IJMMS 25:10 (2001) 679–688
PII. S0161171201002794
http://ijmms.hindawi.com

TAUTNESS AND APPLICATIONS OF THE ALEXANDER-SPANIER COHOMOLOGY OF K-TYPES

ABD EL-SATTAR A. DABBOUR and ROLA A. HIJAZI

(Received 5 February 1999)

Abstract. The aim of the present work is centered around the tautness property for the two K-types of Alexander-Spanier cohomology given by the authors. A version of the continuity property is proved, and some applications are given.

2000 Mathematics Subject Classification. Primary 55N05, 55N20, 55N35.

1. Introduction. It is well known that in the Alexander-Spanier cohomology theory [17, 18] or in the isomorphic theory of Čech [9], if the coefficient group G is topological then either the theory does not take into account the topology on G [9, 18], or considers only the case when G is compact to obtain a compact cohomology [4, 1]. Continuous cohomology naturally arises when the coefficient group of a cohomology theory is topological [2, 3, 11]. The partially continuous Alexander-Spanier cohomology theory [14] can be considered as a variant of the continuous cohomology of a space with two topologies in the sense of Bott-Haefliger [15]; also it is isomorphic to the continuous cohomology of a simplicial space defined by Brown-Szczarba [2].

The idea of K-groups [5, 6], where K is a locally-finite simplicial complex, is used to introduce the K-types of Alexander-Spanier cohomology with coefficients in a pair (G,G') of topological abelian groups [7, 8]; namely, K-Alexander-Spanier and partially continuous K-Alexander-Spanier cohomologies $\bar{H}^*_K, \tilde{H}^*_K$. It is proved that these K-types satisfied the seven Eilenberg-Steenrod axioms [9]; the excision axiom for the second K-type is verified for compact Hausdorff spaces when (G,G') are absolutely retract. Therefore, the uniqueness theorem of the cohomology theory on the category of compact polyhedral pairs [9] asserts that our Alexander-Spanier K-types over a pair of absolute retract coefficient abelian groups are naturally isomorphic.

In the present work, we prove that the K-Alexander-Spanier cohomology of a closed subset in a paracompact space is isomorphic to the direct limit of the K-Alexander-Spanier cohomology of its neighborhoods, and that the partially continuous K-Alexander-Spanier cohomology of a neighborhood retract closed subspace of a Hausdorff space is isomorphic to the direct limit of the partially continuous K-Alexander-Spanier cohomology of its neighborhoods. Also a version of the continuity property is proved. Moreover, we study some applications of the K-type cohomologies.

2. Alexander-Spanier cohomology of K-types. Here we mention the notations which we used throughout [7, 8].

For an object (X,A) of the category Q of the pairs of topological spaces and their continuous maps, denote by $\Omega(X,A)[\tilde{\Omega}(X,A)]$ the set of the pairs $\bar{\alpha} = (\alpha, \alpha')$, where

α is an open covering of X and α' is a subcollection of α covering $A[\alpha' = \alpha \cap A]$; it is directed with respect to the refinement relation $\bar{\alpha} < \bar{\beta}$, that is, $\alpha < \beta$ and $\alpha' < \beta'$ [9]. Denote by $C^{q(\tau)}(\tilde{X})$ the group of functions $\varphi^\tau : \tilde{X}^{q(\tau)+1} \to G$, where τ is a simplex in K, $q(\tau) = q + \dim\tau$, $q \geq 0$, and $\tilde{X}$ denotes either a space X or $\alpha \in \Omega(X)$. Let $C^{q(\tau)}(\tilde{X})$ be the subgroup of the direct product $\prod_{\tau\in K} C^{q(\tau)}(\tilde{X})$ consisting of such $\varphi = \{\varphi^\tau\}$ for which the condition (k) is satisfied, which states that there is a cofinite subset $\check{\tau}(\varphi)$ of K, that is, $K - \check{\tau}(\varphi)$ is finite such that $(\varphi^\tau)^{-1}(G') = \tilde{X}^{q(\tau)+1}$, $\forall \tau \in \check{\tau}(\varphi)$. The coboundary $\delta^q : C^q(\tilde{X}) \to C^{q+1}(\tilde{X})$ is given by

$$(\delta^q\varphi)^\tau = \sum_{i=1}^{q(\tau)+1} (-1)^i \varphi^\tau p_i^{(q(\tau)+1)} + (-1)^{q(\tau)+1} \sum_{\sigma\in \mathrm{st}(\tau)} [\sigma : \tau]\varphi^\sigma, \tag{2.1}$$

where $\mathrm{st}(\tau) = \{\sigma \in K : \tau \text{ is } (\dim\sigma - 1)\text{-face of } \sigma\}$, $p_i^{(\tau)} : X^{\tau+1} \to X^\tau$ is the projection defined by: if $\hat{t}_i$ is the τ-tuple consisting of $t = (x_0, \dots, x_\tau) \in X^{\tau+1}$ with x_i omitted, then $p_1^{(\tau)}(t) = \hat{t}_i$, $0 \leq i \leq \tau$. The cohomology groups of the cochain complex $C^{\neq}(X) = \{C^q(X), \delta^q\}$ is, in general, uninteresting, as shown in the following theorem [8].

Theorem 2.1. *If* $\dim K = 0$, *then* $H^q(C^{\neq}(X)) \cong G^{*K}$ *(the subgroup of* $G^K = \prod_{\tau\in K} G^\tau$, $G^\tau = G$, *consisting of those elements having all but a finite number of their* τ*-coordinates in* G'*), and* $H^q(C^{\neq}(X)) = 0$, *when* $q \neq 0$.

