 3, some relationships among the conical $(WEHP)$, the quasi-$(WBCQ)$, and the $(WBCQ)$ are obtained and some examples illustrating the relationships are given.

2 Notations and preliminaries

The notations used in this paper are standard (*cf.* [17]). In particular, we assume throughout the whole paper that X is a real locally convex space and let X^* denote the dual space of X. For $x \in X$ and $x^* \in X^*$, we write $\langle x^*, x\rangle$ for the value of x^* at x, that is, $\langle x^*, x\rangle := x^*(x)$. Let Z be a set in X. The closure of Z is denoted by $\operatorname{cl} Z$. If $W \subseteq X^*$, then $\operatorname{cl} W$ denotes the weak*-closure of W. For the whole paper, we endow $X^* \times \mathbb{R}$ with the product topology of $w^*(X^*, X)$ and the usual Euclidean topology.

The normal cone of Z at $z_0 \in Z$ is denoted by $N_Z(z_0)$ and is defined by

$$N_Z(z_0) = \{x^* \in X^* : \langle x^*, z - z_0\rangle \le 0 \text{ for all } z \in Z\}.$$

The indicator function δ_Z of Z is defined by

$$\delta_Z(x) := \begin{cases} 0, & x \in Z, \\ +\infty, & \text{otherwise}. \end{cases}$$

Let f be a proper function defined on X. The effective domain, the conjugate function, and the epigraph of f are denoted by $\operatorname{dom} f$, f^*, and $\operatorname{epi} f$, respectively; they are defined by

$$\operatorname{dom} f := \{x \in X : f(x) < +\infty\},$$
$$f^*(x^*) := \sup\{\langle x^*, x\rangle - f(x) : x \in X\}, \quad \text{for each } x^* \in X^*,$$

and

$$\operatorname{epi} f := \{(x, r) \in X \times \mathbb{R} : f(x) \le r\}.$$

It is well known and easy to verify that $\operatorname{epi} f^*$ is weak*-closed. The closure of f is denoted by $\operatorname{cl} f$, which is defined by

$$\operatorname{epi}(\operatorname{cl} f) = \operatorname{cl}(\operatorname{epi} f).$$

Then (*cf.* [17, Theorems 2.3.1]),

$$f^* = (\operatorname{cl} f)^*. \tag{2.1}$$

By [17, Theorem 2.3.4], if $\operatorname{cl} f$ is proper and convex, then the following equality holds:

$$f^{**} = \operatorname{cl} f. \tag{2.2}$$

Let $x \in X$. The subdifferential of f at x is defined by

$$\partial f(x) := \{x^* \in X^* : f(x) + \langle x^*, y - x\rangle \le f(y), \text{for each } y \in X\} \tag{2.3}$$

if $x \in \operatorname{dom} f$, and $\partial f(x) := \emptyset$ otherwise. We also define

$$\operatorname{dom} \partial f = \{x \in X : \partial f(x) \ne \emptyset\},$$

and

$$\operatorname{Im} \partial f = \{x^* \in X^* : x^* \in \partial f(x) \text{ for some } x \in X\}.$$

By [17, Theorems 2.3.1 and 2.4.2(iii)], the Young-Fenchel inequality below holds:

$$f(x) + f^*\bigl(x^*\bigr) \geq \bigl\langle x, x^*\bigr\rangle, \quad \text{for each pair } \bigl(x, x^*\bigr) \in X \times X^*, \tag{2.4}$$

and the Young equality holds:

$$f(x) + f^*\bigl(x^*\bigr) = \bigl\langle x^*, x\bigr\rangle \quad \text{if and only if} \quad x^* \in \partial f(x). \tag{2.5}$$

Furthermore, if g, h are proper functions, then

$$\operatorname{epi} g^* + \operatorname{epi} h^* \subseteq \operatorname{epi}(g+h)^*, \tag{2.6}$$

$$g \leq h \quad \Rightarrow \quad g^* \geq h^* \quad \Leftrightarrow \quad \operatorname{epi} g^* \subseteq \operatorname{epi} h^*, \tag{2.7}$$

and

$$\partial g(a) + \partial h(a) \subseteq \partial(g+h)(a), \quad \text{for each } a \in \operatorname{dom} g \cap \operatorname{dom} h. \tag{2.8}$$

We end this section with the remark that an element $p \in X^*$ can be naturally regarded as a function on X in such way that

$$p(x) := \langle p, x\rangle, \quad \text{for each } x \in X. \tag{2.9}$$

Thus the following fact is clear for any $a \in \mathbb{R}$ and real-valued proper function f:

$$\operatorname{epi}(f + p + a)^* = \operatorname{epi} f^* + (p, -a). \tag{2.10}$$

