Zhou et al. *Advances in Difference Equations* 2013, 2013:314
http://www.advancesindifferenceequations.com/content/2013/1/314

Advances in Difference Equations
a SpringerOpen Journal

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Delay-probability-distribution-dependent stability criteria for discrete-time stochastic neural networks with random delays

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Abstract

The problem of delay-probability-distribution-dependent robust stability for a class of discrete-time stochastic neural networks (DSNNs) with delayed and parameter uncertainties is investigated. The information of the probability distribution of the delay is considered and transformed into parameter matrices of the transferred DSSN model. In the DSSN model, the time-varying delay is characterized by introducing a Bernoulli stochastic variable. By constructing an augmented Lyapunov-Krasovskii functional and introducing some analysis techniques, some novel delay-distribution-dependent mean square stability conditions for the DSSN, which are to be robustly globally exponentially stable, are derived. Finally, a numerical example is provided to demonstrate less conservatism and effectiveness of the proposed methods.

Keywords: discrete-time stochastic neural networks; discrete time-varying delays; delay-probability-distribution-dependent; robust exponential stability; LMIs

1 Introduction

In the past few decades, neural networks (NNs) have received considerable attention owing to their potential applications in a variety of areas such as signal processing [1], pattern recognition [2], static image processing [3], associative memory [4], combinatorial optimization [5] and so on. In recent years, the stability problem of time-delay NNs has become a topic of great theoretic and practical use importance due to the fact that inherent time delays and unavoidable parameter uncertainties are all well known to many biological and artificial NNs because of the finite speed of information processing as well as the NNs parameter fluctuations of the hardware implementation. Various efforts have been achieved in the stability analysis of NNs with time-varying delays and parameter uncertainties, please refer to [6–15] and some following references.

The majority of the existing research results have been limited in continuous-time and deterministic NNs. On the one hand, in implementation and application of the NNs, discrete-time neural networks (DNNs) play a more important role than their continuous-time counter-parts in today's digital world. To be more specific, DNNs can ideally keep the dynamical characteristics, functional similarity, and even the physical of biological reality of the continuous-time NNs under mild restriction. On the other hand, when modeling real NNs systems, stochastic disturbance is probably the main resource of the performance

degradation of the implemented NN. Thus, the research on the dynamical behavior of discrete-time stochastic neural networks (DSNNs) with time-varying delays and parameter uncertainties is necessary. Recently, stability analysis for DSNNs with time-varying delays and parameter uncertainties has received more and more interest. Some stability criteria have been proposed in [16–20]. In [16], Liu with his coauthor have researched a class of DSNNs with time-varying delays and parameter uncertainties and have proposed some delay-dependent sufficient conditions guaranteeing the global robust exponential stability by using the Lyapunov method and the linear matrix inequality technology. Employing the similar technique to that in [16], the result obtained in [16] has been improved by Zhang *et al.* in [17] and Luo with his coauthor in [18].

In practice, the time-varying delay in some NNs often exists in a stochastic fashion [21–26]. That is, the time-varying delay in some NNs may be subject to probabilistic measurement delays. In some NNs, the output signal of the node is transferred to another node by multi-branches with arbitrary time delays, which are random, and its probabilities can often be measured by the statistical methods such as normal distribution, uniform distribution, Poisson distribution, Bernoulli random binary distribution. In most of the existing references for DSNNs, the deterministic time delay case was concerned, and the stability criteria were derived based on the information of variation range of the time delay, [16–20], or the information of variation range of the time delay and time delays themselves [17] and [27]. However, it often occurs in the real systems, where the max value of the delay is very large, but the probability of it to take such a large value is very small. It may lead to a more conservative result if only the information of variation range of time delay is considered. Yet, as far as we know, little attention has been paid to the study of stability of DSNNs with stochastic time delay, when considering the variation range and the probability distribution of the time delay. More recently, in [28], some sufficient conditions on robust globally exponential stability for a class of SDNNs, which is an involved parameter, uncertainties and stochastic delay were derived. What is more, the robust globally exponential stability analysis problem for uncertain DSNNs with random delay has not been adequately investigated and still needs challenge.

In this paper, some new improved delay-probability-distribution-dependent stability criteria, which guarantee the robust global exponential stability for discrete-time stochastic neural networks with time-varying delay are obtained via constructing a novel augmented Lyapunov-Krasovskii functional. These new conditions are less conservative than those obtained in [16–18] and [28]. The numerical example is also provided to illuminate the improvement of the proposed criteria.

The notations are quite standard. Throughout this paper, N^+ stands for the set of nonnegative integers, R^n and $R^{n\times m}$ denote, respectively, the n-dimensioned Euclidean space and the set of all $n \times m$ real matrices. The superscript 'T' denotes the transpose and the notation $X \geq Y$ (respective $X > Y$) means that X and Y are symmetric matrices, and that $X - Y$ is positive semi-definitive (respective positive definite). $\|\cdot\|$ is the Euclidean norm in R^n. I is the identity matrix with appropriate dimensions. If A is a matrix, denote by $\|A\|$ its own operator norm, *i.e.*, $\|A\| = \sup\{\|Ax\| : \|x\| = 1\} = \sqrt{\lambda_{\max}(A^TA)}$, where $\lambda_{\max}(A)$ (respectively, $\lambda_{\min}(A)$) means the largest (respectively, smallest) eigenvalue of A. Moreover, let $(\Omega, F, \{F_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{F_t\}_{t\geq 0}$ to satisfy the usual conditions (*i.e.*, the filtration contains all P-null sets and is right continuous). $E\{\cdot\}$ stands for the mathematical expectation operator with respect to the given probability measure P.

The asterisk $*$ in a matrix is used to denote term that is induced by symmetry. Matrices, if not explicitly, are assumed to have compatible dimensions. $N[a,b] = \{a, a+1, \ldots, b\}$. Sometimes, the arguments of function will be omitted in the analysis when no confusion can arise.

2 Problem formulation and preliminaries

Consider the following n-neurons parameter uncertainties DSNN with time-varying delays:

$$\begin{aligned} x(k+1) &= \big(A + \Delta A(k)\big)x(k) + \big(B + \Delta B(k)\big)f\big(x(k)\big) \\ &\quad + \big(D + \Delta D(k)\big)g\big(x\big(k-\tau(k)\big)\big) + \sigma\big(k, x(k), x\big(k-\tau(k)\big)\big)\omega(k), \end{aligned} \tag{1}$$

where $x(k) = [x_1(k), x_2(k), \ldots, x_n(k)]^T \in R^n$ denotes the state vector associated with the n-neurons, the positive integer $\tau(k)$ denotes the time-varying delay, satisfying $\tau_m \le \tau(k) \le \tau_M$, $k \in N^+$, the τ_m and τ_M are known positive integers. The initial condition associated with model (1) is given by

$$x(k) = \phi(k), \quad k \in [-\tau_M, 0]. \tag{2}$$

The diagonal matrix $A = \operatorname{diag}(a_1, a_2, \ldots, a_n)$ with $|a_i| < 1$ is the state feedback coefficient matrix, $B = (b_{ij})_{n\times n}$ and $D = (d_{ij})_{n\times n}$ are the connection weight matrix and the delayed connection weight matrix, respectively, $f(x(k)) = [f_1(x_1(k)), f_2(x_2(k)), \ldots, f_n(x_n(k))]^T$ and $g(x(k)) = [g_1(x_1(k)), g_2(x_2(k)), \ldots, g_n(x_n(k))]^T$ denote the neuron activation functions, $\sigma(k, x(k), x(k-\tau(k)))$ is the noise intensity function vector, $\Delta A(k)$, $\Delta B(k)$ and $\Delta D(k)$ denote the parameter uncertainties which satisfy the following condition:

$$\big[\Delta A(k)\Delta B(k)\Delta D(k)\big] = MF(k)[E_a E_b E_d], \tag{3}$$

where M, E_a, E_b, E_d are known real constant matrices with appropriate dimensions, and $F(k)$ is an unknown time-varying matrix which satisfies

$$F^T(k)F(k) \le I, \quad k \in N^+. \tag{4}$$

$\omega(k)$ is a scalar Wiener process (Brownian motion) on $(\Omega, F, \{F_t\}_{t\ge 0}, P)$ with

$$E\big(\omega(k)\big) = 0, \qquad E\big(\omega^2(k)\big) = 1, \qquad E\big(\omega(i)\omega(j)\big) = 0, \quad i \ne j. \tag{5}$$