To pass to more interesting cohomology groups, the topology of the space X will be used to define that $\varphi \in C^q(X)$ is said to be K-locally zero on $M \subseteq X$ if there is $\alpha \in \Omega_X(M)$ (the set of external covering of M by open subsets of X) such that φ vanishes on $\alpha \cap M$, that is, each φ^τ vanishes on $(\alpha \cap M)^{q(\tau)+1}$, where $\alpha^\tau = \cup\{u_\alpha^\tau : u_\alpha \in \alpha\}$. The subgroups of $C^q(X)$ consisting of those elements which are K-locally zero on X, A, respectively, are denoted by $C_0^q(X), C^q(X,A)$. The K-Alexander-Spanier cohomology of (X,A) over (G,G'), denoted by $\bar{H}_K^*(X,A)$, is the cohomology of the quotient cochain complex $\bar{C}_K^{\neq}(X,A) = C^{\neq}(X,A)/C_0^{\neq}(X)$. If $f : (X,A) \to (Y,B)$ is in Q, $\bar{\beta} \in \Omega(Y,B)$ and $\bar{\alpha} = f^{-1}(\bar{\beta})$, then f defines a cochain map $\bar{f}^{\neq} : \bar{C}_K^{\neq}(Y,B) \to \bar{C}_K^{\neq}(X,A)$, where $\check{\tau}(f^q\varphi) = \check{\tau}(\varphi)$ for each $\varphi \in C^q(Y)$. In turn, $\bar{f}^{\neq}$ induces the homomorphism $\bar{f}^* : \bar{H}_K^*(Y,B) \to \bar{H}_K^*(X,A)$.

On the other hand, for $\bar{\alpha} \in \Omega(X,A)$, denote by $C_{\bar{\alpha}}^q$ the subgroup of $C_\alpha^q = C^q(\alpha)$ consisting of those φ that vanish on $\alpha' \cap A$. Then we obtain a direct system $\{C_{\bar{\alpha}}^{\neq}\}_{\Omega(X,A)}$ such that any map $f \in Q$ constitutes a map $F : \{C_{\bar{\beta}}^{\neq}\}_{\Omega(Y,B)} \to \{C_{\bar{\alpha}}^{\neq}\}_{\Omega(X,A)}$ [9]; its limit is F^∞.

Theorem 2.2. *The* K*-Alexander-Spanier cohomology functor* $\{\bar{H}_K^*, \bar{f}^*\}$ *is naturally isomorphic to the functor* $\{\varinjlim\{H^*(C_{\bar{\alpha}}^{\neq})\}_{\Omega(X,A)}, F^{\infty *}\}$ [7].

In the previous part, the topology on (G,G) plays no role; to pass to the second cohomology of K-type we characterize an element $\varphi \in C^q(X)$ to be K-partially continuous if it is continuous on some $\alpha \in \Omega(X)$, that is, $\varphi^\tau \mid \alpha^{q(\tau)+1}$ are continuous functions. Let $L^q(X)$ be the group of all such elements, and $M_K^{\neq}(X) = L^{\neq}(X)/C_0^{\neq}(X)$. The subgroup of C_α^q, where $\alpha \in \Omega(X)$, consisting of the K-continuous elements φ, that is, φ^τ are continuous, is denoted by M_α^q. Let $i : A \hookrightarrow X$, define $M_K^{\neq}(X,A)$ to be the mapping cone of $i^{\neq} : M_K^{\neq}(X) \to M_K^{\neq}(A)$, (see [13, 18]), assuming that $M_K^q(X,A) = M_K^q(X) \oplus M_K^{q-1}(A)$, and the coboundary is $\triangle^q(\varphi,\psi) = (-\delta^q\varphi, i^q\varphi+\delta^{q-1}\psi)$. The cohomology of $M_K^{\neq}(X,A)$ is the partially continuous K-Alexander-Spanier cohomology of (X,A) over the topological pair (G,G') of coefficient groups; it is denoted by $\tilde{H}_K^*(X,A)$.

On the other hand, if $\tilde{\alpha} \in \tilde{\Omega}(X,A)$, then i defines a cochain map $i_\alpha^{\neq} : M_\alpha^{\neq} \to M_{\alpha'}^{\neq}$; its mapping cone is denoted by $M_{\tilde{\alpha}}^{\neq}$.

Theorem 2.3. *For a pair $(X,A) \in Q$ with A closed, $M_K^{\neq}(X,A)$ is naturally isomorphic to $\varinjlim\{M_{\tilde{\alpha}}^{\neq}\}_{\tilde{\Omega}(X,A)}$* [7].

Theorem 2.4. *For a discrete space, and $q \geq 0$, $\tilde{H}_K^q(X) \cong \bar{H}_K^q(X)$.*

Proof. Since $X^{q(\tau)+1}$ admits a discrete topology, it follows that each τ-coordinate φ^τ of $\varphi \in C_K^q(X)$ is continuous [16]. Then φ is K-partially continuous with respect to any $\alpha \in \Omega(X)$. Therefore, $L^q(X) = C_K^q(X)$ and $M_K^{\neq}(X) = \bar{C}_K^{\neq}(X)$. □

3. Tautness and continuity properties. This article is devoted to study the tautness property for both Alexander-Spanier cohomology of K-types. One of its applications is the continuity property.

The star of a subset A in a space X with respect to $\alpha \in \Omega(X)$ is

$$\operatorname{st}(A,\alpha) = \cup\{U_\alpha \in \alpha : U_d \cap A \neq \varnothing\}. \tag{3.1}$$

The star of α is

$$\alpha^* = \{\operatorname{st}(U_\alpha,\alpha) : u_\alpha \in \alpha\}. \tag{3.2}$$

Definition 3.1. Let $\alpha,\beta \in \Omega(X)$, then β is a star-refinement of α, written $\alpha <^* \beta$ if $\alpha < \beta^*$.

Denote by $\mathcal{N}(A)$ the collections of neighborhoods $\{N\}$ of A in X; it is directed downward by inclusion. If $N_1 < N_2$, then the inclusion $\pi_{N_1N_2} : N_2 \hookrightarrow N_1$ induces the homomorphisms $\bar{\pi}_{N_1N_2}^* : \bar{H}_K^q(N_1) \to \bar{H}_K^q(N_2)$. Also $i_N : A \hookrightarrow N$ induces $\bar{i}_N^* : \bar{H}_K^q(N) \to \bar{H}_K^q(A)$, and they define a homomorphism

$$I^\infty : \varinjlim\{\bar{H}_K^q(N), \bar{\pi}_{N_1N_2}^*\}_{\mathcal{N}(A)} \longrightarrow \bar{H}_K^q(A). \tag{3.3}$$

Theorem 3.2 (Tautness). *A closed subspace of a paracompact space is a taut subspace relative to the K-Alexander-Spanier cohomology, that is, I^∞ is an isomorphism for each q and any pair (G,G') of coefficient groups.*