3 Relationships among constraint qualifications

Let X be a real locally convex Hausdorff vector space, and $C \subseteq X$ be a convex set. Let T be an index set and let f, g, f_t, g_t, $t \in T$ be proper convex functions such that $f - g$ and $f_t - g_t$, $t \in T$, are proper functions. Here and throughout the whole paper, following [17, p.39], we adapt the convention that $(+\infty) + (-\infty) = (+\infty) - (+\infty) = +\infty$, $0 \cdot (+\infty) = +\infty$, and $0 \cdot (-\infty) = 0$. Then

$$\emptyset \neq \operatorname{dom} f \subseteq \operatorname{dom} g \quad \text{and} \quad \emptyset \neq \operatorname{dom} f_t \subseteq \operatorname{dom} g_t. \tag{3.1}$$

Let $A \neq \emptyset$ be the solution set of the following system with the assumption that $A \cap \operatorname{dom}(f - g)$ is nonempty:

$$x \in C; \quad f_t(x) - g_t(x) \leq 0, \quad \text{for each } t \in T, \tag{3.2}$$

and let A^{cl} be the solution set of the following system:

$$x \in C; \quad f_t(x) - \operatorname{cl} g_t(x) \leq 0, \quad \text{for each } t \in T. \tag{3.3}$$

Then $A^{\mathrm{cl}} \subseteq A$. Following [18], we use $\mathbb{R}^{(T)}$ to denote the space of real tuples $\lambda = (\lambda_t)$ with only finitely many $\lambda_t \neq 0$, and let $\mathbb{R}_+^{(T)}$ denote the nonnegative cone in $\mathbb{R}^{(T)}$, that is,

$$\mathbb{R}_+^{(T)} := \bigl\{\lambda = (\lambda_t) \in \mathbb{R}^{(T)} : \lambda_t \geq 0, \text{for each } t \in T\bigr\}.$$

For simplicity, we denote

$$H^* := \operatorname{dom} g^* \times \prod_{t\in T} \operatorname{dom} g_t^*$$

and

$$\partial H(x) := \partial g(x) \times \prod_{t\in T} \partial g_t(x), \quad \text{for each } x \in X.$$

To make the dual problem considered here well defined, we further assume that $\operatorname{cl} g$ and $\operatorname{cl} g_t$, $t \in T$, are proper. Then $H^* \neq \emptyset$. For the whole paper, any elements $\lambda \in \mathbb{R}^{(T)}$ and $v^* \in \prod_{t\in T} \operatorname{dom} g_t^*$ are understood as $\lambda = (\lambda_t) \in \mathbb{R}^{(T)}$ and $v^* = (v_t^*) \in \prod_{t\in T} \operatorname{dom} g_t^*$, respectively. Following [15], we define the characteristic set K for the DC optimization problem (1.1) by

$$K := \bigcup_{\lambda\in\mathbb{R}_+^{(T)}} \left(\bigcap_{(u^*,v^*)\in H^*} \left(\operatorname{epi}\left(f + \delta_C + \sum_{t\in T} \lambda_t f_t\right)^* - \left(u^*, g^*(u^*)\right) - \sum_{t\in T} \lambda_t \left(v_t^*, g_t^*(v_t^*)\right) \right) \right), \tag{3.4}$$

where we adopt the convention that $\bigcap_{t\in\emptyset} S_t = X$ (see [17, p.2]). Below we will make use of the subdifferential $\partial h(x)$ for a general proper function (not necessarily convex) $h : X \to \overline{\mathbb{R}}$; see (2.3). Clearly, the following equivalence holds:

$$x_0 \text{ is a minimizer of } h \text{ if and only if } 0 \in \partial h(x_0). \tag{3.5}$$

For each $x \in X$, let $T(x)$ be the active index set of system (3.2), that is,

$$T(x) := \left\{ t \in T : f_t(x) - g_t(x) = 0 \right\}.$$

Define $N'(x)$ by

$$N'(x) := \bigcup_{\lambda\in\mathbb{R}_+^{(T)}} \left(\bigcap_{(u^*,v^*)\in H^*} \left(\partial\left(f + \delta_C + \sum_{t\in T(x)} \lambda_t f_t\right)(x) - u^* - \sum_{t\in T(x)} \lambda_t v_t^* \right) \right) \tag{3.6}$$

and define $N_0'(x)$ by

$$N_0'(x) := \bigcup_{\lambda\in\mathbb{R}_+^{(T)}} \left(\bigcap_{(u^*,v^*)\in \partial H(x)} \left(\partial\left(f + \delta_C + \sum_{t\in T(x)} \lambda_t f_t\right)(x) - u^* - \sum_{t\in T(x)} \lambda_t v_t^* \right) \right). \tag{3.7}$$

