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ON THE RADIUS OF UNIVALENCE OF CONVEX COMBINATIONS OF ANALYTIC FUNCTIONS

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ABSTRACT. We consider for $\alpha > 0$, the convex combinations $f(z) = (1-\alpha)F(z) + \alpha zF'(z)$, where F belongs to different subclasses of univalent functions and find the radius for which f is in the same class.

KEY WORDS AND PHRASES. *Univalent functions, alpha-quasi-convex, starlike, close-to-convex functions, convex combinations.*

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1. INTRODUCTION.

Let S, K, S* and C denote the classes of analytic functions in the unit disc $E = \{z: |z| < 1\}$ which are respectively univalent, close-to-convex, starlike, and convex. In [1,2], a new subclass C* of univalent functions was introduced and studied. A function f, analytic in E, belongs to C* if and only if there exists a convex function g such that for $z \in E$,

$$\operatorname{Re} \frac{(zf'(z))'}{g'(z)} > 0. \tag{1.1}$$

The functions in C* are called quasi-convex and $C \subset C^* \subset K \subset S$. It is shown [2] that $f \in C^*$ if and only if $zf' \in K$. Recently the functions called α-quasi-convex have been defined and their properties studied in [3]. A function f, analytic in E, is said to be α-quasi-convex if and only if there exists a convex function g such that, for α real and positive

$$\operatorname{Re}\{(1-\alpha)\frac{f'(z)}{g'(z)} + \alpha\frac{(zf'(z))'}{g'(z)}\} > 0. \tag{1.2}$$

It has been shown [3] that F is α-quasi-convex if and only if f with

$f(z) = (1 - \alpha)F(z) + \alpha zF'(z)$ is close-to-convex. (1.3)

All α-quasi-convex functions are close-to-convex.

2. MAIN RESULTS.

We shall now study the mapping properties of f: $f(z) = (1 - \alpha)F(z) + \alpha zF'(z)$, $\alpha > 0$, when F belongs to different subclasses of univalent functions.

THEOREM 2.1. Let $F \in S^*$ and $\alpha > 0$. The function

$$F(z) = (1 - \alpha)F(z) + \alpha zF'(z) \tag{2.1}$$

is starlike in $|z| < r_o$, where

$$r_o = \frac{1}{2\alpha + \sqrt{4\alpha^2 + 1} - 2\alpha} . \tag{2.2}$$

This result is sharp.

PROOF. We can write (2.1) as

$$f(z) = \alpha z^{2-\frac{1}{\alpha}}(Z^{\frac{1}{\alpha}-1} F(z))',$$

and from this it follows that

$$F(z) = \frac{1}{\alpha} z^{1-\frac{1}{\alpha}} \int_0^z z^{\frac{1}{\alpha}-2} f(z)dz. \tag{2.3}$$

Then

$$\frac{zF'(z)}{F(z)} = \{((1-\frac{1}{\alpha})z^{1-\frac{1}{\alpha}} \int_0^z z^{\frac{1}{\alpha}-2} f(z)dz + f(z))/(z^{1-\frac{1}{\alpha}} \int_0^z z^{\frac{1}{\alpha}-2} f(z)dz)\}$$

$$= \{(1-\frac{1}{\alpha})\int_0^z z^{\frac{1}{\alpha}-2} f(z)dz + z^{\frac{1}{\alpha}-1} f(z)\}/\{\int_0^z z^{\frac{1}{\alpha}-2} f(z)dz\} = h(z), \tag{2.4}$$

where $\text{Re } h(z) > 0$, since $F \in S^*$.