Proof. (1) I^∞ is an epimorphism. Let $h \in \bar{H}_K^q(A)$ with representative $\bar{\varphi} \in \bar{C}_K^q(A)$, written as $h = [\bar{\varphi}]$. Let $\varphi \in C^q(A)$ such that $\varphi \in \bar{\varphi}$. Then there is $\alpha = \{u_\alpha = v_\alpha \cap A : v_\alpha \subseteq X \text{ is open}\} \in \Omega(A)$ such that

$$(\delta^q\varphi) \mid \alpha^{q(\tau)+2} = 0. \tag{3.4}$$

Since A is closed, it follows that $\beta = \{v_\alpha\} \cup \{X-A\} \in \Omega(X)$. The paracompactness of X is equivalent to the existence of such $\gamma \in \Omega(X)$ that $\beta <^* \gamma$, and a neighborhood N of A and an extension $f : N \to A$ (not necessarily continuous) of the identity map id_A of A, that is, $fi_N = \mathrm{id}_A$, such that $f(u_\gamma \cap N) \subseteq \operatorname{st}(u_\gamma,\gamma)$ for each $u_\gamma \in \gamma$ [18]. One can show that f defines a cochain map $f^{\neq} : C^{\neq}(A) \to C^{\neq}(N)$ by $(f^q\varphi)^\tau = \varphi^\tau f^{(q(\tau)+1)}$ with

$\check{\tau}(f^q\varphi) = \check{\tau}(\varphi)$, where $f^{(\tau)} : N^\tau \to A^\tau$ given by $f(x_0,\dots,x_{\tau-1}) = (f(x_0),\dots,f(x_{\tau-1}))$. The relation $\beta < \gamma^*$ yields that for each $u_\gamma \in \gamma$ there is $u_\beta \in \beta$ such that $f(u_\gamma \cap N) \subseteq \operatorname{st}(u_\gamma,\gamma) \subseteq u_\beta$. Because $f(N) = A$, then $f(u_\gamma \cap N) \subseteq u_\beta \cap A \subseteq u_\alpha$ for some $u_\alpha \in \alpha$. By using (3.4), we get $(\delta^q f^q\varphi)^\tau \mid (\gamma\cap N)^{q(\tau)+2} = 0$, that is, $\delta^q(f^q\varphi) \in C_0^{q+1}(N)$. Then $f^q\varphi$ represents a cocycle $\overline{f^q\varphi} \in \bar{C}_K^q(N)$ which, in turn, defines $h_N \in \bar{H}_K^q(N)$, that is, $h_N = [\overline{f^q\varphi}]$. Let $t \in A^{q(\tau)+1}$, then

$$\left(i_N^q(f^q\varphi)\right)^\tau(t) = \varphi^\tau f^{(q(\tau)+1)} i_N^{(q(\tau)+1)}(t) = \varphi^\tau(t), \tag{3.5}$$

and therefore, $\bar{i}_N^* h_N = [\overline{(f i_N)^q\varphi}] = [\bar{\varphi}] = h$.

(2) I^∞ is a monomorphism. Let $h_1 \in \bar{H}_K^q(N_1)$, $\bar{\varphi}_1 \in \bar{C}_K^q(N_1)$ and $\varphi_1 \in C^q(N_1)$ such that $\varphi_1 \in \bar{\varphi}_1$, $\bar{\varphi}_1 \in h_1$, and $[h_1] \in \operatorname{Ker} I^\infty$.

First, one can consider that the neighborhood N_1 of A is a paracompact subset of X. For, if N_1 is not so, then there is a paracompact subset M_1 of X such that $M_1 < N_1$ (e.g., take $M_1 = X$) [10]. The inclusion $\pi_{M_1N_1}$ induces an epimorphism $\bar{\pi}_{M_1N_1}^{\#}$ [8], let $\bar{\pi}_{M_1N_1}^q \bar{\psi}_1 = \bar{\varphi}_1$. Thus the cohomology class of $\bar{H}_K^q(M_1)$ represented by $\bar{\psi}_1$ is $[h_1]$, which shows that N_1 can be taken paracompact.

Now, $\bar{\varphi}_1 \in \operatorname{Ker}\delta^q$, or equivalently, there is $\alpha = \{u_\alpha = v_\alpha \cap N_1 : v_\alpha \subseteq X \text{ is open}\} \in \Omega(N_1)$ such that

$$\left(\delta^q\varphi_1\right)^\tau \mid \alpha^{q(\tau)+2} = 0. \tag{3.6}$$

On the other hand, the assumption $\bar{i}_{N_1}^* h_1 = 0$ asserts that there exists $\bar{\varphi} \in \bar{C}_K^{q-1}(A)$ such that $i_{N_1}^q\varphi_1 - \delta^{q-1}\varphi \in C_0^q(A)$, where $\varphi \in \bar{\varphi}$. This means that there is $\beta = \{u_\beta = \omega_\beta \cap A : \omega_\beta \subseteq X \text{ is open}\} \in \Omega(A)$ such that

$$\left(i_{N_1}^q\varphi_1\right)^\tau = \left(\delta^{q-1}\varphi\right)^\tau \quad \text{on } \beta^{q(\tau)+1}. \tag{3.7}$$

Assume that $\beta_1 = \{u_{\beta_1} = \omega_\beta \cap N_1\} \cup \{N_1 - A\}$. The paracompactness of N_1 asserts the existence of $\gamma_1,\gamma_2 \in \Omega(N_1)$ for which $\alpha <^* \gamma_1$ and $\beta_1 <^* \gamma_2$. The directedness of $\Omega(N_1)$ implies that there is $\gamma \in \Omega(N_1)$ for which $\gamma_1,\gamma_2 < \gamma$; and so for each $u_\gamma \in \gamma$ there are $u_{\gamma_i} \in \gamma_i$, $i = 1,2$ and $u_\alpha \in \alpha$, $u_{\beta_1} \in \beta_1$ such that

$$u_\gamma \subset u_{\gamma_i} \subseteq \operatorname{st}\left(u_{\gamma_i},\gamma_i\right) \subseteq u_\alpha \cap u_{\beta_1}. \tag{3.8}$$