Assumption 1 For each neuron, activation function in system (1), $f_i(\cdot)$ and $g_i(\cdot)$ $i = 1, 2, \ldots, n$ are bounded and satisfy the following conditions: $\forall \xi_1, \xi_2 \in R$, $\xi_1 \ne \xi_2$,

$$\begin{aligned} &\gamma_i^- \le \frac{f_i(\xi_1) - f_i(\xi_2)}{\xi_1 - \xi_2} \le \gamma_i^+, \\ &\varrho_i^- \le \frac{g_i(\xi_1) - g_i(\xi_2)}{\xi_1 - \xi_2} \le \varrho_i^+, \\ &f_i(0) = g_i(0) = 0, \quad i = 1, 2, \ldots, n, \end{aligned} \tag{6}$$

where γ_i^-, γ_i^+, ϱ_i^- and ϱ_i^+ are known constants.

Remark 1 The constants γ_i^-, γ_i^+, ϱ_i^-, ϱ_i^+ in Assumption 1 are allowed to be positive, negative, or zero. Hence, the functions $f(x(k))$ and $g(x(k))$ could be non-monotonic, and are more general than the usual sigmoid functions and the commonly used Lipschitz conditions recently.

Assumption 2 $\sigma(k, x(k), x(k-\tau(k))) : R \times R^n \times R^n \to R^n$ is the continuous function, and is assumed to satisfy

$$\sigma^T \sigma \leq \begin{pmatrix} x(k) \\ x(k-\tau(k)) \end{pmatrix}^T G \begin{pmatrix} x(k) \\ x(k-\tau(k)) \end{pmatrix}, \tag{7}$$

where

$$G = \begin{pmatrix} G_1 & G_2 \\ * & G_3 \end{pmatrix}.$$

Remark 2 Choose $G_1 = \rho_1 I$, $G_2 = 0$, $G_3 = \rho_2 I$, we can find that (7) is reduced to

$$\sigma^T \sigma \leq \rho_1 \|x(k)\|^2 + \rho_2 \|x(k-\tau(k))\|^2, \tag{8}$$

where $\rho_1 > 0$, $\rho_2 > 0$ are known constant scalars. Thus, the assumption condition (8) is a special case of the assumption condition (7). It should be pointed out that the robust delay-distribution-dependent stability criteria for DSNNs with time-varying delay by (7) is generally less conservative than by (8).

Assumption 3 For any $\tau_m \leq \tau_0 < \tau_M$, assume that $\tau(k)$ takes values in $[\tau_m, \tau_0]$ or $(\tau_0, \tau_M]$, considering the information of probability distribution of the time-varying delay, two sets and two mapping functions are defined

$$\Omega_1 = \{k | \tau(k) \in [\tau_m, \tau_0]\}, \qquad \Omega_2 = \{k | \tau(k) \in (\tau_0, \tau_M]\}, \tag{9}$$

$$\tau_1(k) = \begin{cases} \tau(k), & k \in \Omega_1, \\ \tau_m, & \text{else}, \end{cases} \qquad \tau_2(k) = \begin{cases} \tau(k), & k \in \Omega_2, \\ \tau_0, & \text{else}. \end{cases} \tag{10}$$

It is obvious that $\Omega_1 \cup \Omega_2 = N^+$, $\Omega_1 \cap \Omega_2 = \Phi$ (empty set). It is easy to check that $k \in \Omega_1$ implies that the event $\tau(k) \in [\tau_m, \tau_0]$ takes place, and $k \in \Omega_2$ means that $\tau(k) \in (\tau_0, \tau_M]$ happens.

Define a stochastic variable as

$$\alpha(k) = \begin{cases} 1, & k \in \Omega_1, \\ 0, & k \in \Omega_2. \end{cases} \tag{11}$$

Assumption 4 $\alpha(k)$ is a Bernoulli distributed sequence with

$$\operatorname{Prob}\{\alpha(k) = 1\} = \alpha_0, \qquad \operatorname{Prob}\{\alpha(k) = 0\} = \bar{\alpha}_0 = 1 - \alpha_0, \tag{12}$$

where α_0 is a constant.

Remark 3 From Assumption 4, it is easy to see that

$$\begin{aligned} &E\{\alpha(k)\} = \alpha_0, \qquad E\{\alpha(k)\bar{\alpha}(k)\} = 0, \qquad E\{\alpha(k) - \alpha_0\} = 0, \\ &E\{(\alpha(k) - \alpha_0)^2\} = \alpha_0\bar{\alpha_0}, \qquad E\{(\alpha(k))^2\} = \alpha_0. \end{aligned} \tag{13}$$

By Assumptions 3 and 4, system (1) can be rewritten as

$$\begin{aligned} x(k+1) = {} & (A + \Delta A(k))x(k) + (B + \Delta B(k))f(x(k)) \\ & + \alpha(k)(D + \Delta D(k))g(x(k - \tau_1(k))) \\ & + (1 - \alpha(k))(D + \Delta D(k))g(x(k - \tau_2(k))) \\ & + \alpha(k)\sigma(k, x(k), x(k - \tau_1(k)))\omega(k) \\ & + (1 - \alpha(k))\sigma(k, x(k), x(k - \tau_2(k)))\omega(k). \end{aligned} \tag{14}$$

Assumption 5 Assume that for any $k \in N^+$, $\alpha(k)$ is independent of $\omega(k)$.

Remark 4 It is noted that the introduction of binary stochastic variable was first introduced in [23] and then successfully used in [25, 26, 28]. By introducing the new functions $\tau_1(k)$ and $\tau_2(k)$, the stochastic variable sequence $\alpha(k)$, system (1) is transformed into (14). In (14), the probabilistic effects of the time delay have been translated into the parameter matrices of the transformed system. Then, the stochastic stability criteria based on the new model (14) can be derived, which show the relationship between the stability of the system and the variation range of the time delay and the probability distribution parameter.

For brevity of the following analysis, we denote $A + \Delta A(k)$, $B + \Delta B(k)$, $D + \Delta D(k)$ and $1 - \alpha(k)$ by A_k, B_k, D_k, and $\bar{\alpha}(k)$, respectively. Then (14) can be rearranged as

$$\begin{aligned} x(k+1) = {} & A_k x(k) + B_k f(x(k)) + \alpha(k) D_k g(x(k - \tau_1(k))) \\ & + \bar{\alpha}(k) D_k g(x(k - \tau_2(k))) + \alpha(k)\sigma(k, x(k), x(k - \tau_1(k)))\omega(k) \\ & + \bar{\alpha}(k)\sigma(k, x(k), x(k - \tau_2(k)))\omega(k). \end{aligned} \tag{15}$$

It is obvious that $x(k) = 0$ is a trivial solution of DSNN (15).

The following definition and lemmas are needed to conclude our main results.

Definition 2.1 [16] The DSNN (1) is said to be robustly exponentially stable in the mean square if there exist constants $\alpha > 0$ and $\mu \in (0,1)$ such that every solution of the DSNN (1) satisfies that

$$E\{\|x(k)\|^2\} \le \alpha\mu^k \max_{-\tau_M \le i \le 0} E\{\|x(i)\|^2\}, \quad \forall k \in N^+ \tag{16}$$

for all parameter uncertainties satisfying the admissible condition.

