Hindawi
Discrete Dynamics in Nature and Society
Volume 2017, Article ID 9050289, 15 pages
https://doi.org/10.1155/2017/9050289

Research Article

Analysis of a Class of Fractional Nonlinear Multidelay Differential Systems

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Received 31 July 2017; Accepted 30 August 2017; Published 15 October 2017

Academic Editor: Antonio Iannizzotto

We address existence and Ulam-Hyers and Ulam-Hyers-Mittag-Leffler stability of fractional nonlinear multiple time-delays systems with respect to two parameters' weighted norm, which provides a foundation to study iterative learning control problem for this system. Secondly, we design PID-type learning laws to generate sequences of output trajectories to tracking the desired trajectory. Two numerical examples are used to illustrate the theoretical results.

1. Introduction

Fractional differential equations have been used to deal with many problems from physics, engineering, and other fields. For some basic results in the theory of fractional differential equations, one can read the monographs [1–3] or the survey [4] and reference therein. Recently, considerable attention has been given to the control and stability of fractional differential equations; one can refer to [5–25] via Ulam's type stability concepts and the references therein. We also note that there are some contributions on Mittag-Leffler stability of fractional order systems and stabilization [26–29]. We remark that there are some difference between the concept of Mittag-Leffler stability and Ulam-Hyers-Mittag-Leffler stability. The concept of Mittag-Leffler stability of solution follows the idea of stability of zero solution for the classical ODEs and gives an estimate inequality for the norm of solution via Mittag-Leffler function. The concept of Ulam-Hyers-Mittag-Leffler stability follows the idea of Ulam-Hyers stability of functional equations and gives an approximate relation via small parameter and Mittag-Leffler function between the solution of equations and the solution of inequalities, which is a special case of Ulam-Hyers-Rassias stability. That is, we try to find a solution of approximate inequalities close to the solution of the original equations in the sense of Ulam-Hyers-Mittag-Leffler stability. The main idea for this concept will provide an approach to find the explicit solution. However, there are only few works on existence and Ulam's type stability for the nonlinear fractional time-delays differential equations.

Iterative learning control has become a popular strategy in the intelligent control community since it was proposed by Uchiyama [30] and developed by Arimoto et al. [31]. Recently, iterative learning control problems of P-type, D-type, I-type, or their combination schemes have been widely applied to various types of repetitive or batch dynamical systems (see, e.g., [32–38]). The problem on designing an ILC for uncertain plants with time-delays has not been fully investigated, and only a limited number of the results are available so far (see, e.g., [39–41]). However, most of the existing literatures focus on iterative learning control of the nonlinear fractional differential system without time-delays, especially multiple time-delays. Note that PID-type ILC learning algorithm is one of the popular updating laws. The advantage of PID-type ILC learning algorithm is simple and very easy to be realized in tracking problem. The disadvantage of PID-type ILC is that the error characterization for the signal is not the best and there is not a uniform method to design the weighting coefficients.

Delay systems are widely used to model dynamical systems in many scientific and engineering areas, for example, biology, climatology, and economy. Comparing with systems with single delay, systems with multidelay are more realistic models in the interacting complex systems. In fact, dynamics of multifeedback systems are representative examples of the multidelay systems.

Motivated by [15, 42], we firstly discuss existence, Ulam-Hyers stability, and Ulam-Hyers-Mittag-Leffler stability of solutions to fractional order nonlinear Cauchy problems with multiple time-delays of the form:

$$\begin{aligned}
&\left({}^cD_{0+}^{\alpha}x\right)(t) \\
&\quad = f\left(t, x(t), x(t-\tau_1), \ldots, x(t-\tau_m)\right) \\
&\qquad + I\left(t, x(t), x(t-\xi_1), \ldots, x(t-\xi_n)\right)u(t), \\
&\qquad\qquad t \in [0,T],\ 0<\alpha<1, \\
&x(t) = \psi(t), \\
&\qquad t \in [-a,0],\ a = \max\{\tau_1,\ldots,\tau_m,\xi_1,\ldots,\xi_n\},
\end{aligned} \tag{1}$$

where T is a positive constant; ${}^cD_{0+}^{\alpha}$ is the Caputo fractional derivative of order α with the lower limit zero; $u \in C([0,T],\mathbb{R})$; $\tau_1,\ldots,\tau_m,\xi_1,\ldots,\xi_n$ are positive constant time-delays; $x(t) \in \mathbb{R}$; ψ is the initial continuous function of the system in $t \in [-a,0]$; $f \in C([0,T]\times\underbrace{\mathbb{R}\times\cdots\mathbb{R}}_{n},\mathbb{R})$; and $I \in C([0,T]\times\underbrace{\mathbb{R}\times\cdots\mathbb{R}}_{n},\mathbb{R})$.

Secondly, we turn to study PID-type ILC learning algorithm of the following fractional order nonlinear system with output equation:

$$\begin{aligned}
&\left({}^cD_{0+}^{\alpha}x_k\right)(t) \\
&\quad = f\left(t, x_k(t), x_k(t-\tau_1), \ldots, x_k(t-\tau_m)\right) \\
&\qquad + I\left(t, x_k(t), x_k(t-\xi_1), \ldots, x_k(t-\xi_n)\right)u_k(t), \\
&\qquad\qquad t \in [0,T],\ 0<\alpha<1, \\
&x_k(t) = \psi_k(t), \\
&\qquad t \in [-a,0],\ a = \max\{\tau_1,\ldots,\tau_m,\xi_1,\ldots,\xi_n\}, \\
&y_k(t) = g\left(t, x_k(t)\right) + d\int_0^t u_k(s)\,ds, \quad t \in [0,T],
\end{aligned} \tag{2}$$

where k denotes the kth learning iteration; $u_k(t) \in \mathbb{R}$ and $y_k(t) \in \mathbb{R}$ are the states and control input and output of the system, respectively; ψ_k is the initial continuous function of the system in $t \in [-a,0]$; f, I, and u_k are given continuous functions in $[0,T]$; and $g : [0,T]\times\mathbb{R} \to \mathbb{R}$ is a continuous function.

The rest of the paper is organized as follows. Section 2 collects some notations and preparation results. Section 3 presents existence and uniqueness of solutions and shows Ulam-Hyers stability and Ulam-Hyers-Mittag-Leffler stability of solutions by using Picard operator method. Section 4 presents convergence result for PID-type ILC updating law. Section 5 gives two illustrative examples.

2. Preliminaries

Denote $X \coloneqq C([-a,T],\mathbb{R})$ as the Banach space of continuous functions from $[-a,T] \to \mathbb{R}$ endowed with the (λ,α)-norm $\|x\|_{\lambda,\alpha} = \max_{t\in[-a,T]} e^{-\lambda t^{\alpha}}|x(t)|$ ($x \in C([-a,T],\mathbb{R})$, $\lambda > 0$, $0<\alpha<1$).

Definition 1 (see [2]). The Riemann-Liouville fractional integrals $I_{a+}^{\alpha}f$ are defined by

$$\left(I_{a+}^{\alpha}f\right)(x) \coloneqq \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}dt, \quad (x>a;\ \alpha>0), \tag{3}$$

and the Riemann-Liouville fractional derivatives $D_{a+}^{\alpha}f$ are defined by

$$\begin{aligned}
{}^L\left(D_{a+}^{\alpha}f\right)(x) &\coloneqq \left(\frac{d}{dx}\right)^n \left(I_{a+}^{n-\alpha}f\right)(x) \\
&= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}}dt, \\
&\qquad (x>a;\ \alpha>0;\ n=[\alpha]+1),
\end{aligned} \tag{4}$$

where $\Gamma(\cdot)$ is Gamma function.

Definition 2 (see [2]). The Caputo derivative of order γ for a function $f : [a,\infty) \to R$ can be written as

$$\begin{aligned}
{}^cD_{a+}^{\gamma}f(t) &= {}^LD_{a+}^{\gamma}\left(f(t) - \sum_{k=a}^{n-1}\frac{t^k}{k!}f^{(k)}(a)\right), \\
&\qquad t>0,\ n-1<\gamma<n.
\end{aligned} \tag{5}$$

Definition 3 (see [43]). Let (X,d) be a metric space. An $A : X \to X$ is a Picard operator if there exists $x^* \in X$ such that (i) $F_A = x^*$, where $F_A = \{x \in X : A(x) = x\}$ is the fixed point set of A; (ii) the sequence $(A^n(x_0))_{n\in N}$ converges to x^* for all $x_0 \in X$.

Lemma 4 (see [43]). *Let $(X,d,\le)$ be an ordered metric space and $A : X \to X$ be an increasing Picard operator ($F_A = x^*$). Then, for $x \in X$, $x \le A(x)$ implies $x \le x_A^*$.*

The following Gronwall inequalities will be used in the sequel.