Then

$$\operatorname{st}\left(u_\gamma,\gamma\right) \subseteq u_\alpha \cap u_{\beta_1}, \tag{3.9}$$

that is, $\alpha,\beta_1 <^* \gamma$. According to [18, Lemma 6.6.1], there is a neighborhood N_2 of N_1 and $f : N_2 \to A$ (not necessarily continuous) such that $f i_{N_2} = \mathrm{id}_A$, and $u_{\beta_1} \in \beta_1$ such that

$$f\left(u_\gamma \cap N_2\right) \subseteq \operatorname{st}\left(u_\gamma,\gamma\right) \subseteq u_{\beta_1} \subseteq u_{\beta_1} \cap A = u_\beta. \tag{3.10}$$

Then, by (3.7), we get

$$\left(\delta^{q-1} f^{q-1}\varphi\right)^\tau = \left(f^q i_{N_1}^q\varphi_1\right) \quad \text{on } \left(\gamma \cap N_2\right)^{q(\tau)+1}. \tag{3.11}$$

Define $D^q : C^{q+1}(N_1) \to C^q(N_2)$ by

$$\text{if } t = \left(x_0,\dots,x_{q(\tau)}\right) \in N_2^{q(\tau)+1} \text{ and } \psi_1 \in C^{q+1}\left(N_1\right) \tag{3.12}$$

then

$$(D^q\psi_1)^\tau(t) = \sum_{r=0}^{q(\tau)}(-1)^\gamma\psi_1^\tau(y_0,\dots,y_\tau,z_\tau,\dots,z_{q(\tau)}), \tag{3.13}$$

where

$$y_j = \pi_{N_1N_2}(x_j), \qquad z_j = (i_{N_1}f)(x_j) = f(x_j), \tag{3.14}$$

and $\breve{\tau}(D^q\psi_1) = \breve{\tau}(\psi_1)$. By a similar calculation as given in [7], we get

$$(\delta^{q-1}D^{q-1}\varphi_1)^\tau = (f^q i^q_{N_1}\varphi_1)^\tau - (\pi^q_{N_1N_2}\varphi_1)^\tau - (D^q\delta^q\varphi_1)^\tau. \tag{3.15}$$

By (3.9), (3.10) for each $u_\gamma \in \gamma$, there is $u_\alpha \in \alpha$ such that

$$(u_\gamma \cap N_2) \cup f(u_\gamma \cap N_2) \subseteq u_\alpha. \tag{3.16}$$

Then, by (3.6), (3.11), and (3.15) consequently, we have

$$(\delta^{q-1}D^{q-1}\varphi_1)^\tau = (f^q i^q_{N_1}\varphi_1)^\tau - (\pi^q_{N_1N_2}\varphi_1)^\tau \quad \text{on } (\gamma\cap N_2)^{q(\tau)+1}, \tag{3.17}$$

and so

$$(\pi^q_{N_1N_2}\varphi_1)^\tau = (\delta^{q-1}(f^{q-1}\varphi - D^{q-1}\varphi_1))^\tau \quad \text{on } (\gamma\cap N_2)^{q(\tau)+1}. \tag{3.18}$$

Therefore

$$\psi_2 = f^{q-1}\varphi - D^{q-1}\varphi_1 \in C^{q-1}(N_2) \tag{3.19}$$

such that

$$(\pi^q_{N_1N_2}\varphi_1)^\tau = (\delta^{q-1}\psi_2)^\tau \quad \text{on } (\gamma\cap N_2)^{q(\tau)+1}, \tag{3.20}$$

that is, $\bar{\pi}_{N_1N_2}h_1 = 0$ which completes the proof. □

Corollary 3.3. *Any one-point subset of a paracompact is a taut subspace relative to $\bar{H}^*_K$.*

The next part is devoted to studying the tautness property for $\tilde{H}^*_K$, which is also valid for $\bar{H}^*_K$. The idea and results of α-β-contiguous maps, introduced in [7] plays an essential role in this study.

The inclusions $\pi_{N_1N_2} : N_2 \hookrightarrow N_1$, corresponding to the relations $N_1 < N_2$ in $\mathcal{N}(A)$, define the direct system $\{\tilde{H}^q_K(N), \tilde{\pi}^*_{N_1N_2}\}$. Also the inclusion $i_N : A \hookrightarrow N$, where $N \in \mathcal{N}(A)$, defines a map of direct systems [9]:

$$I_N : \{H^q(M^\mp_\alpha), \tilde{\pi}^*_{\alpha\beta}\}_{\Omega(N)} \longrightarrow \{H^q(M^\mp_{\tilde{\alpha}}), \tilde{\pi}^*_{\tilde{\alpha}\tilde{\beta}}\}_{\Omega(A)}, \tag{3.21}$$

where $\alpha \in \Omega(N)$, $\tilde{\alpha} = i_N^{-1}(\alpha) = \alpha \cap A$. On the other hand, $\{\tilde{i}^*_N\}$ defines a homomorphism

$$\tilde{I}^\infty : \varinjlim\{\tilde{H}^q_K(N), \tilde{\pi}^*_{N_1N_2}\}_{\mathcal{N}(A)} \longrightarrow \tilde{H}^q_K A. \tag{3.22}$$

Theorem 3.4 (Tautness). *If A is a closed subset in a Hausdorff space X such that A is a neighborhood retract, then A is a taut subspace relative to the cohomology $\tilde{H}^*_K$.*

Proof. (1) $\tilde{I}^\infty$ is an epimorphism. Let $h \in \tilde{H}^q_K(A)$, without loss of generality, the neighborhood retractness of A in X yields that A has an open neighborhood U(in X) such that $U \subseteq N$ and a retraction $\tau_1 : U \to A$ (if U_1 is an open neighborhood of A of which A is retract but $U_1 \nsubseteq N$, take $U = U_1 \cap \operatorname{Int} N$). Let $i_U : A \hookrightarrow U$ then, $\tilde{I}^\infty[\tilde{\tau}_1^*(h)] = \tilde{i}_U^*(\tilde{\tau}_1^* h) = \widetilde{\mathrm{id}}_A^*(h) = h$.