Lemma 2.1 [28] *Given the constant matrices Ω_1, Ω_2 and Ω_3 with appropriate dimensions, where $\Omega_1 = \Omega_1^T$ and $\Omega_2 = \Omega_2^T > 0$, then $\Omega_1 + \Omega_3^T\Omega_2^{-1}\Omega_3 < 0$ if and only if*

$$\begin{pmatrix} \Omega_1 & \Omega_3^T \\ * & -\Omega_2 \end{pmatrix} < 0 \quad \text{or} \quad \begin{pmatrix} -\Omega_2 & \Omega_3 \\ * & \Omega_1 \end{pmatrix} < 0.$$

Lemma 2.2 [28] *Let $x, y \in R^n$ and $\varepsilon > 0$. Then we have*

$$x^T y + y^T x \le \varepsilon^{-1} x^T x + \varepsilon y^T y.$$

3 Robustly globally square exponentially stable of DSNNs

In this section, we shall establish our main criteria based on the LMI approach. For presentation convenience, in the following, we define

$$\begin{aligned}
&\Gamma_1 = \operatorname{diag}\{\gamma_1^-\gamma_1^+, \gamma_2^-\gamma_2^+, \dots, \gamma_n^-\gamma_n^+\}, \qquad \Gamma_3 = \operatorname{diag}\{\varrho_1^-, \varrho_2^-, \dots, \varrho_n^-\}, \\
&\Gamma_2 = \operatorname{diag}\left\{\frac{\gamma_1^- + \gamma_1^+}{2}, \frac{\gamma_2^- + \gamma_2^+}{2}, \dots, \frac{\gamma_n^- + \gamma_n^+}{2}\right\}, \\
&\Gamma_4 = \operatorname{diag}\{\varrho_1^+, \varrho_2^+, \dots, \varrho_n^+\}, \qquad \Gamma_5 = \operatorname{diag}\{\varrho_1^-\varrho_1^+, \varrho_2^-\varrho_2^+, \dots, \varrho_n^-\varrho_n^+\}, \\
&\Gamma_6 = \operatorname{diag}\left\{\frac{\varrho_1^- + \varrho_1^+}{2}, \frac{\varrho_2^- + \varrho_2^+}{2}, \dots, \frac{\varrho_n^- + \varrho_n^+}{2}\right\}.
\end{aligned}$$

Theorem 3.1 *For given positive integers τ_m, τ_M, $\tau_m \le \tau_0 < \tau_M$, under Assumptions* 1-5, *the DSNN* (15) *is globally exponentially stable in the mean square, if there exist symmetric positive-definite matrices P, Q_1, Q_2, Z_1, Z_2 with appropriate dimensional, positive-definite diagonal matrices H, R, S, T, Λ_1, Λ_2, Λ_3, Λ_4 and two positive constants ε, λ^* such that the following two matrix inequalities hold*:

$$P \le \lambda^* I, \tag{17}$$

$$\Xi = \begin{pmatrix}
\Xi_{11} & \alpha_0\lambda^* G_2 & \bar{\alpha}_0\lambda^* G_5 & \Xi_{14} & \Xi_{15} & \Xi_{16} & \Xi_{17} \\
* & \Xi_{22} & 0 & 0 & 0 & \Xi_{26} & 0 \\
* & * & \Xi_{33} & 0 & 0 & 0 & \Xi_{37} \\
* & * & * & \Xi_{44} & 0 & \Xi_{46} & \Xi_{47} \\
* & * & * & * & \Xi_{55} & 0 & 0 \\
* & * & * & * & * & \Xi_{66} & 0 \\
* & * & * & * & * & * & \Xi_{77}
\end{pmatrix} \le 0, \tag{18}$$

where

$$\begin{aligned}
\Xi_{11} &= A_k^T P A_k - P + \alpha_0\lambda^* G_1 + \bar{\alpha}_0\lambda^* G_4 - 2(\tau_M - \tau_0 + 1)\Gamma_3 H \\
&\quad + 2(\tau_M - \tau_0 + 1)\Gamma_4 R + (\tau_0 - \tau_m + 1)Q_1 + (\tau_M - \tau_0 + 1)Q_2 \\
&\quad - 2(\tau_0 - \tau_m + 1)\Gamma_3 S + 2(\tau_0 - \tau_m + 1)\Gamma_4 T - \Gamma_1\Lambda_1 - \Gamma_5\Lambda_2, \\
\Xi_{15} &= (\tau_M - \tau_0 + 1)H - (\tau_M - \tau_0 + 1)R + (\tau_0 - \tau_m + 1)S \\
&\quad - (\tau_0 - \tau_m + 1)T + \Gamma_6\Lambda_2,
\end{aligned}$$

$$\Xi_{22} = \alpha_0\lambda^* G_3 + 2\Gamma_3 S - 2\Gamma_4 T - Q_1 - \Gamma_5\Lambda_3,$$
$$\Xi_{33} = \bar{\alpha}_0\lambda^* G_6 + 2\Gamma_3 H - 2\Gamma_4 R - Q_2 - \Gamma_5\Lambda_4,$$
$$\Xi_{55} = (\tau_0 - \tau_m + 1)Z_1 + (\tau_M - \tau_0 + 1)Z_2 - \Lambda_2,$$
$$\Xi_{16} = \alpha_0 A_k^T PD_k, \qquad \Xi_{14} = A_k^T PB_k + \Gamma_2\Lambda_1,$$
$$\Xi_{17} = \bar{\alpha}_0 A_k^T PD_k, \qquad \Xi_{26} = -S + T + \Gamma_6\Lambda_3,$$
$$\Xi_{37} = -H + R + \Gamma_6\Lambda_4, \qquad \Xi_{44} = B_k^T PB_k - \Lambda_1,$$
$$\Xi_{46} = \alpha_0 B_k^T PD_k, \qquad \Xi_{47} = \bar{\alpha}_0 B_k^T PD_k,$$
$$\Xi_{66} = \alpha_0 D_k^T PD_k - Z_1 - \Lambda_3, \qquad \Xi_{77} = \bar{\alpha}_0 D_k^T PD_k - Z_2 - \Lambda_4.$$

Proof We construct the following Lyapunov-Krasovskii functional candidate for system (15):

$$V\bigl(k,x(k)\bigr) = \sum_{i=1}^{7} V_i\bigl(k,x(k)\bigr), \tag{19}$$

where

$$V_1\bigl(k,x(k)\bigr) = x^T(k)Px(k),$$
$$V_2\bigl(k,x(k)\bigr) = 2\sum_{i=-\tau_M+1}^{-\tau_0+1}\sum_{j=k+i-1}^{k-1}\bigl\{\bigl[g\bigl(x(j)\bigr) - \Gamma_3 x(j)\bigr]^T Hx(j) + \bigl[\Gamma_4 x(j) - g\bigl(x(j)\bigr)\bigr]^T Rx(j)\bigr\},$$
$$V_3\bigl(k,x(k)\bigr) = 2\sum_{i=-\tau_0+1}^{-\tau_m+1}\sum_{j=k+i-1}^{k-1}\bigl\{\bigl[g\bigl(x(j)\bigr) - \Gamma_3 x(j)\bigr]^T Sx(j) + \bigl[\Gamma_4 x(j) - g\bigl(x(j)\bigr)\bigr]^T Tx(j)\bigr\},$$
$$V_4\bigl(k,x(k)\bigr) = \sum_{i=k-\tau_1(k)}^{k-1} x^T(i)Q_1x(i) + \sum_{i=-\tau_0+1}^{-\tau_m}\sum_{j=k+i}^{k-1} x^T(j)Q_1x(j),$$
$$V_5\bigl(k,x(k)\bigr) = \sum_{i=k-\tau_2(k)}^{k-1} x^T(i)Q_2x(i) + \sum_{i=-\tau_M+1}^{-\tau_0}\sum_{j=k+i}^{k-1} x^T(j)Q_2x(j),$$
$$V_6\bigl(k,x(k)\bigr) = \sum_{i=k-\tau_1(k)}^{k-1} g^T\bigl(x(i)\bigr)Z_1g\bigl(x(i)\bigr) + \sum_{i=\tau_m}^{\tau_0-1}\sum_{j=k-i}^{k-1} g^T\bigl(x(j)\bigr)Z_1g\bigl(x(j)\bigr),$$
$$V_7\bigl(k,x(k)\bigr) = \sum_{i=k-\tau_2(k)}^{k-1} g^T\bigl(x(i)\bigr)Z_2g\bigl(x(i)\bigr) + \sum_{i=\tau_0}^{\tau_M-1}\sum_{j=k-i}^{k-1} g^T\bigl(x(j)\bigr)Z_2g\bigl(x(j)\bigr).$$