Lemma 5 (see [44, Lemma 3.1]). *Let $u(t)$ be a continuous function on $t \in [0,T]$ and let $v(t-\tau)$ be continuous and nonnegative on the triangle $0 \le \tau \le t$. Moreover, let $w(t)$ be a positive continuous and nondecreasing function on $t \in [0,T]$. If*

$$u(t) \le w(t) + \int_0^t v(t-\tau)u(\tau)\,d\tau, \quad t \in [0,T], \tag{6}$$

then

$$u(t) \le w(t)\,e^{\int_0^t v(t-\tau)d\tau}, \quad t \in [0,T]. \tag{7}$$

Lemma 6 (see [44, Lemma 7.1.1]). *Let $z, \omega : [0, T) \to [0, \infty)$ be continuous functions where $T \leq \infty$. If ω is nondecreasing and there are constants $\kappa \geq 0$ and $0 < \alpha < 1$ such that*

$$z(t) \leq \omega(t) + \kappa \int_0^t (t-s)^{\alpha-1} z(s)\, ds, \quad t \in [0, T), \tag{8}$$

then

$$z(t) \leq \omega(t) + \int_0^t \left(\sum_{n=1}^{\infty} \frac{(\kappa \Gamma(\alpha))^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} \omega(s) \right) ds, \quad t \in [0, T). \tag{9}$$

Remark 7 (see [44]). Under the hypothesis of Lemma 6, let $\omega(t)$ be a nondecreasing function on $[0, T)$. Then we have $z(t) \leq \omega(t) E_\alpha(\kappa \Gamma(\alpha) t^\alpha)$.

By [45, Lemma 2.12], one can adopt the similar idea to prove the following result.

Lemma 8. *Let $0 < \alpha < 1$ and $\lambda > 0$. Set*

$$Z := \int_0^t (t-s)^{\alpha-1} e^{\lambda s^\alpha} ds, \quad t \in [0, T], \ T > 0. \tag{10}$$

Then

$$\begin{aligned} Z &\leq \frac{t^\alpha e^{\lambda t^\alpha}}{\alpha([\lambda]+1)^\alpha}, \quad \text{if } [\lambda] = 0, \\ Z &\leq \frac{e^{\lambda t^\alpha}(1+t^\alpha)}{\alpha [\lambda]^\alpha}, \quad \text{if } [\lambda] \geq 1, \end{aligned} \tag{11}$$

where T is a positive number and $[\lambda]$ denotes the integer part of λ.

Proof. For completeness we supply the proofs. Denote

$$\begin{aligned} Z_1 &:= \sum_{k=1}^{[\lambda]} \int_{(k-1)t/([\lambda]+1)}^{kt/([\lambda]+1)} (t-s)^{\alpha-1} e^{\lambda s^\alpha} ds, \\ Z_2 &:= \int_{[\lambda]t/([\lambda]+1)}^{t} (t-s)^{\alpha-1} e^{\lambda s^\alpha} ds. \end{aligned} \tag{12}$$

Case 1. If $[\lambda] = 0$ then $Z = Z_2$. Obviously,

$$\begin{aligned} Z_2 &= \int_{[\lambda]t/([\lambda]+1)}^{t} (t-s)^{\alpha-1} e^{\lambda s^\alpha} ds \\ &\leq e^{\lambda t^\alpha} \int_{[\lambda]t/([\lambda]+1)}^{t} (t-s)^{\alpha-1} ds = \frac{t^\alpha e^{\lambda t^\alpha}}{\alpha([\lambda]+1)^\alpha}. \end{aligned} \tag{13}$$

Case 2. If $[\lambda] \geq 1$ then $Z = Z_1 + Z_2$. Obviously,

$$\begin{aligned} &\int_{(k-1)t/([\lambda]+1)}^{kt/([\lambda]+1)} (t-s)^{\alpha-1} e^{\lambda s^\alpha} ds \\ &\leq \int_{(k-1)t/([\lambda]+1)}^{kt/([\lambda]+1)} \left(\frac{s([\lambda]+1)}{k} - s \right)^{\alpha-1} \\ &\cdot e^{\lambda s^\alpha} ds \left(s \leq \frac{kt}{[\lambda]+1} \Longrightarrow t \geq \frac{s([\lambda]+1)}{k} \right) \\ &\leq \left(\frac{([\lambda]+1)}{k} - 1 \right)^{\alpha-1} \int_{(k-1)t/([\lambda]+1)}^{kt/([\lambda]+1)} s^{\alpha-1} e^{\lambda s^\alpha} ds \\ &\leq \frac{1}{\lambda\alpha} \left(\frac{([\lambda]+1)}{k} - 1 \right)^{\alpha-1} \left(e^{\lambda(kt/([\lambda]+1))^\alpha} \right. \\ &\left. - e^{\lambda((k-1)t/([\lambda]+1))^\alpha} \right). \end{aligned} \tag{14}$$

Hence, we get

$$\begin{aligned} Z_1 &:= \sum_{k=1}^{[\lambda]} \int_{(k-1)t/([\lambda]+1)}^{kt/([\lambda]+1)} (t-s)^{\alpha-1} e^{\lambda s^\alpha} ds \\ &\leq \sum_{k=1}^{[\lambda]} \frac{1}{\lambda\alpha} \left(\frac{([\lambda]+1)}{k} - 1 \right)^{\alpha-1} \left(e^{\lambda(kt/([\lambda]+1))^\alpha} \right. \\ &\left. - e^{\lambda((k-1)t/([\lambda]+1))^\alpha} \right) \leq \frac{1}{\lambda\alpha} \left[[\lambda]^{\alpha-1} \left(e^{\lambda(t/([\lambda]+1))^\alpha} - 1 \right) \right. \\ &+ \left(\frac{([\lambda]+1)}{2} - 1 \right)^{\alpha-1} \left(e^{\lambda(2t/([\lambda]+1))^\alpha} - e^{\lambda(t/([\lambda]+1))^\alpha} \right) \\ &+ \cdots + \left(\frac{([\lambda]+1)}{[\lambda]} - 1 \right)^{\alpha-1} \\ &\left. \cdot \left(e^{\lambda([\lambda]t/([\lambda]+1))^\alpha} - e^{\lambda(([\lambda]-1)t/([\lambda]+1))^\alpha} \right) \right] \\ &\leq \frac{1}{\lambda\alpha} \left[\left(\frac{([\lambda]+1)}{1} - 1 \right)^{\alpha-1} \left(e^{\lambda(t/([\lambda]+1))^\alpha} - 1 \right) \right. \\ &+ \left(\frac{([\lambda]+1)}{2} - 1 \right)^{\alpha-1} \left(e^{\lambda(2t/([\lambda]+1))^\alpha} - 1 \right) + \cdots \\ &\left. + \left(\frac{([\lambda]+1)}{[\lambda]} - 1 \right)^{\alpha-1} \left(e^{\lambda([\lambda]t/([\lambda]+1))^\alpha} - 1 \right) \right] \leq \frac{1}{\lambda\alpha} \\ &\cdot [\lambda] [\lambda]^{1-\alpha} \left(e^{\lambda([\lambda]t/([\lambda]+1))^\alpha} - 1 \right) \\ &\leq \frac{1}{\alpha [\lambda]^{\alpha-1}} \left(e^{\lambda([\lambda]t/([\lambda]+1))^\alpha} - 1 \right) \leq \frac{1}{\alpha [\lambda]^{\alpha-1}} \\ &\cdot e^{\lambda([\lambda]t/([\lambda]+1))^\alpha} \leq \frac{e^{\lambda t^\alpha}}{\alpha [\lambda]^\alpha}. \end{aligned} \tag{15}$$

Furthermore, we can get

$$Z = Z_1 + Z_2 \leq \frac{e^{\lambda t^\alpha}}{\alpha [\lambda]^\alpha} + \frac{t^\alpha e^{\lambda t^\alpha}}{\alpha([\lambda]+1)^\alpha} \leq \frac{e^{\lambda t^\alpha}(1+t^\alpha)}{\alpha [\lambda]^\alpha}. \tag{16}$$

The proof is finished. □

3. Existence and Stability Results

We introduce the following assumptions:

(H_1) Assume that $f \in C([0,T] \times \underbrace{\mathbb{R} \times \cdots \mathbb{R}}_{n}, \mathbb{R})$, $I \in C([0,T] \times \underbrace{\mathbb{R} \times \cdots \mathbb{R}}_{n}, \mathbb{R})$, and $u \in C([0,T], \mathbb{R})$. In addition, set $u_b = \max_{t\in[0,T]} |u(t)|$.

(H_2) For arbitrary $x_1, x_2 \in C([-a,T], \mathbb{R})$, there exist positive constants L_f, L_I such that

$$\begin{aligned} &|f(t, x_2(t), x_2(t-\tau_1), \ldots, x_2(t-\tau_m)) \\ &\quad - f(t, x_1(t), x_1(t-\tau_1), \ldots, x_1(t-\tau_m))| \\ &\quad \le L_f \left(|x_2(t) - x_1(t)| \right. \\ &\quad \left. + \left| \sum_{l=1}^{m} (x_2(t-\tau_l) - x_1(t-\tau_l)) \right| \right), \\ &|I(t, x_2(t), x_2(t-\xi_1), \ldots, x_2(t-\xi_m)) \\ &\quad - I(t, x_1(t), x_1(t-\xi_1), \ldots, x_1(t-\xi_m))| \\ &\quad \le L_I \left(|x_2(t) - x_1(t)| \right. \\ &\quad \left. + \left| \sum_{l=1}^{n} (x_2(t-\xi_l) - x_1(t-\xi_l)) \right| \right). \end{aligned} \tag{17}$$

(H_3) Suppose the following inequalities hold:

$$\begin{aligned} \frac{cT^\alpha}{\Gamma(\alpha+1)} &< 1, \quad [\lambda] = 0, \\ \frac{c(1+T^\alpha)}{\Gamma(\alpha+1)[\lambda]^\alpha} &< 1, \quad [\lambda] \ge 1, \end{aligned} \tag{18}$$

where $c = L_f(m+1) + L_I b_u(n+1)$.