Then, for each $x \in X$,

$$N'(x) \subseteq N_0'(x).$$

Definition 3.1 The family $\{f, g, \delta_C; f_t, g_t : t \in T\}$ is said to satisfy

(a) the lower semi-continuity closure ((LSC)) if

$$\operatorname{epi}(f - g + \delta_A)^* = \operatorname{epi}(f - \operatorname{cl} g + \delta_{A^{\mathrm{cl}}})^*; \tag{3.8}$$

(b) the conical weak epigraph hull property ((*WEHP*)) if

$$\operatorname{epi}(f - g + \delta_A)^* = K; \tag{3.9}$$

(c) the quasi-weakly basic constraint qualification (the quasi-(*WBCQ*)) at $x \in A$ if

$$\partial(f - g + \delta_A)(x) \subseteq N_0'(x); \tag{3.10}$$

(d) the weakly basic constraint qualification (the (*WBCQ*)) at $x \in A$ if

$$\partial(f - g + \delta_A)(x) \subseteq N'(x). \tag{3.11}$$

It is said that the family $\{f, g, \delta_C; f_t, g_t : t \in T\}$ satisfies the quasi-(*WBCQ*) (resp. the (*WBCQ*)) if it satisfies the quasi-(*WBCQ*) (resp. the (*WBCQ*)) at each point $x \in A$.

Remark 3.1

(a) The notions of (*LSC*) and the conical (*WEHP*) were introduced in [15] and the quasi-(*WBCQ*) and the (*WBCQ*) were taken from [16].

(b) Recall from [3, 4] that the family $\{\delta_C; f_t : t \in T\}$ has the conical $(WEHP)_f$ if

$$\operatorname{epi}(f + \delta_A)^* = \bigcup_{\lambda \in R_+^{(T)}} \operatorname{epi}\left(f + \delta_C + \sum_{t\in T} \lambda_t f_t\right)^* \tag{3.12}$$

and has the $(WBCQ)_f$ at $x \in \operatorname{dom} f \cap A$ if

$$\partial(f + \delta_A)(x) = \bigcup_{\substack{\lambda \in R_+^{(T)} \\ \sum_{t\in T} \lambda_t f_t(x) = 0}} \partial\left(f + \delta_C + \sum_{t\in T} \lambda_t f_t\right)(x). \tag{3.13}$$

Thus, in the special case when $g = g_t = 0$, $t \in T$, the conical (*WEHP*) coincides with the conical $(WEHP)_f$ for the family $\{\delta_C; f_t : t \in T\}$ and the quasi-(*WBCQ*) and (*WBCQ*) are reduced to the $(WBCQ)_f$ for the family $\{\delta_C; f_t : t \in T\}$.

Theorems 3.1 and 3.2 characterize the relationships among the quasi-(*WBCQ*), the (*WBCQ*), and the conical (*WEHP*).

Theorem 3.1 *The following implication holds*:

$$\left[\operatorname{epi}(f - g + \delta_A)^* \subseteq K\right] \quad \Longrightarrow \quad \textit{the quasi-}(WBCQ). \tag{3.14}$$

Consequently,

$$\textit{the conical } (WEHP) \quad \Longrightarrow \quad \textit{the quasi-}(WBCQ). \tag{3.15}$$

Proof Suppose that $\operatorname{epi}(f - g + \delta_A)^* \subseteq K$. To show the quasi-(*WBCQ*), let $x_0 \in A$ and let $x^* \in \partial(f - g + \delta_A)(x_0)$. Then, by (2.5),

$$\langle x^*, x_0\rangle - (f - g + \delta_A)(x_0) = (f - g + \delta_A)^*(x^*).$$

This implies that

$$\left(x^*, \langle x^*, x_0\rangle - (f - g + \delta_A)(x_0)\right) \in \operatorname{epi}(f - g + \delta_A)^* \subseteq K.$$