Denote $\mathfrak{X} = \{x(k), x(k-1), \ldots, x(k-\tau(k))\}$. Calculating the difference of $V(k,x(k))$ and taking the mathematical expectation, by (5), and Assumption 4 and Remark 3, we have

$$\begin{aligned}
E\bigl\{\Delta V_1\bigl(k,x(k)\bigr)\bigr\} &= E\bigl\{E\bigl\{V_1\bigl(k+1,x(k+1)\bigr)|\mathfrak{X}\bigr\} - V_1\bigl(k,x(k)\bigr)\bigr\} \\
&= E\bigl\{x^T(k)\bigl(A_k^T PA_k - P\bigr)x(k) + 2x^T(k)A_k^T PB_k f\bigl(x(k)\bigr) \\
&\quad + 2\alpha_0 x^T(k)A_k^T PD_k g\bigl(x\bigl(k-\tau_1(k)\bigr)\bigr)
\end{aligned}$$

$$
\begin{aligned}
&+ 2\bar{\alpha}_0 x^T(k) A_k^T P D_k g\big(x\big(k-\tau_2(k)\big)\big) \\
&+ f^T\big(x(k)\big) B_k^T P B_k f\big(x(k)\big) \\
&+ 2\alpha_0 f^T\big(x(k)\big) B_k^T P D_k g\big(x\big(k-\tau_1(k)\big)\big) \\
&+ 2\bar{\alpha}_0 f^T\big(x(k)\big) B_k^T P D_k g\big(x\big(k-\tau_1(k)\big)\big) \\
&+ \alpha_0 g^T\big(x\big(k-\tau_1(k)\big)\big) D_k^T P D_k g\big(x\big(k-\tau_1(k)\big)\big) \\
&+ \bar{\alpha}_0 g^T\big(x\big(k-\tau_2(k)\big)\big) D_k^T P D_k g\big(x\big(k-\tau_2(k)\big)\big) \\
&+ \alpha_0 \sigma^T\big(k, x(k), x\big(k-\tau_1(k)\big)\big) P \sigma\big(k, x(k), x\big(k-\tau_1(k)\big)\big) \\
&+ \bar{\alpha}_0 \sigma^T\big(k, x(k), x\big(k-\tau_2(k)\big)\big) P \sigma\big(k, x(k), x\big(k-\tau_2(k)\big)\big)\Big\}. \qquad (20)
\end{aligned}
$$

It is very easy to check from Assumption 2 and (17) that

$$
\begin{aligned}
&\alpha_0 \sigma^T\big(k, x(k), x\big(k-\tau_1(k)\big)\big) P \sigma\big(k, x(k), x\big(k-\tau_1(k)\big)\big) \\
&\quad \le \alpha_0 \lambda^* \begin{pmatrix} x(k) \\ x(k-\tau_1(k)) \end{pmatrix}^T \begin{pmatrix} G_1 & G_2 \\ * & G_3 \end{pmatrix} \begin{pmatrix} x(k) \\ x(k-\tau_1(k)) \end{pmatrix}, \qquad (21) \\
&\bar{\alpha}_0 \sigma^T\big(k, x(k), x\big(k-\tau_2(k)\big)\big) P \sigma\big(k, x(k), x\big(k-\tau_2(k)\big)\big) \\
&\quad \le \bar{\alpha}_0 \lambda^* \begin{pmatrix} x(k) \\ x(k-\tau_2(k)) \end{pmatrix}^T \begin{pmatrix} G_4 & G_5 \\ * & G_6 \end{pmatrix} \begin{pmatrix} x(k) \\ x(k-\tau_2(k)) \end{pmatrix}, \qquad (22) \\
&E\big\{\Delta V_2(k)\big\} = E\big\{E\big\{V_2\big(k+1, x(k+1)\big) | \mathfrak{X}\big\} - V_2\big(k, x(k)\big)\big\} \\
&\qquad = 2E\Bigg\{\sum_{i=-\tau_M+1}^{-\tau_0+1} \big\{\big[g\big(x(k)\big) - \Gamma_3 x(k)\big]^T H x(k) + \big[\Gamma_4 x(k) - g\big(x(k)\big)\big]^T R x(k)\big\} \\
&\qquad\quad - \sum_{i=k-\tau_M}^{k-\tau_0} \big\{\big[g\big(x(i)\big) - \Gamma_3 x(i)\big]^T H x(i) \\
&\qquad\quad + \big[\Gamma_4 x(i) - g\big(x(i)\big)\big]^T R x(i)\big\}\Bigg\} \\
&\qquad \le E\big\{2(\tau_M - \tau_0 + 1)\big[g\big(x(k)\big) - \Gamma_3 x(k)\big]^T H x(k) \\
&\qquad\quad + 2(\tau_M - \tau_0 + 1)\big[\Gamma_4 x(k) - g\big(x(k)\big)\big]^T R x(k) \\
&\qquad\quad - 2\big[g\big(x\big(k-\tau_2(k)\big)\big) - \Gamma_3 x\big(k-\tau_2(k)\big)\big]^T H x\big(k-\tau_2(k)\big) \\
&\qquad\quad - 2\big[\Gamma_4 x\big(k-\tau_2(k)\big) - g\big(x\big(k-\tau_2(k)\big)\big)\big]^T R x\big(k-\tau_2(k)\big)\big\}, \qquad (23) \\
&E\big\{\Delta V_3(k)\big\} = E\big\{E\big\{V_3\big(k+1, x(k+1)\big) | \mathfrak{X}\big\} - V_3\big(k, x(k)\big)\big\} \\
&\qquad \le E\big\{2(\tau_0 - \tau_m + 1)\big[g\big(x(k)\big) - \Gamma_3 x(k)\big]^T S x(k) \\
&\qquad\quad + 2(\tau_0 - \tau_m + 1)\big[\Gamma_4 x(k) - g\big(x(k)\big)\big]^T T x(k) \\
&\qquad\quad - 2\big[g\big(x\big(k-\tau_1(k)\big)\big) - \Gamma_3 x\big(k-\tau_1(k)\big)\big]^T S x\big(k-\tau_1(k)\big) \\
&\qquad\quad - 2\big[\Gamma_4 x\big(k-\tau_1(k)\big) - g\big(x\big(k-\tau_1(k)\big)\big)\big]^T T x\big(k-\tau_1(k)\big)\big\}, \qquad (24)
\end{aligned}
$$