From (2.4), we have

$$z^{\frac{1}{\alpha}-1} f(z) - (\frac{1}{\alpha} - 1)\int_0^z z^{\frac{1}{\alpha}-2} f(z)dz = h(z)\int_0^z z^{\frac{1}{\alpha}-2} f(z)dz. \tag{2.5}$$

Differentiating both sides of (2.5), we obtain

$$(\frac{1}{\alpha}-1)z^{\frac{1}{\alpha}-2} f(z) + z^{\frac{1}{\alpha}-1} f'(z) - (\frac{1}{\alpha}-1)z^{\frac{1}{\alpha}-2} f(z) = h'(z)\int_0^z z^{\frac{1}{\alpha}-2} f(z)dz + h(z)z^{\frac{1}{\alpha}-2} f(z).$$

Thus

$$\frac{zf'(z)}{f(z)} = h(z) + \{h'(z)\int_0^z z^{\frac{1}{\alpha}-2} f(z)dz\}/\{z^{\frac{1}{\alpha}-2} f(z)\}.$$

Now, using the well-known result [4], $|h'(z)| \le \{2\mathrm{Re}\, h(z)\}/(1 - r^2)$, $|z| = r$, we have

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)} \ge \mathrm{Re}\, h(z)\{1 - \frac{2}{1 - r^2}\left|\frac{\int_0^z z^{\frac{1}{\alpha}-2} f(z)dz}{z^{\frac{1}{\alpha}-2} f(z)}\right|\}. \tag{2.6}$$

From (2.1) and (2.3), we have

$$\frac{z^{\frac{1}{\alpha}-1} f(z)}{\int_0^z z^{\frac{1}{\alpha}-2} f(z)dz} = \frac{\alpha z(z^{\frac{1}{\alpha}-1} F(z))'}{\alpha(z^{\frac{1}{\alpha}-1} F(z))} = \frac{z\{z^{\frac{1}{\alpha}-1} F'(z) + (\frac{1}{\alpha} - 1)z^{\frac{1}{\alpha}-2} F(z)\}}{(z^{\frac{1}{\alpha}-1} F(z))}$$

$$= \frac{zF'(z)}{F(z)} + (\frac{1}{\alpha} - 1) = h(z) + (\frac{1}{\alpha} - 1),$$

from which it follows that

$$|\{z^{\frac{1}{\alpha}-1} f(z)/\int_0^z z^{\frac{1}{\alpha}-2} f(z)dz\}| \ge \mathrm{Re}\{h(z) + (\frac{1}{\alpha}-1)\} \ge (\frac{1}{\alpha}-1) + \frac{1-r}{1+r}. \tag{2.7}$$

Using (2.7), we have from (2.6)

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)} \ge \mathrm{Re}\, h(z)\{1 - (\frac{2}{1 - r^2})(\frac{r + r^2}{\frac{1}{\alpha} + (\frac{1}{\alpha} - 2)r})\}$$

$$= \mathrm{Re}\, h(z)\{(\frac{1}{\alpha} - 4r - (\frac{1}{\alpha} - 2)r^2)\}/\{(1 - r)(\frac{1}{\alpha} + (\frac{1}{\alpha} - 2r)\}. \tag{2.8}$$

The right hand side of (2.8) is positive for $r < r_o$, where r_o is given by (2.2). This result is sharp as can be seen by

$$f_o(z) = \{\alpha(z(\frac{1}{\alpha} - (\frac{1}{\alpha} - 2)z))\}/(1 - z)^3$$

$$= (1 - \alpha)F_o(z) + \alpha z F_o'(z), \tag{2.9}$$

where

$$F_o(z) = \frac{z}{(1 - z)^2} \in S^*.$$

REMARK 2.1. Let $f \in C$, then f, given by (2.1), is convex for $|z| < r_o$, where r_o is given by (2.2). The proof follows on the same lines as in Theorem 2.1. See also [5] and [6].

REMARK 2.2. In [6], Nikolaeva and Repnina treated the same problem, with a different notation, for the convex and starlike functions of order β. Theorem 2.1 follows from their result when we take $\beta = 0$ for $0 \le \alpha \le 1$. On the other hand, our proof of Theorem 2.1 is much simpler and the result holds for all $\alpha > 0$.

THEOREM 2.2. Let $F \in K$ and $f(z) = (1-\alpha)F(z) + \alpha zF'(z)$, $\alpha > 0$. Then f is close-to-convex in $|z| < r_o$, r_o is given by (2.2). The function f_o in (2.9) shows that this result is sharp.