Hence, there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that, for each $(u^*, v^*) \in \partial H(x_0)$,

$$\left(x^*, \langle x^*, x_0\rangle - (f - g + \delta_A)(x_0)\right) \in \operatorname{epi}\left(f + \delta_C + \sum_{t\in T} \lambda_t f_t\right)^* - \left(u^*, g^*(u^*)\right) - \sum_{t\in T} \lambda_t\left(v_t^*, g_t^*(v_t^*)\right).$$

Let $(u^*, v^*) \in \partial H(x_0)$. There exists $(x_1^*, r_1) \in \operatorname{epi}(f + \delta_C + \sum_{t\in J} \lambda_t f_t)^*$ such that

$$x^* = x_1^* - u^* - \sum_{t\in J} \lambda_t v_t^* \tag{3.16}$$

and

$$\langle x^*, x_0\rangle - (f - g + \delta_A)(x_0) = r_1 - g^*(u^*) - \sum_{t\in J} \lambda_t g_t^*(v_t^*), \tag{3.17}$$

where $J := \{t \in T : \lambda_t \neq 0\}$ is a finite subset of T. Below we only need to show that $x_1^* \in \partial(f + \delta_C + \sum_{t\in J} \lambda_t f_t)(x_0)$ and $J \subseteq T(x_0)$. To do this, note by the definition of epigraph, one has

$$\left(f + \delta_C + \sum_{t\in J} \lambda_t f_t\right)^*(x_1^*) \le r_1. \tag{3.18}$$

Note that $(u^*, v^*) \in \partial H(x_0)$, it follows from (2.5) that

$$g(x_0) + g^*(u^*) = \langle u^*, x_0\rangle \quad \text{and} \quad g_t(x_0) + g_t^*(v_t^*) = \langle v_t^*, x_0\rangle, \quad \text{for each } t \in T. \tag{3.19}$$

This together with (3.16), (3.17), and (3.18) implies that

$$\begin{aligned}
&\left(f + \delta_C + \sum_{t\in J} \lambda_t f_t\right)^*(x_1^*) \\
&\quad \le \langle x^*, x_0\rangle - (f - g + \delta_A)(x_0) + g^*(u^*) + \sum_{t\in J} \lambda_t g_t^*(v_t^*) \\
&\quad \le \left\langle x_1^* - u^* - \sum_{t\in J} \lambda_t v_t^*, x_0\right\rangle - \left(f - g + \delta_C + \sum_{t\in J} \lambda_t(f_t - g_t)\right)(x_0) \\
&\qquad + g^*(u^*) + \sum_{t\in J} \lambda_t g_t^*(v_t^*) \\
&\quad \le \langle x_1^*, x_0\rangle - \left(f + \delta_C + \sum_{t\in J} \lambda_t f_t\right)(x_0) + \left\{g(x_0) - \langle u^*, x_0\rangle + g^*(u^*)\right\} \\
&\qquad + \sum_{t\in J} \lambda_t \left\{g_t(x_0) - \langle v_t^*, x_0\rangle + g_t^*(v_t^*)\right\} \\
&\quad = \langle x_1^*, x_0\rangle - \left(f + \delta_C + \sum_{t\in J} \lambda_t f_t\right)(x_0),
\end{aligned}$$

where the second inequality holds because $x_0 \in A$. Hence,

$$\left(f + \delta_C + \sum_{t\in J} \lambda_t f_t\right)^*(x_1^*) + \left(f + \delta_C + \sum_{t\in J} \lambda_t f_t\right)(x_0) = \langle x_1^*, x_0\rangle$$

since

$$\left(f + \delta_C + \sum_{t\in J} \lambda_t f_t\right)^*(x_1^*) \geq \langle x_1^*, x_0\rangle - \left(f + \delta_C + \sum_{t\in J} \lambda_t f_t\right)(x_0)$$

holds automatically by the Fenchel-Young inequality (2.4). Therefore, by (2.5), $x^* \in \partial(f + \delta_C + \sum_{t\in J} \lambda_t f_t)(x_0)$. To show $J \subseteq T(x_0)$, note that $x_0 \in A$, then

$$\left(f + \delta_C + \sum_{t\in J} \lambda_t f_t\right)^*(x_1^*) \leq \langle x^*, x_0\rangle - f(x_0) + g(x_0) + g^*(u^*) + \sum_{t\in J} \lambda_t g_t^*(v_t^*)$$

and

$$\left(f + \delta_C + \sum_{t\in J} \lambda_t f_t\right)^*(x_1^*) \geq \langle x_1^*, x_0\rangle - f(x_0) - \sum_{t\in J} \lambda_t f_t(x_0).$$