$$
\begin{aligned}
E\{\Delta V_4(k)\} &= E\{E\{V_4(k+1,x(k+1))|\mathfrak{X}\} - V_4(k,x(k))\} \\
&= E\left\{\left(\sum_{i=k+1-\tau_1(k+1)}^{k} - \sum_{i=k-\tau_1(k)}^{k-1}\right)x^T(i)Q_1x(i)\right. \\
&\quad \left. + \sum_{i=-\tau_0+1}^{-\tau_m}\left(\sum_{j=k+i+1}^{k} - \sum_{j=k+i}^{k-1}\right)x^T(j)Q_1x(j)\right\} \\
&= E\left\{x^T(k)Q_1x(k) - x^T(k-\tau_1(k))Q_1x(k-\tau_1(k))\right. \\
&\quad + \left(\sum_{i=k+1-\tau_1(k+1)}^{k-1} - \sum_{i=k-\tau_1(k)+1}^{k-1}\right)x^T(i)Q_1x(i) \\
&\quad \left. + \sum_{i=-\tau_0+1}^{-\tau_m}\left(x^T(k)Q_1x(k) - x^T(k+i)Q_1x(k+i)\right)\right\} \\
&\le E\left\{(\tau_0-\tau_m+1)x^T(k)Q_1x(k) - \sum_{i=k-\tau_0+1}^{k-\tau_m} x^T(i)Q_1x(i)\right. \\
&\quad + \left(\sum_{i=k+1-\tau_0}^{k-1} - \sum_{i=k-\tau_m+1}^{k-1}\right)x^T(i)Q_1x(i) \\
&\quad \left. - x^T(k-\tau_1(k))Q_1x(k-\tau_1(k))\right\} \\
&= E\{(\tau_0-\tau_m+1)x^T(k)Q_1x(k) \\
&\quad - x^T(k-\tau_1(k))Q_1x(k-\tau_1(k))\},
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
E\{\Delta V_5(k)\} &= E\{E\{V_5(k+1,x(k+1))|\mathfrak{X}\} - V_5(k,x(k))\} \\
&\le E\{(\tau_M-\tau_0+1)x^T(k)Q_2x(k) \\
&\quad - x^T(k-\tau_2(k))Q_2x(k-\tau_2(k))\},
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
E\{\Delta V_6(k)\} &= E\{E\{V_6(k+1,x(k+1))|\mathfrak{X}\} - V_6(k,x(k))\} \\
&= E\left\{\left(\sum_{i=k+1-\tau_1(k+1)}^{k} - \sum_{i=k-\tau_1(k)}^{k-1}\right)g^T(x(i))Z_1g(x(i))\right. \\
&\quad \left. + \sum_{i=\tau_m}^{\tau_0-1}\left(\sum_{j=k-i+1}^{k} - \sum_{j=k-i}^{k-1}\right)g^T(x(j))Z_1g(x(j))\right\} \\
&= E\left\{g^T(x(k))Z_1g(x(k)) - g^T(x(k-\tau_1(k)))Z_1g(x(k-\tau_1(k)))\right. \\
&\quad + \left(\sum_{i=k+1-\tau_1(k+1)}^{k-1} - \sum_{i=k-\tau_1(k)+1}^{k-1}\right)g^T(x(i))Z_1g(x(i)) \\
&\quad \left. + \sum_{i=\tau_m}^{\tau_0-1}\left(g^T(x(k))Z_1g(x(k)) - g^T(x(k-i))Z_1g(x(k-i))\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\le E\Bigg\{(\tau_0-\tau_m+1)g^T\big(x(k)\big)Z_1g\big(x(k)\big)\\
&\quad-\sum_{i=k-\tau_0+1}^{k-\tau_m} g^T\big(x(i)\big)Z_1g\big(x(i)\big)\\
&\quad+\left(\sum_{i=k+1-\tau_0}^{k-1}-\sum_{i=k-\tau_m+1}^{k-1}\right)g^T\big(x(i)\big)Z_1g\big(x(i)\big)\\
&\quad-g^T\big(x\big(k-\tau_1(k)\big)\big)Z_1g\big(x\big(k-\tau_1(k)\big)\big)\Bigg\}\\
&=E\big\{(\tau_0-\tau_m+1)g^T\big(x(k)\big)Z_1g\big(x(k)\big)\\
&\quad-g^T\big(x\big(k-\tau_1(k)\big)\big)Z_1g\big(x\big(k-\tau_1(k)\big)\big)\big\},
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
E\big\{\Delta V_7(k)\big\}&=E\big\{E\big\{V_7\big(k+1,x(k+1)\big)|\mathfrak{X}\big\}-V_7\big(k,x(k)\big)\big\}\\
&\le E\big\{(\tau_M-\tau_0+1)g^T\big(x(k)\big)Z_2g\big(x(k)\big)\\
&\quad-g^T\big(x\big(k-\tau_2(k)\big)\big)Z_2g\big(x\big(k-\tau_2(k)\big)\big)\big\}.
\end{aligned}
\tag{28}
$$

From (6), it follows that

$$
\big(f_i\big(x(k)\big)-\gamma_i^+x_i(k)\big)\big(f_i\big(x(k)\big)-\gamma_i^-x_i(k)\big)\le 0,\quad i=1,2,\ldots,n,
$$

which are equivalent to

$$
\begin{pmatrix} x(k)\\ f(x(k))\end{pmatrix}^T\begin{pmatrix}\gamma_i^-\gamma_i^+e_ie_i^T & -\frac{\gamma_i^-+\gamma_i^+}{2}e_ie_i^T\\ * & e_ie_i^T\end{pmatrix}\begin{pmatrix} x(k)\\ f(x(k))\end{pmatrix}\le 0,
\tag{29}
$$

where e_i denotes the unit column vector having one element on its ith row, zeros elsewhere.

Then from (6) and (19), for any matrices $\Lambda_1=\operatorname{diag}\{\lambda_{11},\lambda_{12},\ldots,\lambda_{1n}\}>0$, it follows that

$$
\begin{pmatrix} x(k)\\ f(x(k))\end{pmatrix}^T\begin{pmatrix}-\Gamma_1\Lambda_1 & \Gamma_2\Lambda_1\\ * & -\Lambda_1\end{pmatrix}\begin{pmatrix} x(k)\\ f(x(k))\end{pmatrix}\ge 0.
\tag{30}
$$

Similarly, for any matrices $\Lambda_i=\operatorname{diag}\{\lambda_{i1},\lambda_{i2},\ldots,\lambda_{in}\}>0$, $i=2,3,4$, we get the following inequalities:

$$
\begin{pmatrix} x(k)\\ g(x(k))\end{pmatrix}^T\begin{pmatrix}-\Gamma_5\Lambda_2 & \Gamma_6\Lambda_2\\ * & -\Lambda_2\end{pmatrix}\begin{pmatrix} x(k)\\ g(x(k))\end{pmatrix}\ge 0,
\tag{31}
$$

$$
\begin{pmatrix} x(k-\tau_1(k))\\ g(x(k-\tau_1(k)))\end{pmatrix}^T\begin{pmatrix}-\Gamma_5\Lambda_3 & \Gamma_6\Lambda_3\\ * & -\Lambda_3\end{pmatrix}\begin{pmatrix} x(k-\tau_1(k))\\ g(x(k-\tau_1(k)))\end{pmatrix}\ge 0,
\tag{32}
$$

$$
\begin{pmatrix} x(k-\tau_2(k))\\ g(x(k-\tau_2(k)))\end{pmatrix}^T\begin{pmatrix}-\Gamma_5\Lambda_4 & \Gamma_6\Lambda_4\\ * & -\Lambda_4\end{pmatrix}\begin{pmatrix} x(k-\tau_2(k))\\ g(x(k-\tau_2(k)))\end{pmatrix}\ge 0.
\tag{33}
$$

Then from (19) to (33), we have

$$E\{\Delta V(k)\} \le E\{\zeta^T(k)\Xi\zeta(k)\}, \tag{34}$$

where

$$\begin{aligned}\xi^T(k) = {} & \big[x^T(k), x^T\big(k-\tau_1(k)\big), x^T\big(k-\tau_2(k)\big), f^T\big(x(k)\big), g^T\big(x(k)\big), \\ & g^T\big(x\big(k-\tau_1(k)\big)\big), g^T\big(x\big(k-\tau_2(k)\big)\big)\big].\end{aligned}$$

Since $\Xi < 0$, from (34), we can conclude that

$$E\{\Delta V(k)\} \le \lambda_{\max}(\Xi)E\{\|x(k)\|^2\}. \tag{35}$$

It is easy to derive that

$$E\{V(k)\} \le \mu_1 E\{\|x(k)\|^2\} + \mu_2 \sum_{i=k-\tau_M}^{k-1} E\{\|x(i)\|^2\}, \tag{36}$$

where

$$\begin{aligned}\mu_1 &= \lambda_{\max}(P), \\ \mu_2 &= (\tau_M - \tau_0 + 1)\big[2\gamma^*\big(\lambda_{\max}(H) + \lambda_{\max}(R)\big) + \varrho^*\lambda_{\max}(Z_2)\big] \\ &\quad + (\tau_0 - \tau_m + 1)\big[2\gamma^*\big(\lambda_{\max}(S) + \lambda_{\max}(T)\big) + \varrho^*\lambda_{\max}(Z_1)\big] \\ &\quad + (\tau_0 - \tau_m + 1)\lambda_{\max}(Q_1) + (\tau_M - \tau_0 + 1)\lambda_{\max}(Q_2)\end{aligned}$$

with

$$\gamma^* = \max_{1\le i\le n}\{|\gamma_i^-|, |\gamma_i^+|\}, \qquad \varrho^* = \max_{1\le i\le n}\{|\varrho_i^-|, |\varrho_i^+|\}.$$