(2) $\tilde{I}^\infty$ is a monomorphism. Let $[h] \in \operatorname{Ker} \tilde{I}^\infty$. It is sufficient to construct $V \in \mathcal{N}(A)$ satisfying $N < V$ and $\tilde{\pi}^*_{NV} h = 0$. Since the cohomology functor commutes with the direct limit [18]. Theorem 2.3 asserts that one may assume that h belongs to $\varinjlim\{H^q(M^{\neq}_\alpha), \tilde{\pi}^*_{\alpha\beta}\}_{\Omega(N)}$ with representative $h_\alpha \in H^q(M^{\neq}_\alpha)$, where

$$\alpha = \{u_\alpha = \omega_\alpha \cap N : \omega_\alpha \subseteq X \text{ is open}\} \in \Omega(N). \tag{3.23}$$

Let $\alpha_1 = \{\omega_\alpha\} \cup \{X - A\}$, $\tilde{\alpha} = \alpha_1 \cap A$,

$$\beta = \{u_\beta = \tau_1^{-1}(u_{\tilde{\alpha}}) \cap (u_\alpha \cap U) : \phi \neq u_{\tilde{\alpha}} \in \tilde{\alpha}\}, \tag{3.24}$$

$V = \cup u_\beta$, $\tau = \tau_1 \mid V : V \hookrightarrow A$, and $\alpha' = \alpha_1 \cap V$. Then $\tilde{\alpha} \in \Omega(A)$, $\alpha' = \alpha \cap V \in \Omega(V)$, $u_{\tilde{\alpha}} \subseteq u_\beta$ for each $u_{\tilde{\alpha}} \neq \phi$, β is a family of open subsets in U and so open in X, V is an open neighborhood of A such that $V \subseteq U$, and $\beta \in \Omega(V)$. Since $u_\beta = u_\beta \cap u_\alpha \subseteq V \cap u_\alpha = u_{\alpha'}$, it follows that $\alpha' < \beta$. Also $\alpha' \cap A = \alpha \cap A = \tilde{\alpha}$ and $j^{-1}\beta = \tilde{\alpha}$, where $j : A \hookrightarrow V$. If $\ell : V \hookrightarrow N$, and $[\varphi] \in H^q(M^{\neq}_\alpha)$, then

$$\tilde{j}^*_\beta \tilde{\pi}^*_{\alpha'\beta} \tilde{\ell}^*_\alpha[\varphi] = \tilde{j}^*_\beta[\{(\varphi^\tau \mid \alpha'^{q(\tau)+1}) \mid \beta^{q(\tau)+1}\}] = [\{\varphi^\tau \mid \tilde{\alpha}^{q(\tau)+1}\}], \tag{3.25}$$

that is,

$$\tilde{j}^*_\beta \tilde{\pi}^*_{\alpha'\beta} \tilde{\ell}^*_\alpha = \tilde{i}^*_{N,\alpha}, \tag{3.26}$$

where $\tilde{i}^{\neq}_{N,\alpha} : M^{\neq}_{\tilde{\alpha}} \to M^{\neq}_\alpha$ is induced by $i_N : A \hookrightarrow N$.

On the other hand, $(j\tau)u_\beta \subseteq u_\beta$ and so $j\tau$, $\mathrm{id}_V : V \to V$ are $\beta-\beta$-contiguous [7].

It follows that $(\widetilde{\mathrm{id}}_V)^q_{\beta-\beta}$, $(\widetilde{j\tau})^q_{\beta-\beta} : M^q_\beta \to M^q_\beta$ are cochain homotopic [7]. Then $(\widetilde{\mathrm{id}}_V)^*_{\beta-\beta} = (\widetilde{j\tau})^*_{\beta-\beta} = \tilde{\tau}^*_{\tilde{\alpha}-\beta} \tilde{j}^*_\beta$, which yields that $\tilde{j}^*_\beta$ is a monomorphism. Because $\tilde{i}^*_{N,\alpha} h_\alpha = 0$, equation (3.26) yields that $\tilde{\pi}^*_{\alpha'\beta} \tilde{\ell}^*_\alpha h_\alpha = 0$. Since $\tilde{\ell}^*_\alpha h_\alpha, \tilde{\pi}^*_{\alpha'\beta}(\tilde{\ell}_\alpha h_\alpha)$ represent the zero element of $\varinjlim\{H^q(M^{\neq}_\alpha), \tilde{\pi}^*_{\alpha\beta}\}_{\Omega(N)}$, it follows that $\tilde{\pi}^*_{NV} h = [\tilde{\ell}^*_\alpha h_\alpha] = 0$. □

The rest of this article is centered around a special case of the continuity property for $\bar{H}^*_K$. As an application of the continuity property the cohomology groups satisfy a much stronger form of the excision axiom.

The following results can be deduced from those given in [9].

Lemma 3.5. *Let X be the intersection of a nested system $\{X_\alpha, \pi_{\beta\alpha}\}_\Lambda$, then*

(i) *X and $\varprojlim\{X_\alpha, \pi_{\beta\alpha}\}_\Lambda$ are homeomorphic.*

(ii) *If the nested system consists of compact Hausdorff spaces then X is a closed subset of each X_α.*

(iii) *If N is an open neighborhood of X in X_α (for some $\alpha \in \Lambda$), then there is $\beta > \alpha$ in Λ such that $X_\beta \subseteq N$.*

The inclusions $i_\alpha : X \hookrightarrow X_\alpha$ *define a map*

$$I : \{\bar{H}^q_K(X_\alpha), \bar{\pi}^*_{\alpha\beta}\}_\Lambda \longrightarrow \bar{H}^q_K(X), \tag{3.27}$$

its direct limit is denoted by $\bar{I}^\infty$.

Theorem 3.6 (weak continuity). *If* X *is the intersection of a nested system* $\{X_\alpha, \pi_{\beta\alpha}\}_\Lambda$ *of compact Hausdorff spaces, then* $\bar{I}^\infty$ *is an isomorphism.*

Proof. Since each X_α is a paracompact Hausdorff space [10] and X_α is closed in X (Lemma 3.5), it follows, by Theorem 3.2, that X is a taut subspace in X_α relative to $\bar{H}^*_K$.