Thus, by (3.16) and (3.19), we have

$$\begin{aligned} f(x_0) - g(x_0) - \langle x^*, x_0\rangle &\leq g^*(u^*) + \sum_{t\in J} \lambda_t g_t^*(v_t^*) - \left(f + \delta_C + \sum_{t\in J} \lambda_t f_t\right)^*(x_1^*) \\ &\leq g^*(u^*) + \sum_{t\in J} \lambda_t g_t^*(v_t^*) - \langle x_1^*, x_0\rangle + f(x_0) + \sum_{t\in J} \lambda_t f_t(x_0) \\ &= f(x_0) - g(x_0) - \langle x^*, x_0\rangle + \sum_{t\in J} \lambda_t\big(f_t(x_0) - g_t(x_0)\big) \\ &\leq f(x_0) - g(x_0) - \langle x^*, x_0\rangle. \end{aligned}$$

Since $\lambda_t > 0$ and $f_t(x_0) - g_t(x_0) \leq 0$, for each $t \in J$, it follows that $\lambda_t(f_t(x_0) - g_t(x_0)) = 0$, that is, $f_t(x_0) - g_t(x_0) = 0$, for each $t \in J$. Thus, $J \subseteq T(x_0)$ and hence the quasi-(*WBCQ*) holds. □

Theorem 3.2 *If* $\operatorname{dom}(f - g + \delta_A)^* \subseteq \operatorname{im}\partial(f - g + \delta_A)$, *then*

$$\text{the } (WBCQ) \quad \Longrightarrow \quad \left[\operatorname{epi}(f - g + \delta_A)^* \subseteq K\right]. \tag{3.20}$$

Furthermore, if the (*LSC*) *holds, then*

$$\text{the } (WBCQ) \Longrightarrow \text{ the conical } (WEHP). \tag{3.21}$$

Proof Suppose that $\operatorname{dom}(f-g+\delta_A)^* \subseteq \operatorname{im}\partial(f-g+\delta_A)$ and that the (*WBCQ*) holds. To show $\operatorname{epi}(f-g+\delta_A)^* \subseteq K$, let $(x^*, \alpha) \in \operatorname{epi}(f-g+\delta_A)^*$. Since $x^* \in \operatorname{dom}(f-g+\delta_A)^* \subseteq \operatorname{im}\partial(f-g+\delta_A)$, it follows that there exists $x_0 \in \operatorname{dom}(f - g) \cap A$ such that $x^* \in \partial(f - g + \delta_A)(x_0) \subseteq N'(x_0)$, thanks to the assumed (*WBCQ*). This means that there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that, for each $(u^*, v^*) \in H^*$,

$$x^* \in \partial\left(f + \delta_C + \sum_{t\in J} \lambda_t f_t\right)(x_0) - u^* - \sum_{t\in J} \lambda_t v_t^*$$

for some finite subset $J \subseteq T(x_0)$ and $\{\lambda_t\} \subseteq \mathbb{R}$ with $\lambda_t \geq 0$, for each $t \in J$. Let $(u^*, v^*) \in H^*$. Then there exists $x_1^* \in \partial(f + \delta_C + \sum_{t\in J} \lambda_t f_t)(x_0)$ such that

$$x^* = x_1^* - u^* - \sum_{t\in J} \lambda_t v_t^*. \tag{3.22}$$

By the Young equality (2.5), we have

$$\langle x_1^*, x_0\rangle = \left(f + \delta_C + \sum_{t\in J} \lambda_t f_t\right)^*(x_1^*) + \left(f + \delta_C + \sum_{t\in J} \lambda_t f_t\right)(x_0) \tag{3.23}$$

and

$$\langle x^*, x_0\rangle = (f - g + \delta_A)^*(x^*) + (f - g + \delta_A)(x_0) \leq \alpha + f(x_0) - g(x_0), \tag{3.24}$$

where the last inequality holds because of $(x^*, \alpha) \in \operatorname{epi}(f - g + \delta_A)^*$ and $x_0 \in A$. This together with (3.22) and (3.23) implies that

$$\begin{aligned}\left(f + \delta_C + \sum_{t\in J} \lambda_t f_t\right)^*(x_1^*) &\leq \langle u^*, x_0\rangle + \sum_{t\in J} \lambda_t\langle v_t^*, x_0\rangle + \alpha - g(x_0) - \sum_{t\in J} \lambda_t f_t(x_0) \\ &\leq \alpha + g^*(u^*) + \sum_{t\in J} g_t^*(v_t^*) - \sum_{t\in J} \lambda_t\big(f_t(x_0) - g_t(x_0)\big) \\ &= \alpha + g^*(u^*) + \sum_{t\in J} g_t^*(v_t^*),\end{aligned}$$

where the second inequality holds by the Fenchel-Young inequality and the last equality holds because $J \subseteq T(x_0)$. This means that

$$\left(x_1^*, \alpha + g^*(u^*) + \sum_{t\in J} g_t^*(v_t^*)\right) \in \operatorname{epi}\left(f + \delta_C + \sum_{t\in J} \lambda_t f_t\right)^*.$$