For any $\theta > 1$, it follows from (35) and (36) that

$$\begin{aligned}E\{\theta^{k+1}V(k+1) - \theta^k V(k)\} &= \theta^{k+1}E\{\Delta V(k)\} + \theta^k(\theta-1)E\{V(k)\} \\ &\le \theta^k\big[(\theta-1)\mu_1 + \theta\lambda_{\max}(\Xi)\big]E\{\|x(k)\|^2\} \\ &\quad + (\theta-1)\mu_2 \sum_{i=k-\tau_M}^{k-1} E\{\|x(i)\|^2\}.\end{aligned} \tag{37}$$

Furthermore, for any integer $N \ge \tau_M + 1$, summing up both sides of (37) from 0 to $N-1$ with respect to k, we have

$$\begin{aligned}\theta^N E\{V(N)\} - E\{V(0)\} &\le \big((\theta-1)\mu_1 + \theta\lambda_{\max}(\Xi)\big)\sum_{k=0}^{N-1}\theta^k E\{\|x(k)\|^2\} \\ &\quad + \mu_2(\theta-1)\sum_{k=0}^{N-1}\sum_{i=k-\tau_M}^{k-1}\theta^k E\{\|x(i)\|^2\}.\end{aligned} \tag{38}$$

Note that for $\tau_M \geq 1$, it is easy to compute that

$$\begin{aligned}\sum_{k=0}^{N-1}\sum_{i=k-\tau_M}^{k-1}\theta^k E\{\|x(i)\|^2\} &\leq \left(\sum_{i=-\tau_M}^{-1}\sum_{k=0}^{i+\tau_M}+\sum_{i=0}^{N-1-\tau_M}\sum_{k=i+1}^{i+\tau_M}+\sum_{i=N-\tau_M}^{N-1}\sum_{k=i+1}^{N-1}\right)\mu^k E\{\|x(i)\|^2\} \\ &\leq \tau_M\theta^{\tau_M}\sup_{-\tau_M\leq i\leq 0}E\{\|x(i)\|^2\}+\tau_M\theta^{\tau_M}\sum_{i=0}^{N-1}\theta^i E\{\|x(i)\|^2\}. \qquad (39)\end{aligned}$$

Then from (38) and (39), one has

$$\begin{aligned}\theta^N E\{V(N)\} &\leq E\{V(0)\}+\tau_M\theta^{\tau_M}(\theta-1)\mu_2\sup_{-\tau_M\leq i\leq 0}E\{\|x(i)\|^2\} \\ &\quad +\left[(\theta-1)\mu_1+\theta\lambda_{\max}(\Xi)+\tau_M\theta^{\tau_M}(\theta-1)\mu_2\right]\sum_{k=0}^{N-1}\theta^k E\{\|x(k)\|^2\}. \qquad (40)\end{aligned}$$

Let $\mu^* = \max\{\mu_1, \mu_2\}$. From (36), it is obvious that

$$E\{V(0)\} \leq \mu^*\sup_{-\tau_M\leq i\leq 0}E\{\|x(i)\|^2\}. \qquad (41)$$

In addition, by (19), we can get

$$E\{V(N)\} \geq \lambda_{\min}(P)E\{\|x(N)\|^2\}. \qquad (42)$$

In addition, it can be verified that there exists a scalar $\theta_0 > 1$ such that

$$(\theta-1)\mu_1+\theta\lambda_{\max}(\Xi)+\tau_M\theta^{\tau_M}(\theta-1)\mu_2 = 0. \qquad (43)$$

Substituting (41)-(43) into (40), we obtain

$$E\{\|x(N)\|^2\} \leq \frac{\mu^*+\tau_M\theta_0^{\tau_M}(\theta_0-1)\mu_2}{\lambda_{\min}(P)}\left(\frac{1}{\theta_0}\right)^N\sup_{-\tau_M\leq i\leq 0}E\{\|x(i)\|^2\}. \qquad (44)$$

By Definition 2.1, the DSNN (15) is globally exponentially stable in the mean square. This completes the proof. □

Remark 5 In Theorem 3.1, free-weighting matrices R, H, S, T are introduced by constructing a new Lyapunov functional (19). On the one hand, in (19), the useful information of the time delays is considered sufficiently. On the other hand, the terms $V_2(k, x(k))$, $V_3(k, x(k))$ are introduced and make full use of the information of the activation function $g(x(k))$. Which make this stability criterion generally less conservative than those obtained in [16–18, 28]. However, because of the parameter uncertainties contained in (18), it is difficult to use Theorem 3.1 directly to determine the stability of the DSNN (15). Thus, it is necessary for us to give another criterion as follows.

Theorem 3.2 *For given positive integers τ_m, τ_M, $\tau_m \leq \tau_0 < \tau_M$, under Assumptions* 1-5, *the DSNN* (15) *is robustly globally exponentially stable in the mean square if there exist symmetric positive-definite matrices P, Q_1, Q_2, Z_1, Z_2 with appropriate dimensional, positive-definite diagonal matrices H, R, S, T, Λ_1, Λ_2, Λ_3, Λ_4 and positive constants ε, λ^* such that*

the following two LMIs hold:

$$P \le \lambda^* I, \tag{45}$$

$$\tilde{\Xi} = \begin{pmatrix} \tilde{\Xi}_{11} & \alpha_0\lambda^* G_2 & \bar{\alpha}_0\lambda^* G_5 & \tilde{\Xi}_{14} & \tilde{\Xi}_{15} & \tilde{\Xi}_{16} & \tilde{\Xi}_{17} & A^T P & 0 \\ * & \tilde{\Xi}_{22} & 0 & 0 & 0 & \tilde{\Xi}_{26} & 0 & 0 & 0 \\ * & * & \tilde{\Xi}_{33} & 0 & 0 & 0 & \tilde{\Xi}_{37} & 0 & 0 \\ * & * & * & \tilde{\Xi}_{44} & 0 & \tilde{\Xi}_{46} & \tilde{\Xi}_{47} & B^T P & 0 \\ * & * & * & * & \tilde{\Xi}_{55} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \tilde{\Xi}_{66} & 0 & \alpha_0 D^T P & 0 \\ * & * & * & * & * & * & \tilde{\Xi}_{77} & \bar{\alpha}_0 D^T P & 0 \\ * & * & * & * & * & * & * & -P & PM \\ * & * & * & * & * & * & * & * & -\varepsilon I \end{pmatrix} \le 0, \tag{46}$$