Theorem 9. *Assume that* (H_1), (H_2), *and* (H_3) *are satisfied. Then problem (1) has a unique solution in* $C([-a,T], \mathbb{R})$.

Proof. Define an operator $A : X \to X$ $(X := C([-a,T], \mathbb{R}))$ as follows:

$$A(x)(t) = \begin{cases} \psi(t), & t \in [-a, 0], \\ \psi(0) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} (f(s, x(s), x(s-\tau_1), \ldots, x(s-\tau_m)) + I(s, x(s), x(s-\xi_1), \ldots, x(s-\xi_n)) u(s)) \, ds, & t \in [0, T]. \end{cases} \tag{19}$$

Next, we show that A defined in (19) is a contraction mapping on X with respect to the previous (λ, α)-norm $\|\cdot\|_{\lambda,\alpha}$.

For all $t \in [-a, 0]$ and $x(t), z(t) \in X$, we have $\|A(x)(t) - A(z)(t)\| = 0$. This yields that $\|A(x) - A(z)\|_{\lambda,\alpha} = 0$.

For any $t \in [0,T]$ and $x, z \in X$, according to (H_2), we have

$$\begin{aligned} |A(x)(t) - A(z)(t)| &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\ &\quad \cdot |f(s, x(s), x(s-\tau_1), \ldots, x(s-\tau_m)) \\ &\quad - f(s, z(s), z(s-\tau_1), \ldots, z(s-\tau_m))| \\ &\quad + |(I(s, x(s), x(s-\xi_1), \ldots, x(s-\xi_n)) \\ &\quad - I(s, z(s), z(s-\xi_1), \ldots, z(s-\xi_n)))| \, |u(s)| \, ds \\ &\le \frac{L_f}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(|x(s) - z(s)| \right. \\ &\quad \left. + \left| \sum_{l=1}^{m} (x(s-\tau_l) - z(s-\tau_l)) \right| \right) ds + \frac{L_I b_u}{\Gamma(\alpha)} \\ &\quad \cdot \int_0^t (t-s)^{\alpha-1} \left(|x(s) - z(s)| \right. \\ &\quad \left. + \left| \sum_{l=1}^{n} (x(s-\xi_l) - z(s-\xi_l)) \right| \right) ds. \end{aligned} \tag{20}$$

Let

$$G(\theta) = |x(t-\theta) - z(t-\theta)|, \quad \theta \in \{\tau_1, \ldots, \tau_m, \xi_1, \ldots, \xi_n\}. \tag{21}$$

Therefore, (20) can be written as

$$\begin{aligned} |A(x)(t) - A(z)(t)| &\le \frac{L_f}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\ &\quad \cdot \left(|x(s) - z(s)| + \left| \sum_{l=1}^{m} G(\tau_l) \right| \right) ds + \frac{L_I b_u}{\Gamma(\alpha)} \\ &\quad \cdot \int_0^t (t-s)^{\alpha-1} \left(|x(s) - z(s)| + \left| \sum_{l=1}^{n} G(\xi_l) \right| \right) ds \\ &\le \frac{L_f}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\ &\quad \cdot e^{\lambda s^\alpha} e^{-\lambda s^\alpha} \left(|x(s) - z(s)| + \sum_{l=1}^{m} G(\tau_l) \right) ds \end{aligned}$$

$$+\frac{L_I b_u}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \cdot e^{\lambda s^\alpha} e^{-\lambda s^\alpha}\left(|x(s)-z(s)|+\sum_{l=1}^{n} G(\xi_l)\right) ds. \quad (22)$$

Consider $G(\cdot)$ defined in (21); we have

$$\int_0^t (t-s)^\alpha e^{\lambda s^\alpha} e^{-\lambda s^\alpha} G(\theta)\, ds = \int_0^t (t-s)^\alpha e^{\lambda s^\alpha} e^{-\lambda s^\alpha} |x(s-\theta)-z(s-\theta)|\, ds \le \begin{cases} \int_0^t (t-s)^\alpha e^{\lambda s^\alpha} e^{-\lambda s^\alpha} |\psi(s)-\psi(s)|\, ds = 0, & t\in[0,\theta], \\ \int_0^t (t-s)^\alpha e^{\lambda s^\alpha} \|x-z\|_{\lambda,\alpha}\, ds, & t\in[\theta,T], \end{cases} \le \int_0^t (t-s)^\alpha e^{\lambda s^\alpha} ds\, \|x-z\|_{\lambda,\alpha}. \quad (23)$$

Substituting (23) into (22), using Lemma 8, we can obtain

$$|A(x)(t)-A(z)(t)| \le \frac{L_f}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \cdot e^{\lambda s^\alpha} e^{-\lambda s^\alpha}\left(|x(s)-z(s)|+\sum_{l=1}^{m} G(\tau_l)\right) ds + \frac{L_I b_u}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \cdot e^{\lambda s^\alpha} e^{-\lambda s^\alpha}\left(|x(s)-z(s)|+\sum_{l=1}^{n} G(\xi_l)\right) ds \le \frac{L_f(m+1)+L_I b_u(n+1)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} e^{\lambda s^\alpha} ds\, \|x-z\|_{\lambda,\alpha} \le \begin{cases} \frac{cT^\alpha e^{\lambda t^\alpha}}{\Gamma(\alpha+1)(1+[\lambda])^\alpha}\|x-z\|_{\lambda,\alpha}, & [\lambda]=0, \\ \frac{c(1+T^\alpha)e^{\lambda t^\alpha}}{\Gamma(\alpha+1)[\lambda]^\alpha}\|x-z\|_{\lambda,\alpha}, & [\lambda]\ge 1. \end{cases} \quad (24)$$

This implies that

$$\|A(x)-A(z)\|_{\lambda,\alpha} \le \begin{cases} \frac{cT^\alpha}{\Gamma(\alpha+1)(1+[\lambda])^\alpha}\|x-z\|_{\lambda,\alpha}, & [\lambda]=0, \\ \frac{c(1+T^\alpha)}{\Gamma(\alpha+1)[\lambda]^\alpha}\|x-z\|_{\lambda,\alpha}, & [\lambda]\ge 1, \end{cases} \quad (25)$$

where c is defined in (H_3).

Due to (H_3), we can derive that A is a contraction via the (λ,α)-norm $\|\cdot\|_{\lambda,\alpha}$ on X. The rest of the proof follows from the Banach contraction principle. □

Let $\epsilon>0$. Consider (1) and the following inequality:

$$\left|{}^cD_{0,t}^\alpha z(t) - f(t,z(t),z(t-\tau_1),\ldots,z(t-\tau_m)) - I(t,z(t),z(t-\xi_1),\ldots,z(t-\xi_n))u(t)\right| \le \epsilon, \quad t\in[-a,T],\ 0<\alpha<1,\ z\in X. \quad (26)$$

Definition 10. Equation (1) is Ulam-Hyers stable if there exists $\hat{c}>0$ such that for each $\epsilon>0$ and for each solution $\overline{z}\in X$ of the inequality (26) there exists a solution $x\in X$ of (1) with

$$|\overline{z}(t)-x(t)| \le \hat{c}\epsilon, \quad t\in[-a,T]. \quad (27)$$

Remark 11. A function $\overline{z}\in X$ is a solution of inequality (26) if and only if there exists a function $\overline{w}\in X$ (which depend on $\overline{z}$) such that

(i) $|\overline{w}(t)|\le\epsilon$, $t\in[-a,T]$.
(ii) ${}^cD_{0,t}^\alpha \overline{z}(t) = f(t,\overline{z}(t),\overline{z}(t-\tau_1),\ldots,\overline{z}(t-\tau_m)) + I(t,\overline{z}(t),\overline{z}(t-\xi_1),\ldots,\overline{z}(t-\xi_n))u(t) + \overline{w}(t)$, $t\in[0,T]$, $0<\alpha<1$.

