

ON THE q -KONHAUSER BIORTHOGONAL POLYNOMIALS

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(Received May 8, 1986 and in revised form July 18, 1986)

ABSTRACT. Recently, Al-Salam and Verma discussed two polynomial sets $\{Z_n^{(\alpha)}(x, k|q)\}$ and $\{Y_n^{(\alpha)}(x, k|q)\}$, which are biorthogonal on $(0, \infty)$ with respect to a continuous or discrete distribution function. For the polynomials $Y_n^{(\alpha)}(x, k|q)$ the operational formula is derived.

KEY WORDS AND PHRASES. q -Konhauser polynomials, Biorthogonality, q -derivative, q -binomial theorem, q -Laguerre polynomials, Operational formula.

1980 AMS SUBJECT CLASSIFICATION CODE. 33A65.

1. INTRODUCTION.

For $|q| < 1$, let

$$(a; q)_{\infty} = \prod_{j=0}^{\infty} (1 - aq^j)$$

and for arbitrary complex n ,

$$(a; q)_n = (a; q)_{\infty} / (aq^n; q)_{\infty},$$

so that, we have

$$(a; q)_n = \begin{cases} 1, & \text{if } n=0, \\ (1-a)(1-aq)\dots(1-aq^{n-1}), & \text{if } n = 1, 2, \dots \end{cases}$$

For convenience, we shall write $[a]_n$ to mean $(a; q)_n$. If the base is not q but, say p , then we shall mention it explicitly as $(a; p)_n$.

Let δ be the q -derivative defined by means of the following

$$\delta f(x) = \{f(x) - f(qx)\}/x$$

By induction it is fairly easy to verify the relation

$$(x^{k+1}\delta)^n x^{\alpha} = (q^{\alpha}; q^k)_n x^{\alpha+nk}. \quad (1.1)$$

Using the q -binomial theorem (Slater [1]),

$$\sum_{n=0}^{\infty} \frac{[a]_n}{[q]_n} x^n = \frac{[ax]_{\infty}}{[x]_{\infty}},$$

One can easily show that

$$\sum_{n=0}^{\infty} \frac{x^n}{[q]_n} q^{n(n-1)/2} = [-x]_{\infty} ; \text{ (see Askey [2].) } \tag{1.2}$$

Al-Salam and Verma [3] introduced the following pair of biorthogonal polynomials.

$$\begin{aligned} & Z_n^{(\alpha)}(x, k|q) \\ &= \frac{[q^{1+\alpha}]_{nk}}{(q^k; q^k)_n} \sum_{j=0}^n \frac{(q^{-nk}; q^k)_j}{(q^k; q^k)_j [q^{1+\alpha}]_{kj}} q^{(1/2)kj(kj-1)+kj(n+\alpha+1)} , \end{aligned} \tag{1.3}$$

$$\begin{aligned} & Y_n^{(\alpha)}(x, k|q) \\ &= \frac{1}{[q]_n} \sum_{r=0}^n \frac{x^r}{[q]_r} q^{r(r-1)/2} \sum_{j=0}^r \frac{[q^{-r}]_j}{[q]_j} q^j (q^{1+\alpha+j}; q^k)_n . \end{aligned} \tag{1.4}$$

For $k=1$, both $Z_n^{(\alpha)}(x, k|q)$ and $Y_n^{(\alpha)}(x, k|q)$ get reduced to the q -Laguerre polynomials $L_n^{(\alpha)}(x|q)$ discovered by Hahn [4].

2. OPERATIONAL FORMULA.

In order to obtain operational representation for the polynomials $Y_n^{(\alpha)}(x, k|q)$, we can write from (1.4)

$$\begin{aligned} & Y_n^{(\alpha)}(x, k|q) \\ &= \frac{1}{[q]_n} \sum_{r=0}^{\infty} \sum_{s=0}^r \frac{[q^{-r}]_s}{[q]_r [q]_s} x^r (q^{1+\alpha+s}; q^k)_n q^{(1/2)r(r-1)+s} \\ &= \frac{1}{[q]_n} \sum_{r=0}^{\infty} \frac{x^r}{[q]_r} q^{r(r-1)/2} \sum_{s=0}^{\infty} \frac{(-x)^s}{[q]_s} (q^{1+\alpha+s}; q^k)_n \\ &= \frac{[-x]_{\infty}}{[q]_n} \sum_{s=0}^{\infty} \frac{(-x)^s}{[q]_s} (q^{1+\alpha+s}; q^k)_n . \end{aligned}$$

This may be put in the form

$$Y_n^{(\alpha)}(x, k|q) = \frac{[-x]_{\infty}}{[q]_n} \sum_{s=0}^{\infty} \frac{(-x)^s}{[q]_s} x^{-1-\alpha-s-nk} (x^{k+1}\delta)^n x^{1+\alpha+s}$$

where property (1.1) of the operator δ is used. Finally, we shall have

$$Y_n^{(\alpha)}(x, k|q) = \frac{1}{[q]_n} x^{-1-\alpha-nk} [-x]_{\infty} (x^{k+1}\delta)^n \left\{ \frac{x^{1+\alpha}}{[-x]_{\infty}} \right\} . \tag{2.1}$$

More generally, one can obtain

$$\begin{aligned} & (x^{k+1}\delta)^m \left\{ \frac{x^{1+\alpha+nk}}{[-x]_{\infty}} Y_n^{(\alpha)}(x, k|q) \right\} \\ &= [q^{n+1}]_m \frac{x^{1+\alpha+nk+mk}}{[-x]_{\infty}} Y_{m+n}^{(\alpha)}(x, k|q) . \end{aligned} \tag{2.2}$$

For $m=1$, this reduces to a recurrence relation

$$(1-q^{n+1}) Y_{n+1}^{(\alpha)}(x, k|q) = Y_n^{(\alpha)}(x, k|q) - q^{1+\alpha+nk} (1+x) Y_n^{(\alpha)}(xq, k|q). \quad (2.3)$$

One notes that (2.1), (2.2) and (2.3) reduce, when $k=1$, to corresponding properties for the q-Laguerre polynomials $L_n^{(\alpha)}(x|q)$.

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