Hence,

$$\begin{aligned}(x^*, \alpha) &= \left(x_1^*, \alpha + g^*(u^*) + \sum_{t\in J} g_t^*(v_t^*)\right) - \big(u^*, g^*(u^*)\big) - \sum_{t\in J} \lambda_t\big(v_t^*, g_t^*(v_t^*)\big) \\ &\in \operatorname{epi}\left(f + \delta_C + \sum_{t\in J} \lambda_t f_t\right)^* - \big(u^*, g^*(u^*)\big) - \sum_{t\in J} \lambda_t\big(v_t^*, g_t^*(v_t^*)\big)\end{aligned}$$

and so $(x^*, \alpha) \in K$ by the arbitrary of $(u^*, v^*) \in H^*$. Therefore,

$$\operatorname{epi}(f - g + \delta_A)^* \subseteq K. \tag{3.25}$$

Furthermore, we assume that the (*LSC*) holds. Then (3.8) holds. By [15, Lemma 3.1], we see that

$$K = \bigcup_{\lambda\in\mathbb{R}_+^{(T)}} \left(f - \operatorname{cl} g + \delta_C + \sum_{t\in T} \lambda_t(f_t - \operatorname{cl} g_t)\right)^*; \tag{3.26}$$

while by [3, (3.5)],

$$\bigcup_{\lambda\in\mathbb{R}_+^{(T)}} \left(f - \operatorname{cl} g + \delta_C + \sum_{t\in T} \lambda_t(f_t - \operatorname{cl} g_t)\right)^* \subseteq \operatorname{epi}(f - \operatorname{cl} g + \delta_{A^{\mathrm{cl}}})^*. \tag{3.27}$$

Combining (3.26), (3.27) with (3.8), we have

$$K \subseteq \operatorname{epi}(f - g + \delta_A)^*. \tag{3.28}$$

Hence, by (3.25), the conical (*WEHP*) holds and the proof is complete. □

Remark 3.2 By [16, Remark 3.2], we see that

$$\text{the } (WBCQ) \quad \Longrightarrow \quad \text{the quasi-}(WBCQ)$$

and by Theorems 3.1 and 3.2, we get

$$\begin{aligned}&\big[\text{the } (WBCQ)\ \&\ \operatorname{dom}(f - g + \delta_A)^* \subseteq \operatorname{im}\partial(f - g + \delta_A)\ \&\ \text{the } (LSC)\big]\\ &\quad\Longrightarrow \quad \text{the conical } (WEHP) \quad \Longrightarrow \quad \text{the quasi-}(WBCQ).\end{aligned}$$

By Theorems 3.1 and 3.2, we get the following corollary directly, which was given in [4, Proposition 3.1]. Note that the conical $(WEHP)_f$ and the $(WBCQ)_f$ for the family $\{\delta_C; f_t : t \in T\}$ were introduced in [3, 4]; see also Remark 3.1(ii).

Corollary 3.1 *For the family $\{\delta_C; f_t : t \in T\}$, the following implication holds*:

$$\textit{the conical } (WEHP)_f \quad \Longrightarrow \quad \textit{the quasi-}(WBCQ)_f$$

and

$$\textit{the conical } (WEHP)_f \quad \Longleftrightarrow \quad \textit{the quasi-}(WBCQ)_f$$

if $\operatorname{dom}(f + \delta_A)^* \subseteq \operatorname{im}\partial(f + \delta_A)$.

The following example illustrates (3.14) and shows that the quasi-(*WBCQ*) in (3.14) cannot be replaced by the (*WBCQ*).