PROOF. Since $F \in K$, there exists a $G \in S^*$ such that, for $z \in E$, $\text{Re}\,\frac{zF'(z)}{G(z)} > 0$. Now let $g(z) = (1-\alpha)G(z) + \alpha zG'(z)$. Then by Theorem 2.1, g is starlike for $|z| < r_o$, r_o is defined by (2.2). Using the same technique of Theorem 2.1, we can easily show that $\text{Re}\,\frac{zf'(z)}{g(z)} > 0$ for $|z| < r_o$.

REMARK 2.3. For $\alpha = \frac{1}{2}$, this result has been proved in [7].

As an easy consequence of (1.3) and Theorem 2.2, we have the following.

COROLLARY 2.1. Let $F \in K$ and $f(z) = (1-\alpha)F(z) + \alpha zF'(z)$, $\alpha > 0$. Then F is α-quasi-convex in $|z| < r_o$. This means that the radius of α-quasi-convexity for close-to-convex functions is given by (2.2).

THEOREM 2.3. Let $F \in C^*$ and $\alpha > 0$. Let $f(z) = (1-\alpha)F(z) + \alpha zF'(z)$. Then f is in C^*, for $|z| < r_o$, r_o is given by (2.2).

PROOF. Since $F \in C^*$, there exists a $G \in C$ such that for $z \in E$, $\text{Re}\,\frac{(zF'(z))'}{G'(z)} > 0$. Now let $g(z) = (1-\alpha)G(z) + \alpha zG'(z)$, then g is convex in $|z| < r_o$. We can write

$$f(z) = (1-\alpha)F(z) + \alpha zF'(z) = z^{2-\frac{1}{\alpha}}(z^{\frac{1}{\alpha}-1}F(z))'$$

and

$$g(z) = (1-\alpha)G(z) + \alpha zG'(z) = z^{2-\frac{1}{\alpha}}(z^{\frac{1}{\alpha}-1}G(z))'.$$

Thus

$$\frac{(zf'(z))'}{g'(z)} = ((z(z^{2-\frac{1}{\alpha}}(z^{\frac{1}{\alpha}-1}F(z))')')')/(z^{2-\frac{1}{\alpha}}((z^{\frac{1}{\alpha}-1}G(z))')'). \tag{2.10}$$

Now

$$(z(z^{2-\frac{1}{\alpha}}(z^{\frac{1}{\alpha}-1}F(z))')')' = (z((\tfrac{1}{\alpha}-1)F(z) + zF'(z))')' = (\tfrac{1}{\alpha}zF'(z) + z^2F''(z))'$$

$$= (z^{2-\frac{1}{\alpha}}(\tfrac{1}{\alpha}z^{\frac{1}{\alpha}-1}F'(z) + z^{\frac{1}{\alpha}}F''(z)))' = (z^{2-\frac{1}{\alpha}}(z^{\frac{1}{\alpha}-1}zF'(z))')'.$$

Let $zF'(z) = H(z)$, then from (2.10), we have

$$\frac{(zf'(z))'}{g'(z)} = (z^{2-\frac{1}{\alpha}}(z^{\frac{1}{\alpha}-1}H(z))')'/(z^{2-\frac{1}{\alpha}}(z^{\frac{1}{\alpha}-1}G(z))')'$$

Since from Theorem 2.2, the function $(1-\alpha)H(z) + zH'(z) = z^{2-\frac{1}{\alpha}}(z^{\frac{1}{\alpha}-1}H(z))'$ belongs to K with respect to a convex function g: $g(z) = (1-\alpha)G(z) + \alpha zG'(z)$ in

$|z| < r_o$, so f is in C* for $|z| < r_o$, where r_o is given by (2.2).

REMARK 2.4. For $F \in C^*$ and $\alpha = \frac{1}{2}$, Theorem 2.3 has been proved in [1].

We now deal with a generalized form of (1.1) by taking g to be starlike and prove the following.