Theorem 12. *Assume that (H_1), (H_2), and (H_3) are satisfied; then (1) is Ulam-Hyers stable.*

Proof. Let $\overline{z}\in C([-a,T],\mathbb{R})$ be a solution of inequality (26) and $x(t)$ be a solution of

$$\left({}^cD_{0+}^\alpha x\right)(t) = f(t,x(t),x(t-\tau_1),\ldots,x(t-\tau_m)) + I(t,x(t),x(t-\xi_1),\ldots,x(t-\xi_n))u(t), \quad t\in[0,T],\ 0<\alpha<1, \quad (28)$$
$$x(t)=\overline{z}(t), \quad t\in[-a,0],\ a=\max\{\tau_1,\ldots,\tau_m,\xi_1,\ldots,\xi_n\}.$$

Then

$$x(t) = \begin{cases} \overline{z}(t), & t\in[-a,0], \\ \overline{z}(0)+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[f(s,x(s),x(s-\tau_1),\ldots,x(s-\tau_m))+I(s,x(s),x(s-\xi_1),\ldots,x(s-\xi_n))u(s)\right] ds, & t\in[0,T]. \end{cases} \quad (29)$$

Obviously,

$$\begin{aligned}{}^{c}D_{0,t}^{\alpha}\bar{z}(t) &= f(t,\bar{z}(t),\bar{z}(t-\tau_1),\ldots,\bar{z}(t-\tau_m)) \\ &\quad + I(t,\bar{z}(t),\bar{z}(t-\xi_1),\ldots,\bar{z}(t-\xi_n))u(t) \\ &\quad + \bar{w}(t), \quad t\in[0,T],\end{aligned} \tag{30}$$

which can be turned to

$$\begin{aligned}\bar{z}(t) &= \bar{z}(0)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1} \\ &\quad \cdot[f(s,\bar{z}(s),\bar{z}(s-\tau_1),\ldots,\bar{z}(s-\tau_m)) \\ &\quad + I(s,\bar{z}(s),\bar{z}(s-\xi_1),\ldots,\bar{z}(s-\xi_n))u(s) \\ &\quad + \bar{w}(s)]\,ds, \quad t\in[0,T],\end{aligned} \tag{31}$$

where $\bar{w}\in X$ (which depend on $\bar{z}$).

According to Remark 11, one has

$$\begin{aligned}&\Big|\bar{z}(t)-\bar{z}(0)-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1} \\ &\quad\cdot[f(s,\bar{z}(s),\bar{z}(s-\tau_1),\ldots,\bar{z}(s-\tau_m)) \\ &\quad - I(s,\bar{z}(s),\bar{z}(s-\xi_1),\ldots,\bar{z}(s-\xi_n))u(s)]\,ds\Big| \\ &\quad\le\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|\bar{w}(s)|\,ds\le\frac{\epsilon}{\Gamma(\alpha)}\int_0^t(t \\ &\quad - s)^{\alpha-1}\,ds\le c_1\epsilon, \quad c_1=\frac{T^{\alpha}}{\Gamma(\alpha+1)}.\end{aligned} \tag{32}$$

Therefore,

$$\begin{aligned}|\bar{z}(t)-x(t)| &\le \Big|\bar{z}(t)-\bar{z}(0)-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}(f(s,\bar{z}(s),\bar{z}(s-\tau_1),\ldots,\bar{z}(s-\tau_m)) \\ &\quad + I(s,\bar{z}(s),\bar{z}(s-\xi_1),\ldots,\bar{z}(s-\xi_n))u(s)+\bar{w}(s))\,ds\Big|+\frac{1}{\Gamma(\alpha)}\Big|\int_0^t(t-s)^{\alpha-1} \\ &\quad\times[(f(s,x(s),x(s-\tau_1),\ldots,x(s-\tau_m))+I(s,x(s),x(s-\xi_1),\ldots,x(s-\xi_n))u(s)) \\ &\quad - (f(s,\bar{z}(s),\bar{z}(s-\tau_1),\ldots,\bar{z}(s-\tau_m))+I(s,\bar{z}(s),\bar{z}(s-\xi_1),\ldots,\bar{z}(s-\xi_n))u(s))]\,ds\Big|\le c_1\epsilon+\int_0^t(t-s)^{\alpha-1} \\ &\quad\cdot\left[\frac{L_f}{\Gamma(\alpha)}\left(|\bar{z}(s)-x(s)|+\left|\sum_{l=1}^{m}(\bar{z}(s-\tau_l)-x(s-\tau_l))\right|\right)+\frac{L_I b_u}{\Gamma(\alpha)}\left(|\bar{z}(s)-x(s)|\right.\right. \\ &\quad\left.\left.+\left|\sum_{l=1}^{n}(\bar{z}(s-\xi_l)-x(s-\xi_l))\right|\right)\right]ds\le\begin{cases}c_1\epsilon+\dfrac{cT^{\alpha}}{\Gamma(\alpha+1)(1+[\lambda])^{\alpha}}\|\bar{z}-x\|_{\lambda,\alpha}, & [\lambda]=0,\ t\in[0,T],\\ c_1\epsilon+\dfrac{c(1+T^{\alpha})}{\Gamma(\alpha+1)[\lambda]^{\alpha}}\|\bar{z}-x\|_{\lambda,\alpha}, & [\lambda]\ge1,\ t\in[0,T].\end{cases}\end{aligned} \tag{33}$$

Now multiplying by the fact $e^{-\lambda t^{\alpha}}$ on both side of the above inequalities, one can derive that

$$\begin{aligned}&\left(1-\frac{cT^{\alpha}}{\Gamma(\alpha+1)(1+[\lambda])^{\alpha}}\right)\|\bar{z}-x\|_{\lambda,\alpha}\le c_1\epsilon, \\ &\qquad [\lambda]=0,\ t\in[0,T], \\ &\left(1-\frac{c(1+T^{\alpha})}{\Gamma(\alpha+1)[\lambda]^{\alpha}}\right)\|\bar{z}-x\|_{\lambda,\alpha}\le c_1\epsilon, \\ &\qquad [\lambda]\ge1,\ t\in[0,T].\end{aligned} \tag{34}$$

So, we obtain

$$\begin{aligned}|\bar{z}(t)-x(t)| &\le \frac{c_1\epsilon e^{\lambda T^{\alpha}}}{1-cT^{\alpha}/\Gamma(\alpha+1)(1+[\lambda])^{\alpha}} \\ &= \frac{c_1\epsilon\Gamma(\alpha+1)(1+[\lambda])^{\alpha}e^{\lambda T^{\alpha}}}{\Gamma(\alpha+1)(1+[\lambda])^{\alpha}-cT^{\alpha}}, \\ &\qquad [\lambda]=0,\ t\in[0,T],\end{aligned}$$

$$\begin{aligned}|\bar{z}(t)-x(t)| &\le \frac{c_1\epsilon e^{\lambda T^{\alpha}}}{1-c(1+T^{\alpha})/\Gamma(\alpha+1)[\lambda]^{\alpha}} \\ &= \frac{c_1\epsilon\Gamma(\alpha+1)[\lambda]^{\alpha}e^{\lambda T^{\alpha}}}{\Gamma(\alpha+1)[\lambda]^{\alpha}-c(1+T^{\alpha})}, \\ &\qquad [\lambda]\ge1,\ t\in[0,T].\end{aligned} \tag{35}$$

Furthermore, according to (H_3), combined with the fact of $|\bar{z}(t)-x(t)|=0,\ t\in[-a,0]$, we can get

$$|\bar{z}(t)-x(t)|\le\hat{c}\epsilon, \quad t\in[-a,T], \tag{36}$$

where

$$\hat{c}=\max\left\{\frac{c_1\Gamma(\alpha+1)(1+[\lambda])^{\alpha}e^{\lambda T^{\alpha}}}{\Gamma(\alpha+1)(1+[\lambda])^{\alpha}-cT^{\alpha}},\ \frac{c_1\Gamma(\alpha+1)[\lambda]^{\alpha}e^{\lambda T^{\alpha}}}{\Gamma(\alpha+1)[\lambda]^{\alpha}-c(1+T^{\alpha})}\right\}. \tag{37}$$

Therefore, (1) is Ulam-Hyers stable. □

Different from the above stability result in the Theorem 12, in the following part, we will discuss Ulam-Hyers-Mittag-Leffler stability of (1) on the time interval $[-a, T]$. So we first introduce the following Ulam-Hyers-Mittag-Leffler stability definition.

Consider problem (1) with

$$\left|{}^{c}D_{0,t}^{\alpha} z(t) - f(t, z(t), z(t-\tau_1), \ldots, z(t-\tau_m)) - I(t, z(t), z(t-\xi_1), \ldots, z(t-\xi_n)) u(t)\right| \le \epsilon E_{\alpha}(t^{\alpha}), \quad t \in [0, T], \ 0 < \alpha < 1, \tag{38}$$

where E_α is the Mittag-Leffler function [2] defined by

$$E_{\alpha}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)}, \quad z \in C, \quad \operatorname{Re}(\alpha) > 0. \tag{39}$$

Definition 13 ([46]). Equation (1) is Ulam-Hyers-Mittag-Leffler stable with respect to $E_\alpha(t^\alpha)$ if there exists $c_{E_\alpha} > 0$ such that for each $\epsilon > 0$ and for each solution $z \in X$ to (38), there exists a solution $x \in X$ to (1) with

$$|z(t) - x(t)| \le c_{E_\alpha} \epsilon E_{\alpha}(t^{\alpha}), \quad t \in [-a, T]. \tag{40}$$

Remark 14 ([46]). A function $z \in X$ is a solution of inequality (38) if and only if there exists a function $w \in X$ (which depends on z) such that

(i) $|w(t)| \le \epsilon E_\alpha(t^\alpha)$ for all $t \in [0, T]$;

(ii) ${}^{c}D_{0,t}^{\alpha} z(t) = f(t, z(t), z(t-\tau_1), \ldots, z(t-\tau_m)) + I(t, z(t), z(t-\xi_1), \ldots, z(t-\xi_n)) u(t) + w(t)$, $t \in [0, T]$, $0 < \alpha < 1$.