(1) $\bar{I}^\infty$ is an epimorphism. Let $h \in \bar{H}^q_K(X)$, then, according to Theorem 3.2, there exists an open neighborhood N of X in X_α and $h_N \in \bar{H}^q_K(N)$, such that $\bar{i}^*_N(h_N) = h$. By Lemma 3.5, there is $\beta > \alpha$ in Λ such that $X_\beta \subseteq N$. Let $i_\beta : X \hookrightarrow X_\beta$, $j_\beta : X_\beta \hookrightarrow N$. Because $\bar{i}^*_\beta(\bar{j}^*_\beta h_N) = (\overline{j_\beta i_\beta})^* h_N = \bar{i}^*_N h_N = h$, then $\bar{I}^\infty[\bar{j}^*_\beta h_N] = h$.

(2) $\bar{I}^\infty$ is a monomorphism. Let $[h_\alpha] \in \operatorname{Ker} \bar{I}^\infty$, that is, $\bar{i}^*_\alpha h_\alpha = 0$. The tautness of X in X_α yields, by Theorem 3.2, an open neighborhood N of X in X_α such that h_N is the unique element for which $\bar{i}'^*_N h_N = 0$, where $i'_N : X \hookrightarrow N$. Because $\bar{i}'^*_N(\bar{i}^*_N h_\alpha) = \bar{i}^*_\alpha h_\alpha = 0$, then $\bar{i}^*_N h_\alpha = 0$. Let $\beta > \alpha$ in Λ such that $X_\beta \subseteq N$, then $\bar{\pi}^*_{\alpha\beta} h_\alpha = (\overline{i_N i_\beta})^* h_\alpha = \bar{j}^*_\beta(\bar{i}^*_N h_\alpha) = 0$, that is, $[h_\alpha] = 0$. □

4. Applications. One of the good applications of the Alexander-Spanier cohomology of K-types is the study of the 0-dimensional cohomology groups and their relation with the connectedness of the space [7]. In this article, two applications are given. In a next work, we hope to give more applications. As a first application, we define the partially continuous K-Alexander-Spanier cohomology of an excision map and calculate its value for some dimensions.

Let $\tilde{f}^\neq : M^\neq_K(Y,B) \to M^\neq_K(X,A)$ be the cochain map induced by the map f in Q. Define $M^\neq_K(f)$ to be the mapping cone of $\tilde{f}^\neq$ by

$$\begin{aligned} M^q_K(f) &= M^q_K(Y,B) \oplus M^{q-1}_K(X,A) \\ &= M^q_K(Y) \oplus M^{q-1}_K(B) \oplus M^{q-1}_K(X) \oplus M^{q-2}_K(A), \end{aligned} \tag{4.1}$$

and the coboundary is

$$\begin{aligned} &\tilde{\triangle}^q(\varphi_2,\psi_2,\varphi_1,\psi_1) \\ &\quad = \left(-\tilde{\triangle}^q(\varphi_2,\psi_2), \triangle^q(\varphi_1,\psi_1) + \tilde{f}^q(\varphi_2,\psi_2)\right) \\ &\quad = \left(\delta^q\varphi_2, -\tilde{i}^q\varphi_2 - \delta^{q-1}\psi_2, -\delta^{q-1}\varphi_1 + \tilde{f}^q\varphi_2, \tilde{i}^{q-1}\varphi_1 + \delta^{q-2}\psi_1 + \widetilde{f\mid A}\right)^{q-1}\psi_2. \end{aligned} \tag{4.2}$$

Then there is a short exact sequence

$$0 \longrightarrow \overset{+}{M}{}^\neq_K(X,A) \xrightarrow{\lambda^\neq} M^\neq_K(f) \xrightarrow{\chi^\neq} \bar{M}^\neq_K(Y,B) \longrightarrow \underline{O}_2, \tag{4.3}$$

where $\lambda^\neq, \chi^\neq$ are injection, projection, respectively; $\overset{+}{M}{}^\neq(X,A)$ is the complex $M^\neq_K(X,A)$ with the dimensions all raised by one, and $\bar{M}^\neq(Y,B)$ is the complex $M^\neq(Y,B)$ with the sign of the coboundary changed [12]. Note that $H^q(\bar{M}_K^{\neq}(Y,B)) = \tilde{H}_K^q(Y,B)$. Let V be an open subset of X such that $\bar{V} \subseteq \operatorname{Int} A$, $B = X - V$, and $C = A - V$. Put the excision map $e : (B,C) \hookrightarrow (X,A)$ in (4.3) instead of f, and then apply the cohomology functor to get the long exact sequence

$$\cdots \longrightarrow \tilde{H}_K^q(e) \xrightarrow{\tilde{\chi}^*} \tilde{H}_K^q(X,A) \xrightarrow{\tilde{e}^*} \tilde{H}_K^q(B,C) \xrightarrow{\tilde{\lambda}^*} \tilde{H}_K^{q+1}(e) \longrightarrow \cdots. \tag{4.4}$$

Thus the groups $\tilde{H}_K^q(e), \tilde{H}_K^{q+1}(e)$ measure how much the cohomological groups deviate from the excision axiom.

Theorem 4.1. *If $\dim K = 0$, $e : (B,C) \hookrightarrow (X,A)$ is an excision map, where A is closed and (G,G') any pair of topological abelian groups, then $\tilde{H}_K^q(e) = 0$ when $q = 0$ or $q = 1$.*

Proof. (1) Case $q = 0$. We have

$$M_K^0(e) = M_K^0(X,A) = M_K^0(X) = L_K^0(X). \tag{4.5}$$

Let $\varphi \in M_K^0(e)$ such that $\tilde{\triangle}_\varphi = 0$, then $\tilde{i}^0\varphi = 0$, $\tilde{e}\varphi = 0$. Then $\varphi = 0$ [7], which means that $\operatorname{Ker} \tilde{\triangle}^0 = 0$.