where

$$\begin{aligned}
\tilde{\Xi}_{11} &= \varepsilon E_a^T E_a - P + \alpha_0\lambda^* G_1 + \bar{\alpha}_0\lambda^* G_4 - 2(\tau_M - \tau_0 + 1)\Gamma_3 H \\
&\quad + 2(\tau_M - \tau_0 + 1)\Gamma_4 R + (\tau_0 - \tau_m + 1)Q_1 + (\tau_M - \tau_0 + 1)Q_2 \\
&\quad - 2(\tau_0 - \tau_m + 1)\Gamma_3 S + 2(\tau_0 - \tau_m + 1)\Gamma_4 T - \Gamma_1\Lambda_1 - \Gamma_5\Lambda_2, \\
\tilde{\Xi}_{15} &= (\tau_M - \tau_0 + 1)H - (\tau_M - \tau_0 + 1)R + (\tau_0 - \tau_m + 1)S \\
&\quad - (\tau_0 - \tau_m + 1)T + \Gamma_6\Lambda_2, \\
\tilde{\Xi}_{22} &= \alpha_0\lambda^* G_3 + 2\Gamma_3 S - 2\Gamma_4 T - Q_1 - \Gamma_5\Lambda_3, \\
\tilde{\Xi}_{33} &= \bar{\alpha}_0\lambda^* G_6 + 2\Gamma_3 H - 2\Gamma_4 R - Q_2 - \Gamma_5\Lambda_4, \\
\tilde{\Xi}_{55} &= (\tau_0 - \tau_m + 1)Z_1 + (\tau_M - \tau_0 + 1)Z_2 - \Lambda_2, \\
\tilde{\Xi}_{14} &= \varepsilon E_a^T E_b + \Gamma_2\Lambda_1, \qquad \tilde{\Xi}_{16} = \alpha_0\varepsilon E_a^T E_d, \qquad \tilde{\Xi}_{17} = \bar{\alpha}_0\varepsilon E_a^T E_d, \\
\tilde{\Xi}_{37} &= -H + R + \Gamma_6\Lambda_4, \qquad \tilde{\Xi}_{17} = \bar{\alpha}_0\varepsilon E_a^T E_d, \qquad \tilde{\Xi}_{26} = -S + T + \Gamma_6\Lambda_3, \\
\tilde{\Xi}_{44} &= \varepsilon E_b^T E_b - \Lambda_1, \qquad \tilde{\Xi}_{46} = \alpha_0\varepsilon E_b^T E_d, \qquad \tilde{\Xi}_{47} = \bar{\alpha}_0\varepsilon E_b^T E_d, \\
\tilde{\Xi}_{66} &= \alpha_0\varepsilon E_d^T E_d - Z_1 - \Lambda_3, \qquad \tilde{\Xi}_{77} = \bar{\alpha}_0\varepsilon E_d^T E_d - Z_2 - \Lambda_4.
\end{aligned}$$

Proof We show that $\Xi < 0$ in (18) implies that $\Xi^* + \eta^T P^{-1}\eta < 0$, where

$$\Xi^* = \begin{pmatrix} \Xi_{11}^* & \alpha_0\lambda^* G_2 & \bar{\alpha}_0\lambda^* G_5 & \Gamma_2\Lambda_1 & \Xi_{15} & 0 & 0 \\ * & \Xi_{22} & 0 & 0 & 0 & \Xi_{26} & 0 \\ * & * & \Xi_{33} & 0 & 0 & 0 & \Xi_{37}^* \\ * & * & * & -\Lambda_1 & 0 & 0 & 0 \\ * & * & * & * & \Xi_{55} & 0 & 0 \\ * & * & * & * & * & -Z_1 - \Lambda_3 & 0 \\ * & * & * & * & * & * & -Z_2 - \Lambda_4 \end{pmatrix} \le 0,$$

$$\begin{aligned}
\Xi_{11}^* &= -P + \alpha_0\lambda^* G_1 + \bar{\alpha}_0\lambda^* G_4 - 2(\tau_M - \tau_0 + 1)\Gamma_3^T H \\
&\quad + 2(\tau_M - \tau_0 + 1)\Gamma_4^T R + (\tau_0 - \tau_m + 1)Q_1 + (\tau_M - \tau_0 + 1)Q_2, \\
\eta &= [PA_k, 0, 0, PB_k, 0, \alpha_k PD_k, \bar{\alpha}_k PD_k].
\end{aligned}$$

According to Lemma 2.1, $\Xi^* + \eta^T P^{-1}\eta < 0$ is equivalent to

$$\begin{pmatrix} \Xi^* & \eta^T \\ \eta & -P \end{pmatrix} = \begin{pmatrix} \Xi^* & \eta_1^T \\ \eta_1 & -P \end{pmatrix} + \begin{pmatrix} 0 & \eta_2^T \\ \eta_2 & 0 \end{pmatrix} < 0 \tag{47}$$

with

$$\eta_1 = [PA, 0, 0, PB, 0, \alpha_0 PD, \bar{\alpha}_0 PD],$$
$$\eta_2 = \bigl[P\Delta A(k), 0, 0, P\Delta B(k), 0, \alpha_0 P\Delta D(k), \bar{\alpha}_0 P\Delta D(k)\bigr].$$

From Lemma 2.2, we can get

$$\begin{aligned} \begin{pmatrix} 0 & \eta_2^T \\ \eta_2 & 0 \end{pmatrix} &= \varpi_1 MF(k)\varpi_2 + \varpi_2^T F^T(k) M^T \varpi_1^T \\ &\le \varepsilon^{-1}\varpi_1 MM^T \varpi_1^T + \varepsilon \varpi_2^T \varpi_2, \end{aligned} \tag{48}$$

where

$$\varpi_1 = [0, 0, 0, 0, 0, 0, 0, P],$$
$$\varpi_2 = [E_a, 0, 0, E_b, 0, \alpha_0 E_d, \bar{\alpha}_0 E_d, 0].$$

Combining (47) with (48), we have

$$\begin{pmatrix} \begin{pmatrix} \Xi^* & \eta_1^T \\ \eta_1 & -P \end{pmatrix} + \varepsilon \varpi_2^T \varpi_2 & \varpi_1 M \\ * & -\varepsilon I \end{pmatrix} \le 0,$$

which implies that (46) holds. This completes the proof. □

Remark 6 When $\alpha_k \equiv 1$, the DSNN (15) reduce to (1), which has been well investigated in [16–18]. By setting $G_i = \rho_i I$, $i = 1, 3, 4, 6$ and $G_2 = G_5 = 0$ in Theorem 3.2 and deleting the fifth rows and the corresponding fifth columns of (46), we can obtain the stability condition for system (1), which can be easily seen to be equivalent to Theorem 3.2 in [28].

If the stochastic term and parameter uncertainties are removed in (15), then (15) reduces to

$$x(k+1) = Ax(k) + Bf\bigl(x(k)\bigr) + \alpha(k)Dg\bigl(x\bigl(k-\tau_1(k)\bigr)\bigr) + \bar{\alpha}(k)Dg\bigl(x\bigl(k-\tau_2(k)\bigr)\bigr), \tag{49}$$

then we get the following results.

Corollary 3.1 *For given positive integers τ_m, τ_M, $\tau_m \le \tau_0 < \tau_M$, under Assumptions* 1-5, *the DSNN* (15) *is globally exponentially stable in the mean square if there exist symmetric positive-definite matrices P, Q_1, Q_2, Z_1, Z_2 with appropriate dimensional, positive-definite diagonal matrices H, R, S, T, Λ_1, Λ_2, Λ_3, Λ_4 and a positive constant ε such that the fol-*

lowing LMI holds:

$$\check{\Xi} = \begin{pmatrix} \check{\Xi}_{11} & 0 & 0 & \check{\Xi}_{14} & \Xi_{15} & \check{\Xi}_{16} & \check{\Xi}_{17} \\ * & \check{\Xi}_{22} & 0 & 0 & 0 & \Xi_{26} & 0 \\ * & * & \check{\Xi}_{33} & 0 & 0 & 0 & \Xi_{37} \\ * & * & * & \check{\Xi}_{44} & 0 & \check{\Xi}_{46} & \check{\Xi}_{47} \\ * & * & * & * & \Xi_{55} & 0 & 0 \\ * & * & * & * & * & \check{\Xi}_{66} & 0 \\ * & * & * & * & * & * & \check{\Xi}_{77} \end{pmatrix} \leq 0, \tag{50}$$

where

$$\begin{aligned} \check{\Xi}_{11} &= A^T PA - P - 2(\tau_M - \tau_0 + 1)\Gamma_3 H + 2(\tau_M - \tau_0 + 1)\Gamma_4 R \\ &\quad + (\tau_0 - \tau_m + 1)Q_1 + (\tau_M - \tau_0 + 1)Q_2 - \Gamma_1\Lambda_1 - \Gamma_5\Lambda_2 \\ &\quad - 2(\tau_0 - \tau_m + 1)\Gamma_3 S + 2(\tau_0 - \tau_m + 1)\Gamma_4 T, \\ \check{\Xi}_{22} &= 2\Gamma_3 S - 2\Gamma_4 T - Q_1 - \Gamma_5\Lambda_3, \qquad \check{\Xi}_{66} = \alpha_0 D^T PD - Z_1 - \Lambda_3, \\ \check{\Xi}_{33} &= 2\Gamma_3 H - 2\Gamma_4 R - Q_2 - \Gamma_5\Lambda_4, \qquad \check{\Xi}_{77} = \bar{\alpha}_0 D^T PD - Z_2 - \Lambda_4, \\ \check{\Xi}_{14} &= A^T PB + \Gamma_2\Lambda_1, \qquad \check{\Xi}_{16} = \alpha_0 A^T PD, \qquad \check{\Xi}_{17} = \bar{\alpha}_0 A^T PD, \\ \check{\Xi}_{44} &= B^T PB - \Lambda_1, \qquad \check{\Xi}_{46} = \alpha_0 B^T PD, \qquad \check{\Xi}_{47} = \bar{\alpha}_0 B^T PD. \end{aligned}$$