Theorem 15. *Assume that (H_1), (H_2), and (H_3) are satisfied; then (1) is Ulam-Hyers-Mittag-Leffler stable.*

Proof. Let $z \in X$ be a solution to (38) and $x \in X$ be the unique solution of the following problem:

$$\begin{aligned} {}^{c}D_{0,t}^{\alpha} x(t) &= f(t, x(t), x(t-\tau_1), \ldots, x(t-\tau_m)) \\ &\quad + I(t, x(t), x(t-\xi_1), \ldots, x(t-\xi_n)) u(t), \quad t \in [0, T], \\ x(t) &= z(t), \quad t \in [-a, 0]. \end{aligned} \tag{41}$$

Obviously,

$$x(t) = \begin{cases} z(t), & t \in [-a, 0], \\ z(0) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} [f(s, x(s), x(s-\tau_1), \ldots, x(s-\tau_m)) + I(s, x(s), x(s-\xi_1), \ldots, x(s-\xi_n)) u(s) ds, & t \in [0, T]. \end{cases} \tag{42}$$

On the other hand, from [47, Remark 2] via Remark 14, we can know that z satisfied the following inequality:

$$\begin{aligned} &\left| z(t) - z(0) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \cdot [f(s, z(s), z(s-\tau_1), \ldots, z(s-\tau_m)) + I(s, z(s), z(s-\xi_1), \ldots, z(s-\xi_n)) u(s)] ds \right| \\ &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |w(s)| ds \le \frac{\epsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \cdot E_\alpha(s^\alpha) ds \le \epsilon E_\alpha(t^\alpha), \quad t \in [0, T]. \end{aligned} \tag{43}$$

Then, for $t \in [0, T]$, according to (H_2), we have

$$\begin{aligned} |z(t) - x(t)| &\le \left| z(t) - z(0) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \cdot (f(s, z(s), z(s-\tau_1), \ldots, z(s-\tau_m)) + I(s, z(s), z(s-\xi_1), \ldots, z(s-\xi_n)) u(s)) ds \right| \\ &\quad + \frac{1}{\Gamma(\alpha)} \left| \int_0^t (t-s)^{\alpha-1} \cdot (f(s, z(s), z(s-\tau_1), \ldots, z(s-\tau_m)) + I(s, z(s), z(s-\xi_1), \ldots, z(s-\xi_n)) u(s)) ds \right. \\ &\quad \left. - \int_0^t (t-s)^{\alpha-1} \cdot (f(s, x(s), x(s-\tau_1), \ldots, x(s-\tau_m)) + I(s, x(s), x(s-\xi_1), \ldots, x(s-\xi_n)) u(s)) ds \right| \\ &\le \epsilon E_\alpha(t^\alpha) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (|(f \cdot (s, z(s), z(s-\tau_1), \ldots, z(s-\tau_m)) - f(s, x(s), x(s-\tau_1), \ldots, x(s-\tau_m))| \\ &\quad + |I(s, z(s), z(s-\xi_1), \ldots, z(s-\xi_n)) - I(s, x(s), x(s-\xi_1), \ldots, x(s-\xi_n))| \end{aligned}$$

$$\cdot |u(s)|)\, ds \le \epsilon E_\alpha(t^\alpha) + \frac{L_f}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \cdot \Bigg(|x(s) - z(s)| + \left|\sum_{l=1}^m (x(s-\tau_l) - z(s-\tau_l))\right| \Bigg) ds + \frac{L_I b_u}{\Gamma(\alpha)} \cdot \int_0^t (t-s)^{\alpha-1} \Bigg(|x(s) - z(s)| + \left|\sum_{l=1}^n (x(s-\xi_l) - z(s-\xi_l))\right| \Bigg) ds. \tag{44}$$

Note the fact $|z(t) - x(t)| = 0,\ t \in [-a, 0]$. Consider the operator $F : C([-a, T], \mathbb{R}_+) \to C([-a, T], \mathbb{R}_+)$ defined by

$$F(\overline{x})(t) = \begin{cases} 0, & t \in [-a, 0], \\ \epsilon E_\alpha(t^\alpha) + \dfrac{L_f}{\Gamma(\alpha)} \left(\displaystyle\int_0^t (t-s)^{\alpha-1} \overline{x}(s)\, ds + \int_0^t (t-s)^{\alpha-1} \sum_{l=1}^m (\overline{x}(s-\tau_l)) \right) ds + \dfrac{L_I b_u}{\Gamma(\alpha)} \left(\displaystyle\int_0^t (t-s)^{\alpha-1} \overline{x}(s)\, ds + \int_0^t (t-s)^{\alpha-1} \sum_{l=1}^n (\overline{x}(s-\xi_l)) \right) ds, & t \in [0, T], \end{cases} \tag{45}$$

where $\overline{x} \in C([-a, T], \mathbb{R}_+)$.

Next, we verify that F is a Picard operator. In fact, for all $t \in [0, T]$ and arbitrary $\overline{x}, \overline{z} \in C([-a, T], \mathbb{R}_+)$, it follows the proof in Theorem 9; one can show that F is a contraction via the (λ, α)-norm on $C([-a, T], \mathbb{R}_+)$ due to (H_3).

Applying the Banach contraction principle to F, we derive that F is a Picard operator and $G_F = \{x^*\}$. Then, we have $x^* = 0$, for $t \in [-a, 0]$ and

$$\begin{aligned} x^*(t) &= \epsilon E_\alpha(t^\alpha) + \frac{L_f}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{\alpha-1} x^*(s)\, ds + \int_0^t (t-s)^{\alpha-1} \sum_{l=1}^m (x^*(s-\tau_l))\, ds \right) \\ &\quad + \frac{L_I b_u}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{\alpha-1} x^*(s)\, ds + \int_0^t (t-s)^{\alpha-1} \sum_{l=1}^n (x^*(s-\xi_l))\, ds \right), \quad t \in [0, T]. \end{aligned} \tag{46}$$

We go on to verify that the solution x^* is increasing. Now, denote $m_1 := \min_{s\in[0,T]} [x^*(s) + \sum_{l=1}^m x^*(s-\tau_l)] \in \mathbb{R}_+$ and $m_2 := \min_{s\in[0,T]} [x^*(s) + \sum_{l=1}^n x^*(s-\xi_l)] \in \mathbb{R}_+$.

Then, for $0 \le t_1 \le t_2 \le T$, we have

$$\begin{aligned} x^*(t_2) - x^*(t_1) &= \epsilon [E_\alpha(t_2^\alpha) - E_\alpha(t_1^\alpha)] + \frac{L_f}{\Gamma(\alpha)} \left(\int_0^{t_1} [(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}] \cdot \left(x^*(s) + \sum_{l=1}^m x^*(s-\tau_l) \right) ds \right) \\ &\quad + \frac{L_I b_u}{\Gamma(\alpha)} \left(\int_0^{t_1} [(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}] \cdot \left(x^*(s) + \sum_{l=1}^n x^*(s-\xi_l) \right) ds \right) \\ &\quad + \frac{L_f}{\Gamma(\alpha)} \left(\int_{t_1}^{t_2} (t_2-s)^{\alpha-1} \cdot \left(x^*(s) + \sum_{l=1}^m x^*(s-\tau_l) \right) ds \right) \\ &\quad + \frac{L_I b_u}{\Gamma(\alpha)} \left(\int_{t_1}^{t_2} (t_2-s)^{\alpha-1} \cdot \left(x^*(s) + \sum_{l=1}^n x^*(s-\xi_l) \right) ds \right) \ge \epsilon [E_\alpha(t_2^\alpha) - E_\alpha(t_1^\alpha)] \\ &\quad + \left(\frac{L_f m_1}{\Gamma(\alpha)} + \frac{L_I b_u m_2}{\Gamma(\alpha)} \right) \int_0^{t_1} [(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}]\, ds + \left(\frac{L_f m_1}{\Gamma(\alpha)} + \frac{L_I b_u m_2}{\Gamma(\alpha)} \right) \\ &\quad \cdot \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\, ds = \epsilon [E_\alpha(t_2^\alpha) - E_\alpha(t_1^\alpha)] + \left(\frac{L_f m_1}{\Gamma(\alpha+1)} + \frac{L_I b_u m_2}{\Gamma(\alpha+1)} \right) (t_2^\alpha - t_1^\alpha) > 0. \end{aligned} \tag{47}$$

So, x^* is increasing. Thus, $x^*(t-\theta) \le x^*(t)$ due to $t - \theta \le t$ $(\theta \in \{\tau_1, \ldots, \tau_m, \xi_1, \ldots, \xi_n\})$ and

$$\begin{aligned} x^*(t) &\le \epsilon E_\alpha(t^\alpha) + \frac{L_f(1+m)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x^*(s)\, ds + \frac{L_I b_u (n+1)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x^*(s)\, ds \le \epsilon E_\alpha(t^\alpha) \\ &\quad + \frac{L_f(m+1) + L_I b_u(n+1)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x^*(s)\, ds. \end{aligned} \tag{48}$$

Using Lemma 6, Remark 7, and the fact of $x^* = 0$, for $t \in [-a, 0]$, we obtain

$$\begin{aligned} x^*(t) &\le \epsilon E_\alpha(t^\alpha) E_\alpha\left(\left(L_f(m+1) + L_I b_u(n+1)\right) T^\alpha\right) \\ &\le c_{E_\alpha} \epsilon E_\alpha(t^\alpha), \quad t \in [-a, T], \end{aligned} \tag{49}$$

where $c_{E_\alpha} = E_\alpha((L_f(m+1) + L_I b_u(n+1))T^\alpha)$.