Example 3.1 Let $X = C := \mathbb{R}$ and let $T = \{1\}$. Define $f, g, f_1, g_1 : \mathbb{R} \to \overline{\mathbb{R}}$, respectively, by

$$f(x) := \begin{cases} x, & x \geq 0, \\ +\infty, & x < 0, \end{cases} \qquad g(x) := \begin{cases} 0, & x > 0, \\ 1, & x = 0, \\ +\infty, & x < 0, \end{cases} \quad \text{for each } x \in \mathbb{R},$$

$f_1 := \delta_{[0,+\infty)}$ and $g_1 := 0$. Then f, g, f_1, and g_1 are proper convex functions and $A = [0, +\infty)$. Note that, for each $x \in \mathbb{R}$,

$$(f - g + \delta_A)(x) = \begin{cases} x, & x > 0, \\ -1, & x = 0, \\ +\infty, & x < 0, \end{cases}$$

and $f + \delta_C + \lambda f_1 = f$ holds, for each $\lambda \geq 0$. Then, for each $x^* \in \mathbb{R}$, $g^* = \delta_{(-\infty,0]}$,

$$(f - g + \delta_A)^*(x^*) = \begin{cases} 1, & x^* \leq 1, \\ +\infty, & x^* > 1, \end{cases}$$

and, for each $\lambda \geq 0$,

$$(f + \delta_C + \lambda f_1)^*(x^*) = \begin{cases} 0, & x^* \leq 1, \\ +\infty, & x^* > 1. \end{cases}$$

This means that $\operatorname{dom} g^* = (-\infty, 0]$,

$$\operatorname{epi}(f - g + \delta_A)^* = (-\infty, 1] \times [1, +\infty)$$

and

$$\operatorname{epi}(f + \delta_C + \lambda f_1)^* = (-\infty, 1] \times [0, +\infty), \quad \text{for each } \lambda \geq 0.$$

Hence

$$K = \bigcup_{\lambda \geq 0} \left(\bigcap_{u^* \in (-\infty,0]} \left(\operatorname{epi}(f + \delta_C + \lambda f_1)^* - \left(u^*, g^*(u^*)\right)\right) \right) = (-\infty, 1] \times [0, +\infty).$$

This implies that $\operatorname{epi}(f - g + \delta_A)^* \subseteq K$. Moreover, it is easy to see that, for each $x \in A$,

$$\partial g(x) = \begin{cases} \{0\}, & x > 0, \\ \emptyset, & x = 0, \end{cases}$$

and, for each $\lambda \geq 0$,

$$\partial(f - g + \delta_A)(x) = \partial(f + \delta_C + \lambda f_1)(x) = \begin{cases} 1, & x > 0, \\ (-\infty, 1], & x = 0. \end{cases}$$

Hence, for each $x \in A$,

$$N_0'(x) = \bigcup_{\lambda \geq 0} \left(\bigcap_{u^* \in \partial g(x)} \left(\partial(f + \delta_C + \lambda_1 f_1)(x) - u^* \right) \right) = \begin{cases} 1, & x > 0, \\ \mathbb{R}, & x = 0, \end{cases}$$

and

$$N'(x) = \bigcup_{\lambda \geq 0} \left(\bigcap_{u^* \in \operatorname{dom} g^*} \left(\partial(f + \delta_C + \lambda_1 f_1)(x) - u^* \right) \right) = \begin{cases} \emptyset, & x > 0, \\ (-\infty, 1], & x = 0. \end{cases}$$

This means that $\partial(f-g+\delta_A)(x) \subseteq N_0'(x)$ but $\partial(f-g+\delta_A)(x) \not\subseteq N'(x)$, for each $x \in A$. Thus, the quasi-($WBCQ$) holds but not the ($WBCQ$).

Example 3.2 illustrates Theorem 3.2 and Example 3.3 shows that the condition (LSC) is essential for (3.21) to hold.

Example 3.2 Let $X = C := \mathbb{R}$. Define $f, g, f_1, g_1 : \mathbb{R} \to \overline{\mathbb{R}}$, respectively, by $f = f_1 = g := \delta_{(-\infty,0]}$, $g_1 := 0$. Then f, g, f_1, and g_1 are proper convex functions. Consider the system (3.2) with $T := \{1\}$. Then one sees that

$$A = \{x \in \mathbb{R} : f_1(x) - g_1(x) \le 0\} = (-\infty, 0].$$

It is easy to see that

$$f - g + \delta_A = \delta_A \quad \text{and} \quad (f - g + \delta_A)^* = \delta_{[0,+\infty)}.$$

Hence,

$$\operatorname{dom}(f - g + \delta_A)^* = [0, +\infty),$$

and, for each $x \in A$,

$$\partial(f - g + \delta_A)(x) = N_A(x) = \begin{cases} \{0\}, & x < 0, \\ [0, +\infty), & x = 0. \end{cases}$$