(2) Case $q = 1$. We have

$$M_K^1(e) = M_K'(X) \oplus L^0(A) \oplus L^0(B). \tag{4.6}$$

It is sufficient to show that $\operatorname{Ker} \tilde{\triangle}^1 \subseteq \operatorname{Im} \tilde{\triangle}^0$. Let $(\varphi_2, \psi_2, \varphi_1, 0) \in \operatorname{Ker} \tilde{\triangle}^1$, then

$$\delta^1\varphi = 0, \qquad \tilde{i}'\varphi_2 = -\delta^0\psi_2, \tag{4.7}$$

$$\tilde{e}^1\varphi_2 = \delta^0\varphi_1, \tag{4.8}$$

$$\tilde{e}_1^0(-\psi_2) = \tilde{j}\varphi_1, \tag{4.9}$$

where $i : A \hookrightarrow X$, $j : C \hookrightarrow B$ and $e_1 = e \mid C$.

By (4.9), there exists $\varphi \in M_K^0(X)$ [7] such that

$$\tilde{i}^0\varphi = -\psi_2, \qquad \tilde{e}^0\varphi = \varphi_1. \tag{4.10}$$

By (4.8), (4.9), and (4.10), we get

$$\tilde{i}^1(\delta^0\varphi - \varphi_2) = 0, \qquad \tilde{e}^1(\delta^0\varphi - \varphi_2) = 0. \tag{4.11}$$

Then $\delta^0\varphi = \varphi_2$ [7], which together with (4.11) yield $(\varphi,0,0,0) \in M_K^0(e)$ such that $\tilde{\triangle}^0(\varphi,0,0,0) = (\varphi_2,\psi_2,\varphi_1,0)$. □

Combining the sequence (4.4) and the above theorem, we get the following result.

Corollary 4.2. *Under the assumptions of Theorem 4.1, the map $\tilde{e}^{*0} : \tilde{H}_K^0(X,A) \to \tilde{H}_K^0(B,C)$ is an isomorphism but $\tilde{e}^{*1}$ is a monomorphism.*

Next we give a second application to the work introduced in this paper.

Let $\eta : (G,G') \to (F,F')$ be a homeomorphism of pairs of (discrete) abelian groups, which is an epimorphism, $(L,L') = \operatorname{Ker} \eta$ and $\lambda : (L,L') \hookrightarrow (G,G')$. Then for each $\bar{\alpha} \in \Omega(X,A)$, the maps η, λ define naturally a short exact sequence

$$0 \longrightarrow C^q(\bar{\alpha}, L, L') \longrightarrow C^q(\bar{\alpha}; G, G') \longrightarrow C^q(\bar{\alpha}; F, F') \longrightarrow 0; \tag{4.12}$$

its cohomology is a long exact sequence [12] denoted by $S_{\bar{\alpha}}$. One can show that $\{S_{\bar{\alpha}}\}_{\Omega(X,A)}$ is a direct system, its direct limit [7, 8] is

$$\begin{aligned} \cdots \longrightarrow \bar{H}_K^{q-1}(X,A;F,F') \longrightarrow \bar{H}_K^{q}(X,A';L,L') \longrightarrow \bar{H}_K^{q}(X,A;G,G') \\ \longrightarrow \bar{H}_K^{q}(X,A,F,F') \longrightarrow \bar{H}_K^{q+1}(X,A;L,L') \longrightarrow \cdots. \end{aligned} \tag{4.13}$$

Now instead of F take the factor group G/G' and so F' will be the null subgroup of G/G'. Then the above sequence yields the following result.

Theorem 4.3. *Consider (X,A) has a trivial $(q-1)$-dimensional space K-Alexander-Spanier cohomology group with finite cochains, and a trivial $(q+1)$-dimensional K-Alexander-Spanier cohomology with infinite cochains, taken over the coefficient groups G/G' and G', respectively. Then the group $\bar{H}_K^q(X,A;G,G')$ defined over an arbitrary pair (G,G') of coefficient groups is the extension of the cohomology group $\bar{H}_K^q(X,A;G')$ with infinite cochains over G' by the group $\bar{H}_K^q(X,A,G/G')$ with finite cochains over G/G'.*

Acknowledgement. The first author would like to thank the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy, for hospitality.

References

[1] N. Berikashvili, *On axiomatic theory of spectra and duality laws for arbitrary space*, Trudy Tbilisi Mat. Inst. **24** (1957).

[2] E. H. Brown Jr. and R. H. Szczarba, *Continuous cohomology and real homotopy type*, Trans. Amer. Math. Soc. **311** (1989), no. 1, 57–106. MR 89f:55005. Zbl 671.55006.

[3] ______, *Continuous cohomology and real homotopy type. II*, Astérisque (1990), no. 191, 45–70, International Conference on Homotopy Theory (Marseille, Luminy, 1988). MR 92d:55004. Zbl 731.55008.

[4] G. S. Chogoshvili, *Ueber Homologieapproximationen und Dualitaetssaetze fuer beliebige Mengen* [*On homological approximation and duality laws for arbitrary space*], Mat. Sb. (N.S.) **28** (1951), no. 70, 89–118 (Russian). Zbl 042.16903.

[5] A. E. A. Dabbour, *Ueber die Baladzeschen Homologiegruppen kompakter Raeume* [*On The Baladze homology groups of compact spaces*], Soobshch. Akad. Nauk Gruzin. SSR **77** (1975), no. 3, 541–544 (Russian). MR 52#9213. Zbl 323.55012.

[6] ______, *On the K-Kolmogorov homology and cohomology construction*, Simon Stevin **66** (1992), no. 3-4, 357–387. MR 94k:55013. Zbl 780.55005.

[7] A. E. A. Dabbour and Hijazi R. A., *Axiomatic characterization of Alexander-Spanier K-types*, ICTP preprint IC/96/152, Bull. Pure Appl. Sci. Sec. E Math. **17** (1998).

[8] ______, *The K-Alexander-Spanier cohomology*, ICTP preprint IC/95/231, Bull. Pure Appl. Sci. Sec. E Math. **17** (1998).

[9] S. Eilenberg and N. Steenrod, *Foundations of Algebraic Topology*, Princeton Mathematical Series No.15, Princeton University Press, Princeton, 1952. MR 14,398b. Zbl 047.41402.

[10] R. Engelking, *General Topology*, PWN—Polish Scientific Publishers, Warsaw, 1977, translated from the Polish by the author. Monografie Matematyczne, Tom 60. [Mathematical Monographs, vol. 60]. MR 58#18316b. Zbl 373.54002.