In particular, if $\overline{x}(t) = |z(t) - x(t)|$, from (44), $\overline{x} \le F\overline{x}$ and applying the Lemma 4, we obtain $\overline{x} \le x_F^*$, where F is a Picard and an increasing operator. As a result, we know

$$|z(t) - x(t)| \le c_{E_\alpha} \epsilon E_\alpha(t^\alpha), \quad t \in [-a, T]. \tag{50}$$

Thus, (2) is Ulam-Hyers-Mittag-Leffler stable. □

Remark 16. One can find that we use Gronwall's inequality method to derive asymptotic stability of the corresponding systems instead of using the Lyapunov direct method in [28, 29]. We do not need to assume that Lyapunov function satisfies some certain condition, for example, [28, Theorem 5, (12)-(13)]. Next, note that $E_\alpha(z) \le 1$ if $z \le 0$; then the definition of Mittag-Leffler stability can be turned to stability of zero solutions. However, the concept of Ulam-Hyers-Mittag-Leffler is more general since $E_\alpha(t^\alpha)$ is not necessarily less than 1 on the whole interval $[-a, T]$.

4. PID-Type ILC

In this section, we consider the open-loop and close-loop PID-type ILC updating laws of fractional order nonlinear system with multiple time-delays (2) via (λ, α)-weighted norm $\|\cdot\|_{\lambda,\alpha}$.

4.1. Open-Loop Case. For system (2), consider the open-loop PID-type ILC updating law with initial state learning:

$$\begin{aligned} x_{k+1}(0) &= x_k(0) + \varrho e_k(0), \\ u_{k+1}(t) &= u_k(t) + \eta_1 e_k(t) + \eta_2 \dot{e}_k(t) + \eta_3 \int_0^t e_k(s)\, ds, \end{aligned} \tag{51}$$

where ϱ and η_i $(i = 1, 2, 3)$ are unknown parameters to be determined.

For the sake of brevity, the following notations will be used:

$$\begin{aligned} f_k(t) &= f\left(t, x_k(t), x_k(t-\tau_1), \dots, x_k(t-\tau_m)\right), \\ I_k(t) &= I\left(t, x_k(t), x_k(t-\xi_1), \dots, x_k(t-\xi_n)\right), \\ g_k(t) &= g\left(t, x_k(t)\right). \end{aligned} \tag{52}$$

And we denote that $\Delta u_k(t) = u_{k+1}(t) - u_k(t)$, $\Delta x_k(t) = x_{k+1}(t) - x_k(t)$, and $e_k(t) = y_d(t) - y_k(t)$ which is called tracking error at kth repetition, where $y_d(t)$ denotes desired output bounded trajectory satisfying

$$\begin{aligned} \left({}^cD_{0+}^\alpha x_d\right)(t) &= f\left(t, x_d(t), x_d(t-\tau_1), \dots, x_d(t-\tau_m)\right) \\ &\quad + I\left(t, x_d(t), x_d(t-\xi_1), \dots, x_d(t-\xi_n)\right) u_d(t), \\ &\qquad t \in [0, T], \ 0 < \alpha < 1, \\ x_d(t) &= \psi_d(t), \\ &\qquad t \in [-a, 0], \ a = \max\{\tau_1, \dots, \tau_m, \xi_1, \dots, \xi_n\}, \\ y_d(t) &= g\left(t, x_d(t)\right) + d\int_0^t u_d(s)\, ds, \quad t \in [0, T], \end{aligned} \tag{53}$$

and u_d is a desired control.

In this section, we imposed the following assumptions on the class of system described by (2).

(H_4) The function g is continuous and differentiable for all x and t with partial derivatives g_x and g_t. For the constant β_j, $j = 1, 2$, set

$$0 < \beta_1 \le g_x(\cdot, x) := \frac{\partial g(\cdot, x)}{\partial x} \le \beta_2. \tag{54}$$

(H_5) The function $I(t, x(t), x(t-\xi_1), \dots, x(t-\xi_m))$ is uniformly bounded on $[0, T]$; that is, there exists $b_I > 0$ such that $|I(t, x(t), x(t-\xi_1), \dots, x(t-\xi_m))| \le b_I$ for any $t \in [0, T]$.

Theorem 17. *Assume that (H_1)–(H_5) hold. If*

$$\max\{|1 - \beta_1\varrho|, |1 - \beta_2\varrho|\} \le r_0 < 1, \tag{55}$$

$$r_1 := |1 - d\gamma_2| + |d\gamma_1 T| + \frac{|d\gamma_3| T^2}{2} < 1, \tag{56}$$

then the nonlinear fractional multiple time-delays differential system (2) with the open-loop PID-type ILC updating law (51) guarantees that y_k tends to y_d as $k \to \infty$ in the sense of (λ, α)-norm for all $t \in [0, T]$, where $\psi_d(t)$ is the desired initial state on $[-a, 0]$ and $y_d(t)$ is the desired output trajectory.

Proof. Note that

$$\begin{aligned} e_{k+1}(t) &= e_k(t) + g_k(t) - g_{k+1}(t) \\ &= e_k(t) - g_x\left(\mu_k(t), t\right) \Delta x_k(t) \\ &\quad - d\int_0^t \Delta u_k(s)\, ds, \end{aligned} \tag{57}$$

where $\mu_k(t)$ lies in the segment with end point $x_k(t)$ and $x_{k+1}(t)$ for $t \in [0, T]$.

In what follows, we show that $\|e_k\|_{\lambda,\alpha} \to 0$ as $k \to \infty$ for a.e. $t \in [0, T]$. By using mean value theorem we have

$$\begin{aligned} e_{k+1}(0) &= e_k(0) + y_k(0) - y_{k+1}(0) \\ &= e_k(0) - g_x\left(\mu_k(0), 0\right) \Delta x_k(0). \end{aligned} \tag{58}$$

Substituting (51) into (58) and taking the absolute value, we have

$$\begin{aligned}\left|e_{k+1}(0)\right| &= \left|e_k(0) + y_k(0) - y_{k+1}(0)\right| \\ &= \left|e_k(0) - g_x(\mu_k(0), 0)\,\Delta x_k(0)\right| \\ &= \left|e_k(0) - g_x(\mu_k(0), 0)\,\varrho e_k(0)\right| \\ &\le \left|1 - \varrho g_x(\mu_k(0), 0)\right|\left|e_k(0)\right| \le r_0\left|e_k(0)\right|.\end{aligned} \tag{59}$$

It follows from (55) that

$$\lim_{k\to\infty}\left\|e_k(0)\right\|_{\lambda,\alpha} = 0. \tag{60}$$

Now we turn to give an estimation for the upper bound of Δx_k. One has

$$\begin{aligned}\Delta x_k(t) &= x_{k+1}(t) - x_k(t) = \Delta x_k(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t \\ &- s)^{\alpha-1}\left(f_{k+1}(s) + I_{k+1}(s)\,u_{k+1}(s) - f_k(s)\right. \\ &\left. + I_k(s)\,u_k(s)\right) ds = \Delta x_k(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t \\ &- s)^{\alpha-1}\left(\left(f_{k+1}(s) - f_k(s)\right)\right. \\ &+ \left(I_{k+1}(s) - I_k(s)\right)u_{k+1}(s) \\ &\left. + I_k(s)\left(u_{k+1}(s) - u_k(s)\right)\right) ds.\end{aligned} \tag{61}$$

Repeating the same procedure in Theorem 9, we can get

$$\begin{aligned}\left|\Delta x_k(t)\right| &= \left|x_{k+1}(t) - x_k(t)\right| \le \left|\Delta x_k(0)\right| + \frac{1}{\Gamma(\alpha)}\int_0^t (t \\ &- s)^{\alpha-1}\left|f_{k+1}(s) + I_{k+1}(s)\,u_{k+1}(s) - f_k(s)\right. \\ &\left. - I_k(s)\,u_k(s)\right| ds \le \left|\Delta x_k(0)\right| + \frac{1}{\Gamma(\alpha)}\int_0^t (t \\ &- s)^{\alpha-1}\left(\left|f_{k+1}(s) - f_k(s)\right|\right. \\ &+ \left|I_{k+1}(s) - I_k(s)\right|\left|u_{k+1}(s)\right| \\ &\left. + \left|I_k(s)\right|\left|u_{k+1}(s) - u_k(s)\right|\right) ds \le \left|\Delta x_k(0)\right| \\ &+ \frac{L_f}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(\left|x_{k+1}(s) - x_k(s)\right|\right. \\ &\left. + \left|\sum_{l=1}^{m}\left(x_{k+1}(s-\tau_l) - x_k(s-\tau_l)\right)\right|\right) ds \\ &+ \frac{L_I b_u}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(\left|x_{k+1}(s) - x_k(s)\right|\right. \\ &\left. + \left|\sum_{l=1}^{n}\left(x_{k+1}(s-\xi_l) - x_k(s-\xi_l)\right)\right|\right) ds \\ &+ \frac{b_I}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|\Delta u_k(s)\right| ds \le \left|\Delta x_k(0)\right| \\ &+ \frac{L_f}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(L_f(m+1) + L_I b_u(n+1)\right) \\ &\cdot \left|\Delta x_k(s)\right| ds + \frac{b_I}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|\Delta u_k(s)\right| ds,\end{aligned} \tag{62}$$

where b_I, b_u are defined in (H_6).