This implies that $\operatorname{dom}(f - g + \delta_A)^* \subseteq \operatorname{im} \partial(f - g + \delta_A)$. Note that $g_1^* = \delta_{\{0\}}$, $g^* = \delta_{[0,+\infty)}$, and $(f + \lambda f_1)^* = \delta_{[0,+\infty)}$, for each $\lambda \ge 0$. It follows that, for each $x \in A$,

$$N'(x) = \bigcup_{\lambda \ge 0} \left(\bigcap_{u^* \in [0,+\infty)} \left(N_A(x) - u^*\right) \right) = \begin{cases} \{0\}, & x < 0, \\ [0, +\infty), & x = 0. \end{cases}$$

Thus, $\partial(f - g + \delta_A)(x) = N'(x)$ and the ($WBCQ$) holds. Therefore, by Theorem 3.1, we see that $\operatorname{epi}(f - g + \delta_A)^* \subseteq K$. Moreover, since g is lsc, it follows that the (LSC) holds. Therefore, by (3.21), one sees that the conical ($WEHP$) holds. In fact, it is easy to see that

$$\operatorname{epi}(f - g + \delta_A)^* = [0, +\infty) \times [0, +\infty)$$

and

$$K = \bigcup_{\lambda \ge 0} \left(\bigcap_{u^* \in [0,+\infty)} \left(\operatorname{epi}(f + \lambda f_1)^* - \left(u^*, g^*\left(u^*\right)\right)\right) \right) = [0, +\infty) \times [0, +\infty).$$

Example 3.3 Let $X = C := \mathbb{R}$. Define $f, g, f_1, g_1 : \mathbb{R} \to \overline{\mathbb{R}}$ as in [15, Example 3.1], that is, $f = f_1 := \delta_{(-\infty,0]}$, $g_1 := 0$ and, for each $x \in \mathbb{R}$,

$$g(x) := \begin{cases} 0, & x < 0, \\ 1, & x = 0, \\ +\infty, & x > 0. \end{cases}$$

Then f, g, f_1, and g_1 are proper convex functions. Consider the system (3.2) with $T := \{1\}$. Then one sees that

$$A = \{x \in \mathbb{R} : f_1(x) - g_1(x) \leq 0\} = (-\infty, 0].$$

It is easy to see that, for each $x \in \mathbb{R}$,

$$(f - g + \delta_A)(x) = \begin{cases} 0, & x < 0, \\ -1, & x = 0, \\ +\infty, & x > 0, \end{cases}$$

and, for each $x^* \in \mathbb{R}$,

$$(f - g + \delta_A)^*(x^*) = \begin{cases} 1, & x^* \geq 0, \\ +\infty, & x^* < 0. \end{cases}$$

Moreover, for each $x \in A$, we see that

$$\partial(f - g + \delta_A)(x) = \begin{cases} \emptyset, & x < 0, \\ [0, +\infty), & x = 0. \end{cases}$$

Thus, $\operatorname{dom}(f - g + \delta_A)^* \subseteq \operatorname{im} \partial(f - g + \delta_A)$. Note that $g_1^* = \delta_{\{0\}}$, $g^* = \delta_{[0,+\infty)}$, and $(f + \lambda f_1)^* = \delta_{[0,+\infty)}$, for each $\lambda \geq 0$. It follows that, for each $x \in A$,

$$N'(x) = \bigcup_{\lambda \geq 0} \left(\bigcap_{u^* \in [0,+\infty)} \left(N_A(x) - u^*\right) \right) = \begin{cases} \{0\}, & x < 0, \\ [0, +\infty) & x = 0. \end{cases}$$

Therefore, the (*WBCQ*) holds. However, the conical (*WEHP*) does not hold as shown in Example 3.1 in [15]. Actually, the family $\{f, g, \delta_C; f_t, g_t : t \in T\}$ does not satisfy the (*LSC*), since

$$\operatorname{epi}(f - g + \delta_A)^* = [0, +\infty) \times [1, +\infty);$$

but

$$\operatorname{epi}(f - \operatorname{cl} g + \delta_A)^* = [0, +\infty) \times [0, +\infty).$$

Competing interests

The author declares that they have no competing interests.

Acknowledgements

The author is grateful to both reviewers for their many helpful suggestions and remarks, which improved the quality of the paper. This work was supported in part by the National Natural Science Foundation of China (grant 11461027) and supported in part by the Scientific Research Fund of Hunan Provincial Education Department (grant 13B095).

Received: 23 October 2014 Accepted: 14 January 2015 Published online: 03 February 2015

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