THEOREM 2.4. Let F be analytic in E and let for $z \in E$, $\text{Re}\,\frac{(zF'(z))'}{G'(z)} > 0$, $G \in S^*$. Let $f(z) = (1-\alpha)F(z) + \alpha zF'(z)$ and $g(z) = (1-\alpha)G(z) + \alpha zG'(z)$, with $\alpha > 0$. Then $\text{Re}\,\frac{(zf'(z))'}{g'(z)} > 0$ for $|z| < r_1$, where

$$r_1 = \frac{1}{3\alpha + \sqrt{9\alpha^2 + 1 - 2\alpha}}$$

For $\alpha = \frac{1}{2}$, the problem has been solved in [8].

PROOF. From (2.3), we can write

$$F(z) = \frac{1}{\alpha} z^{1-\frac{1}{\alpha}} \int_0^z z^{\frac{1}{\alpha}-2} f(z)\,dz$$

$$zF'(z) = \frac{1}{\alpha} z^{1-\frac{1}{\alpha}}\left((1-\frac{1}{\alpha})\int_0^z z^{\frac{1}{\alpha}-2} f(z)\,dz + z^{\frac{1}{\alpha}-1} f(z)\right)$$

$$= \frac{1}{\alpha} z^{1-\frac{1}{\alpha}}\left(\int_0^z z^{\frac{1}{\alpha}-1} f'(z)\,dz\right).$$

Thus

$$\frac{(zF'(z))'}{G'(z)} = \frac{\left(z^{\frac{1}{\alpha}} f'(z) - (\frac{1}{\alpha}-1)\int_0^z z^{\frac{1}{\alpha}-1} f'(z)\,dz\right)}{\int_0^z z^{\frac{1}{\alpha}-1} g'(z)\,dz} = h(z), \tag{2.11}$$

where $\text{Re}\,h(z) > 0$, $z \in E$.

From (2.11), we write

$$z^{\frac{1}{\alpha}} f'(z) - (\frac{1}{\alpha}-1)\int_0^z z^{\frac{1}{\alpha}-1} f'(z)\,dz = h(z)\int_0^z z^{\frac{1}{\alpha}-1} g'(z)\,dz.$$

Differentiating both sides, and simplifying, we obtain

$$\frac{(zf'(z))'}{g'(z)} = h(z) + \frac{h'(z)\left(\int_0^z z^{\frac{1}{\alpha}-1} g'(z)\,dz\right)}{z^{\frac{1}{\alpha}-1} g'(z)}. \tag{2.12}$$

Using $|h'(z)| \le \frac{2\text{Re}\,h(z)}{1-r^2}$, (2.12) gives

$$\mathrm{Re}\,\frac{(zf'(z))'}{g'(z)} \geq \mathrm{Re}\, h(z)[1 - \frac{2}{1-r^2}\left|\left(\int_0^z z^{\frac{1}{\alpha}-1} g'(z)dz\right)/\left(z^{\frac{1}{\alpha}-1} g'(z)dz\right)\right|]. \quad (2.13)$$

Now

$$\left(z^{\frac{1}{\alpha}} g'(z)\right)/\left(\int_0^z z^{\frac{1}{\alpha}-1} g'(z)dz\right) = \frac{(1/\alpha)G'(z) + zG''(z)}{G'(z)} = \left(\frac{1}{\alpha}-1\right) + \frac{(zG'(z))'}{G'(z)}. \quad (2.14)$$

Since $G \in S^*$, so

$$\left|\frac{(zG'(z))'}{G'(z)}\right| \geq \frac{1-4r+r^2}{1-r^2}. \quad (2.15)$$

From (2.13), (2.14) and (2.15), we obtain

$$\mathrm{Re}\,\frac{(zf'(z))'}{g'(z)} \geq \mathrm{Re}\, h(z)[1 - \frac{2}{1-r^2}\ \frac{r(1-r^2)}{\frac{1}{\alpha} - 4r - (\frac{1}{\alpha}-2)r^2}]$$

$$= \mathrm{Re}\, h(z)\,\frac{1 - 6\alpha r - (1-2\alpha)r^2}{1 - 4\alpha r - (1-2\alpha)r^2},$$

and this positive for $|z| < r_1$, where

$$r_1 = \frac{1}{3\alpha + \sqrt{9\alpha^2 + 1 - 2\alpha}}.$$

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