Using Lemma 5, we get

$$\begin{aligned}&\left|\Delta x_k(t)\right| \\ &\le \left(\left|\Delta x_k(0)\right| + \frac{b_I}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|\Delta u_k(s)\right| ds\right) \\ &\cdot e^{c/\Gamma(\alpha)\int_0^t (t-s)^{\alpha-1} ds} \\ &\le \left(\left|\Delta x_k(0)\right| + \frac{b_I}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|\Delta u_k(s)\right| ds\right) \\ &\cdot e^{cT^\alpha/\Gamma(\alpha+1)},\end{aligned} \tag{63}$$

where $c = L_f(m+1) + L_I b_u(n+1)$.

We know

$$\begin{aligned}e_{k+1}(t) &= e_k(t) - \left(y_{k+1}(t) - y_k(t)\right) \\ &= e_k(t) - g_x(\mu_k(t), t)\,\Delta x_k(t) \\ &\quad - d\int_0^t \Delta u_k(s)\, ds.\end{aligned} \tag{64}$$

Taking (51) into (64), one obtains

$$\begin{aligned}e_{k+1}(t) &= e_k(t) - \left(y_{k+1}(t) - y_k(t)\right) = e_k(t) \\ &- g_x(\mu_k(t), t)\,\Delta x_k(t) - d\int_0^t \Delta u_k(s)\, ds = e_k(t) \\ &- g_x(\mu_k(t), t)\,\Delta x_k(t) - d\left(\eta_1\int_0^t e_k(s)\, ds\right. \\ &\left. + \eta_2\int_0^t \dot{e}_k(s)\, ds + \eta_3\int_0^t\int_0^s e_k(\tau)\, d\tau\, ds\right) = e_k(t) \\ &- g_x(\mu_k(t), t)\,\Delta x_k(t) - d\left(\eta_1\int_0^t e_k(s)\, ds\right. \\ &\left. + \eta_2\left(e_k(t) - e_k(0)\right) + \eta_3\int_0^t\int_\tau^t e_k(\tau)\, ds\, d\tau\right) = (1\end{aligned}$$

$$-d\eta_2)e_k(t)-g_x(\mu_k(t),t)\Delta x_k(t)-d\left(\eta_1\int_0^t e_k(s)\,ds-\eta_2 e_k(0)+\eta_3\int_0^t(t-s)e_k(s)\,ds\right)=(1-d\eta_2)e_k(t)+d\eta_2 e_k(0)-g_x(\mu_k(t),t)\Delta x_k(t)-d\eta_1\int_0^t e_k(s)\,ds-d\eta_3\int_0^t(t-s)e_k(s)\,ds. \quad (65)$$

Combining with (64) and Lemma 8, we can get

$$\begin{aligned}|e_{k+1}(t)|&\leq|1-d\eta_2||e_k(t)|+|d\eta_2||e_k(0)|+\xi|\Delta x_k(t)|+|d\eta_1|\int_0^t|e_k(s)|\,ds+|d\eta_3|\int_0^t(t-s)\|e_k(s)\|\,ds\\&\leq|1-d\eta_2||e_k(t)|+|d\eta_2||e_k(0)|+\xi\left(|\Delta x_k(0)|+\frac{b_I}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|\Delta u_k(s)|\,ds\right)e^{cT^\alpha/\Gamma(\alpha+1)}+|d\eta_1|\int_0^t|e_k(s)|\,ds+|d\eta_3|\int_0^t(t-s)|e_k(s)|\,ds\\&\leq\begin{cases}|1-d\eta_2||e_k(t)|+\left(|d\eta_2|+\xi\varrho e^{cT^\alpha/\Gamma(\alpha+1)}\right)|e_k(0)|+\dfrac{b_I\xi e^{cT^\alpha/\Gamma(\alpha+1)}}{\Gamma(\alpha)}\cdot\dfrac{e^{\lambda t^\alpha}t^\alpha}{\alpha([\lambda]+1)^\alpha}\|\Delta u_k\|_{\lambda,\alpha}+|d\eta_1|te^{\lambda t^\alpha}\|e_k\|_{\lambda,\alpha}+|d\eta_3|\cdot\dfrac{t^2}{2}e^{\lambda t^\alpha}\|e_k\|_{\lambda,\alpha}, & [\lambda]=0,\\|1-d\eta_2||e_k(t)|+\left(|d\eta_2|+\xi\varrho e^{cT^\alpha/\Gamma(\alpha+1)}\right)|e_k(0)|+\dfrac{b_I\xi e^{cT^\alpha/\Gamma(\alpha+1)}}{\Gamma(\alpha)}\cdot\dfrac{e^{\lambda t^\alpha}(1+t^\alpha)}{\alpha[\lambda]^\alpha}\|\Delta u_k\|_{\lambda,\alpha}+|d\eta_1|te^{\lambda t^\alpha}\|e_k\|_{\lambda,\alpha}+|d\eta_3|\cdot\dfrac{t^2}{2}e^{\lambda t^\alpha}\|e_k\|_{\lambda,\alpha}, & [\lambda]\geq1,\end{cases}\end{aligned} \quad (66)$$

where $\xi=\max\{|\beta_1|,|\beta_2|\}$.

Multiplying $e^{-\lambda t^\alpha}$ on both sides of (66) and taking the maximum value on $[0,T]$, we can get

$$\|e_{k+1}\|_{\lambda,\alpha}\leq\begin{cases}\left(|1-d\eta_2|+|d\eta_1|T+|d\eta_3|\cdot\dfrac{T^2}{2}\right)\|e_k\|_{\lambda,\alpha}+\left(|d\eta_2|+\xi\varrho e^{cT^\alpha/\Gamma(\alpha+1)}\right)\|e_k(0)\|_{\lambda,\alpha}+\dfrac{b_I\xi e^{cT^\alpha/\Gamma(\alpha+1)}\Gamma(\alpha)\cdot T^\alpha}{\alpha(1+[\lambda])^\alpha}\|\Delta u_k\|_{\lambda,\alpha}, & [\lambda]=0,\\\left(|1-d\eta_2|+|d\eta_1|T+|d\eta_3|\cdot\dfrac{T^2}{2}\right)\|e_k\|_{\lambda,\alpha}+\left(|d\eta_2|+\xi\varrho e^{cT^\alpha/\Gamma(\alpha+1)}\right)\|e_k(0)\|_{\lambda,\alpha}+\dfrac{b_I\xi e^{cT^\alpha/\Gamma(\alpha+1)}\Gamma(\alpha)\cdot(1+T^\alpha)}{\alpha[\lambda]^\alpha}\|\Delta u_k\|_{\lambda,\alpha}, & [\lambda]\geq1.\end{cases} \quad (67)$$

Set

$$\mu=\max\left\{\frac{b_I\xi e^{cT^\alpha/\Gamma(\alpha+1)}\Gamma(\alpha)\cdot T^\alpha}{\alpha(1+[\lambda])^\alpha},\frac{b_I\xi e^{cT^\alpha/\Gamma(\alpha+1)}\Gamma(\alpha)\cdot(1+T^\alpha)}{\alpha[\lambda]^\alpha}\right\}. \quad (68)$$

There exists a sufficiently large λ such that μ is very small and using (56) and (60) we can derive $\lim_{k\to\infty}\|e_k\|_{\lambda,\alpha}=0$. The proof is completed. □

4.2. Closed-Loop Case. For system (2), consider the close-loop PID-type ILC updating law with initial state learning:

$$\begin{aligned}x_{k+1}(0)&=x_k(0)+\zeta e_{k+1}(0),\\u_{k+1}(t)&=u_k(t)+\kappa_1 e_{k+1}(t)+\kappa_2\dot{e}_{k+1}(t)+\kappa_3\int_0^t e_{k+1}(s)\,ds,\end{aligned} \quad (69)$$

where ϱ and κ_i $(i=1,2,3)$ are unknown parameters to be determined.

Theorem 18. *Assume that (H_1)–(H_5) hold. If*

$$\min\{|1+\beta_1\zeta|,|1+\beta_2\zeta|\}\geq\overline{r}_0>1, \quad (70)$$

$$r_2:=|1+d\kappa_2|-|d\kappa_1|-\frac{|d\kappa_3|T^2}{2}>1, \quad (71)$$

then the nonlinear multiple time-delays system (2) with the close-loop PID-type ILC updating law (69) guarantees that y_k tends to y_d as $k\to\infty$ in the sense of λ,α-norm for all $t\in[0,T]$, where $\psi_d(t)$ is the desired initial state on $[-a,0]$ and $y_d(t)$ is the desired output trajectory.

Proof. Note that

$$\begin{aligned}e_{k+1}(t)&=e_k(t)+g_k(t)-g_{k+1}(t)\\&=e_k(t)-g_x(\mu_k(t),t)\Delta x_k(t)-d\int_0^t\Delta u_k(s)\,ds,\end{aligned} \quad (72)$$

where $\mu_k(t)$ lies in the segment with end points $x_k(t)$ and $x_{k+1}(t)$ for $t\in[0,T]$.