4 Example

In this section, a numerical example will be presented to show the effectiveness of the main results derived in Section 3. For the convenience of comparison, let us consider the DSNN (15) with the following parameters:

$$\begin{aligned} &A = \begin{pmatrix} -0.1 & 0 \\ 0 & 0.4 \end{pmatrix}, \qquad B = \begin{pmatrix} 0.1 & -0.2 \\ 0 & -0.1 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & 0.2 \\ 0.2 & -0.1 \end{pmatrix}, \\ &M = \begin{pmatrix} -0.02 & 0 \\ 0.1 & 0.01 \end{pmatrix}, \qquad E_a = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.01 \end{pmatrix}, \qquad E_b = \begin{pmatrix} -0.01 & 0.1 \\ 0.02 & 0 \end{pmatrix}, \\ &E_d = \begin{pmatrix} 0.1 & 0 \\ 0.01 & -0.05 \end{pmatrix}, \qquad G_1 = G_3 = G_4 = G_6 = \begin{pmatrix} 0.16 & 0 \\ 0 & 0.16 \end{pmatrix}, \\ &G_2 = G_5 = 0_{2\times 2}, \\ &f_1(s) = \sin(0.2s) - 0.6\cos(s), \qquad f_2(s) = \tanh(-0.4s), \\ &g_1(s) = \tanh(0.83s) + 0.6\cos(s), \qquad g_2(s) = \tanh(0.2s). \end{aligned}$$

It is easy to verify that

$$\begin{aligned} &\Gamma_1 = \begin{pmatrix} -0.64 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 0 & 0 \\ 0 & -0.2 \end{pmatrix}, \qquad \Gamma_3 = \begin{pmatrix} -0.6 & 0 \\ 0 & 0 \end{pmatrix}, \\ &\Gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & 0.2 \end{pmatrix}, \qquad \Gamma_5 = \begin{pmatrix} -0.6 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \Gamma_6 = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.1 \end{pmatrix}. \end{aligned}$$

Table 1 For given $\tau_m = 1$, $\tau_0 = 2$, allowable upper bounds τ_M with different probability distribution of time delay

	0.5	0.6	0.7	0.72	0.8	0.89	0.95	0.99	1
By [28]	3	3	3	4	5	8	16	22	$+\infty$
By Theorem 3.2	5	5	6	6	7	12	24	117	$+\infty$

For $\tau_1 = 1$, $\tau_0 = 2$, $\tau_M = 12$ and $\alpha_0 = 0.89$, by using Matlab LMI toolbox, we can solve a set of feasible solutions for the LMIs (45) and (46) in Theorem 3.2, which are listed as follows:

$$P = \begin{pmatrix} 161.8985 & -0.5616 \\ -0.5616 & 161.5996 \end{pmatrix}, \qquad Z_1 = \begin{pmatrix} 8.5888 & -2.3625 \\ -2.3625 & 14.7045 \end{pmatrix},$$

$$Z_2 = \begin{pmatrix} 0.6886 & 0.3231 \\ 0.3231 & 3.0148 \end{pmatrix}, \qquad Q_1 = \begin{pmatrix} 9.8745 & -0.0872 \\ -0.0872 & 13.4307 \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} 1.2293 & -0.0546 \\ -0.0546 & 1.0868 \end{pmatrix}, \qquad H = \begin{pmatrix} 0.6389 & 0 \\ 0 & 4.4731 \end{pmatrix},$$

$$R = \begin{pmatrix} 1.0193 & 0 \\ 0 & 5.0362 \end{pmatrix}, \qquad S = \begin{pmatrix} 7.2292 & 0 \\ 0 & 25.9788 \end{pmatrix},$$

$$T = \begin{pmatrix} 6.6790 & 0 \\ 0 & 25.6866 \end{pmatrix}, \qquad \Lambda_1 = \begin{pmatrix} 5.9270 & 0 \\ 0 & 71.5861 \end{pmatrix},$$

$$\Lambda_2 = \begin{pmatrix} 27.6014 & 0 \\ 0 & 78.8797 \end{pmatrix}, \qquad \Lambda_3 = \begin{pmatrix} 11.6106 & 0 \\ 0 & 28.3651 \end{pmatrix},$$

$$\Lambda_4 = \begin{pmatrix} 0.9885 & 0 \\ 0 & 7.1557 \end{pmatrix}, \qquad \varepsilon = 65.4665, \qquad \lambda^* = 162.6924.$$

Therefore, for all admissible parameter uncertainties and external perturbations, the DSNN (15) is globally exponentially stable in the mean square sense. For $\tau_1 = 1$, $\tau_0 = 2$ and $\alpha_0 = 0.89$, by [28], the upper bound of the time-varying delay is 8, and by Theorem 3.2 in this paper, we obtain $\tau_M = 12$. What is more, when $\tau_1 = 1$, $\tau_0 = 2$, and $\alpha_0 = 0.1$, $\alpha_0 = 0.2$, $\alpha_0 = 0.3$, $\alpha_0 = 0.4$, and by Theorem 3.2 in this paper, we can get that the upper bound of the time-varying delay τ_M is 4, 4, 4, 4, respectively. While the LMIs (31), (32) in [28] have no feasible solutions. The further comparison is listed in Table 1, from which one can see that the criterion proposed in Theorem 3.2 is less conservative than those obtained in [28]. One can see that the criterion proposed in Theorem 3.2 is less conservative than those obtained in [16–18] when the probability distribution of the time delay is ignored.

Remark 7 From this example, we can see that stability conditions in this paper are dependent on time delays themselves, the variation interval and the distribution probability of the delay, that is, not only dependent on the time-delay interval, which distinguishes them from the traditional delay-dependent stability conditions.

5 Conclusions

In this paper, the robust delay-probability-distribution-dependent stochastic stability problem for a class of DSNNs with parameter uncertainties has been studied. In terms of

LMIs technique, and combined with Lyapunov stable theory, a new augmented Lyapunov-Krasovskii functional has been constructed, and some novel sufficient conditions ensuring robustly globally exponentially stable in the mean square sense have been derived. Compared with some previous works established in the literature cited therein, the new criteria derived in this paper are less conservative. The numerical example has been demonstrated to show the validity of these new sufficient conditions.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
XZ carried out the stability of system (1) studies, obtained the stability conditions and gave a numerical example to show the validity of these conditions and drafted the manuscript. SZ conceived of the study, and participated in its design and coordination. RR checked on the process of reasoning and compute. All authors read and approved the final manuscript.

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Acknowledgements
The authors would like to thank the editor, the associate editor and the anonymous referees for their detailed comments and valuable suggestions which considerably improved the presentation of this paper. The work of Xia Zhou is supported by the National Natural Science Foundation of China (No. 11226140) and the Anhui Provincial Colleges and Universities Natural Science Foundation (No. KJ2013Z267). The work of Yong Ren is supported by the National Natural Science Foundation of China (No. 10901003 and 11126238), the Distinguished Young Scholars of Anhui Province (No. 1108085J08), the Key Project of Chinese Ministry of Education (No. 211077) and the Anhui Provincial Natural Science Foundation (No. 10040606Q30).

Received: 20 June 2013 Accepted: 10 September 2013 Published: 08 Nov 2013

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10.1186/1687-1847-2013-314
Cite this article as: Zhou et al.: Delay-probability-distribution-dependent stability criteria for discrete-time stochastic neural networks with random delays. *Advances in Difference Equations* 2013, 2013:314