[11] S.-t. Hu, *Cohomology theory in topological groups*, Michigan Math. J. **1** (1952), 11–59. MR 14,449b. Zbl 048.25802.

[12] S. Mac Lane, *Homology*, Classics in Mathematics, Springer-Verlag, Berlin, 1995, Reprint of the 1975 edition. MR 96d:18001. Zbl 818.18001.

[13] W. S. Massey, *A Basic Course in Algebraic Topology*, Graduate Texts in Mathematics, 127, Springer-Verlag, New York, 1991. MR 92c:55001. Zbl 725.55001.

[14] L. M. Mdzinarishvili, *Partially continuous Alexander-Spanier cohomology theory*, **130** (1996), 1–39, Mathematical Institute of University Heidelberg.

[15] M. A. Mostow, *Continuous cohomology of spaces with two topologies*, Mem. Amer. Math. Soc. **7** (1976), no. 175, x+142. MR 54#1253. Zbl 343.55004.

[16] C. W. Patty, *Foundations of Topology*, The Prindle, Weber and Schmidt Series in Advanced Mathematics, PWS-KENT Publishing Co., Boston, 1993. MR 94d:54002. Zbl 931.54001.

[17] E. H. Spanier, *Cohomology theory for general spaces*, Ann. of Math. (2) **49** (1948), 407–427. MR 9,523d. Zbl 035.24801.

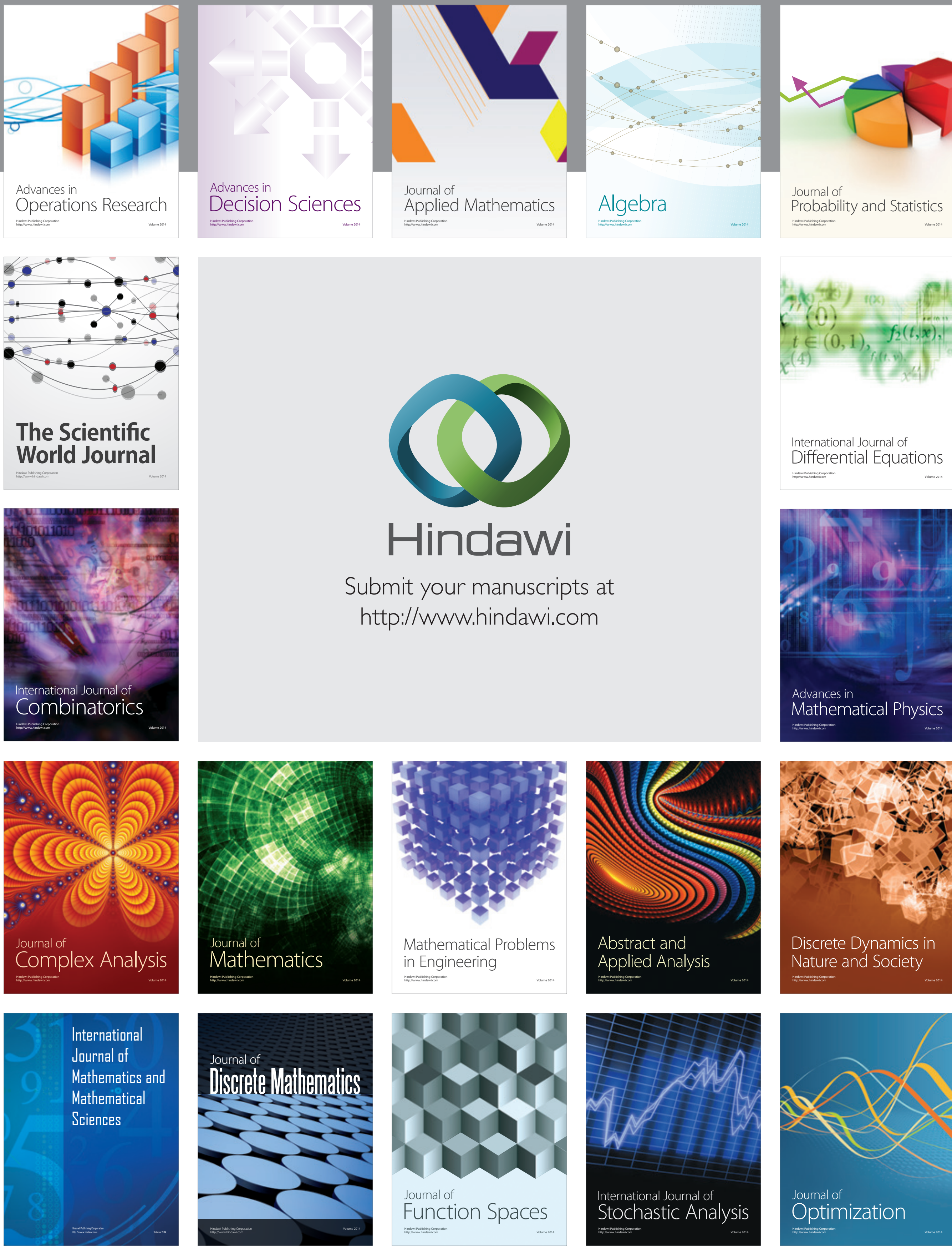
[18] ———, *Algebraic Topology*, McGraw-Hill Series in Higher Mathematics, McGraw-Hill Book, New York, 1966. MR 35#1007. Zbl 145.43303.

Abd El-Sattar A. Dabbour: Department of Mathematics, Faculty of Science, Ain-Shams University, Abbasia, Cairo, Egypt

Current address: The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy

E-mail address: sherif@sunet.shams.eg

Rola A. Hijazi: Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 14466, Jeddah 21424, Saudi Arabia


Advances in Operations Research
Advances in Decision Sciences
Journal of Applied Mathematics
Algebra
Journal of Probability and Statistics
The Scientific World Journal
International Journal of Differential Equations
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of Combinatorics
Advances in Mathematical Physics
Journal of Complex Analysis
Journal of Mathematics
Mathematical Problems in Engineering
Abstract and Applied Analysis
Discrete Dynamics in Nature and Society
International Journal of Mathematics and Mathematical Sciences
Journal of Discrete Mathematics
Journal of Function Spaces
International Journal of Stochastic Analysis
Journal of Optimization