In what follows, we show that $\|e_k\|_{\lambda,\alpha}\to0$ as $k\to\infty$ for a.e. $t\in[0,T]$. By using mean value theorem we have

$$\begin{aligned}e_{k+1}(0)&=e_k(0)+y_k(0)-y_{k+1}(0)\\&=e_k(0)-g_x(\mu_k(0),0)\Delta x_k(0).\end{aligned} \quad (73)$$

Substituting (69) into (73) and taking the absolute value, we have

$$\left|1 + g_x\left(\mu_k\left(0\right), 0\right)\zeta\right|\left|e_{k+1}\left(0\right)\right| = \left|e_k\left(0\right)\right|. \tag{74}$$

It follows from (70) that

$$\lim_{k\to\infty}\left\|e_k\left(0\right)\right\|_{\lambda,\alpha} = 0. \tag{75}$$

Similar to the proof of Theorem 17, we can get

$$\begin{aligned}
e_{k+1}\left(t\right) &= e_k\left(t\right) - \left(y_{k+1}\left(t\right) - y_k\left(t\right)\right) = e_k\left(t\right) \\
&\quad - g_x\left(\mu_k\left(t\right), t\right)\Delta x_k\left(t\right) - d\int_0^t \Delta u_k\left(s\right) ds = e_k\left(t\right) \\
&\quad - g_x\left(\mu_k\left(t\right), t\right)\Delta x_k\left(t\right) - d\left(\kappa_1\int_0^t e_{k+1}\left(s\right) ds\right. \\
&\quad \left. + \kappa_2\int_0^t \dot{e}_{k+1}\left(s\right) ds + \kappa_3\int_0^t\int_0^s e_{k+1}\left(\tau\right) d\tau\, ds\right) \\
&= e_k\left(t\right) - g_x\left(\mu_k\left(t\right), t\right)\Delta x_k\left(t\right) \\
&\quad - d\left(\kappa_1\int_0^t e_{k+1}\left(s\right) ds + \kappa_2\left(e_{k+1}\left(t\right) - e_{k+1}\left(0\right)\right)\right. \\
&\quad \left. + \kappa_3\int_0^t\int_\tau^t e_k\left(\tau\right) ds\, d\tau\right) = e_k\left(t\right) - d\kappa_2 e_{k+1}\left(t\right) \\
&\quad + d\kappa_2 e_{k+1}\left(0\right) - g_x\left(\mu_k\left(t\right), t\right)\Delta x_k\left(t\right) \\
&\quad - d\kappa_1\int_0^t e_{k+1}\left(s\right) ds - d\kappa_3\int_0^t \left(t - s\right) e_{k+1}\left(s\right) ds.
\end{aligned} \tag{76}$$

Then, one has

$$\begin{aligned}
&\left|1 + d\kappa_2\right|\left|e_{k+1}\left(t\right)\right| \le \left|e_k\left(t\right)\right| + \left|d\kappa_2\right|\left|e_{k+1}\left(0\right)\right| + \xi\left|\Delta x_k\left(t\right)\right| + \left|d\kappa_1\right|\int_0^t \left|e_{k+1}\left(s\right)\right| ds + \left|d\kappa_3\right|\int_0^t \left(t - s\right)\left|e_{k+1}\left(s\right)\right| ds \\
&\le \left|e_k\left(t\right)\right| + \left|d\kappa_2\right|\left|e_{k+1}\left(0\right)\right| + \xi\left(\left|\Delta x_k\left(0\right)\right| + \frac{b_I}{\Gamma\left(\alpha\right)}\int_0^t \left(t - s\right)^{\alpha-1}\left|\Delta u_k\left(s\right)\right| ds\right) e^{cT^\alpha/\Gamma(\alpha+1)} + \left|d\kappa_1\right|\int_0^t \left|e_{k+1}\left(s\right)\right| ds + \left|d\kappa_3\right|\int_0^t \left(t - s\right)\left|e_{k+1}\left(s\right)\right| ds \\
&\le \left|e_k\left(t\right)\right| + \left(\left|d\kappa_2\right| + \xi\zeta e^{cT^\alpha/\Gamma(\alpha+1)}\right)\left|e_{k+1}\left(0\right)\right| + \xi\cdot e^{cT^\alpha/\Gamma(\alpha+1)}\frac{b_I}{\Gamma\left(\alpha\right)}\int_0^t \left(t - s\right)^{\alpha-1} e^{\lambda s^\alpha} ds\left\|\Delta u_k\right\|_{\lambda,\alpha} + \left|d\kappa_1\right|\int_0^t e^{\lambda s^\alpha} ds\left\|e_{k+1}\right\|_{\lambda,\alpha} \\
&\quad + \left|d\kappa_3\right|\int_0^t \left(t - s\right) e^{\lambda s^\alpha} ds\left\|e_{k+1}\right\|_{\lambda,\alpha} \\
&\le \begin{cases}
\left|e_k\left(t\right)\right| + \left(\left|d\kappa_2\right| + \xi\zeta e^{cT^\alpha/\Gamma(\alpha+1)}\right)\left|e_{k+1}\left(0\right)\right| + \dfrac{b_I\xi e^{cT^\alpha/\Gamma(\alpha+1)}}{\Gamma\left(\alpha\right)}\cdot\dfrac{e^{\lambda t^\alpha} t^\alpha}{\alpha\left(1 + [\lambda]\right)^\alpha}\left\|\Delta u_k\right\|_{\lambda,\alpha} + \left|d\kappa_1\right| t e^{\lambda t^\alpha}\left\|e_{k+1}\right\|_{\lambda,\alpha} + \left|d\kappa_3\right|\cdot\dfrac{t^2}{2} e^{\lambda t^\alpha}\left\|e_{k+1}\right\|_{\lambda,\alpha}, & [\lambda] = 0, \\
\left|e_k\left(t\right)\right| + \left(\left|d\kappa_2\right| + \xi\zeta e^{cT^\alpha/\Gamma(\alpha+1)}\right)\left|e_{k+1}\left(0\right)\right| + \dfrac{b_I\xi e^{cT^\alpha/\Gamma(\alpha+1)}}{\Gamma\left(\alpha\right)}\cdot\dfrac{e^{\lambda t^\alpha}\left(1 + t^\alpha\right)}{\alpha[\lambda]^\alpha}\left\|\Delta u_k\right\|_{\lambda,\alpha} + \left|d\kappa_1\right| t e^{\lambda t^\alpha}\left\|e_{k+1}\right\|_{\lambda,\alpha} + \left|d\kappa_3\right|\cdot\dfrac{t^2}{2} e^{\lambda t^\alpha}\left\|e_{k+1}\right\|_{\lambda,\alpha}, & [\lambda] \ge 1,
\end{cases}
\end{aligned} \tag{77}$$

where $\xi = \max\{|\beta_1|, |\beta_2|\}$.

Multiplying $e^{-\lambda t^\alpha}$ on both sides of (77) and taking λ,α-norm, we can derive

$$\begin{aligned}
&\left|1 + d\kappa_2\right|\left\|e_{k+1}\right\|_{\lambda,\alpha} \\
&\le \begin{cases}
\left\|e_k\right\|_{\lambda,\alpha} + \left(\left|d\kappa_2\right| + \xi\zeta e^{cT^\alpha/\Gamma(\alpha+1)}\right)\left\|e_{k+1}\left(0\right)\right\|_{\lambda,\alpha} + \dfrac{b_I\xi e^{cT^\alpha/\Gamma(\alpha+1)}}{\Gamma\left(\alpha\right)}\cdot\dfrac{T^\alpha}{\alpha\left(1 + [\lambda]\right)^\alpha}\left\|\Delta u_k\right\|_{\lambda,\alpha} + \left|d\kappa_1\right| T\left\|e_{k+1}\right\|_{\lambda,\alpha} + \left|d\kappa_3\right|\cdot\dfrac{T^2}{2}\left\|e_{k+1}\right\|_{\lambda,\alpha}, & [\lambda] = 0, \\
\left\|e_k\right\|_{\lambda,\alpha} + \left(\left|d\kappa_2\right| + \xi\zeta e^{cT^\alpha/\Gamma(\alpha+1)}\right)\left\|e_{k+1}\left(0\right)\right\|_{\lambda,\alpha} + \dfrac{b_I\xi e^{cT^\alpha/\Gamma(\alpha+1)}}{\Gamma\left(\alpha\right)}\cdot\dfrac{\left(1 + T^\alpha\right)}{\alpha[\lambda]^\alpha}\left\|\Delta u_k\right\|_{\lambda,\alpha} + \left|d\kappa_1\right| T\left\|e_{k+1}\right\|_{\lambda,\alpha} + \left|d\kappa_3\right|\cdot\dfrac{T^2}{2}\left\|e_{k+1}\right\|_{\lambda,\alpha}, & [\lambda] \ge 1.
\end{cases}
\end{aligned} \tag{78}$$

Set

$$\begin{aligned}
\widehat{\mu} = \max&\left\{\frac{b_I\xi e^{cT^\alpha/\Gamma(\alpha+1)}}{\Gamma\left(\alpha\right)}\cdot\frac{T^\alpha}{\alpha\left(1 + [\lambda]\right)^\alpha}, \frac{b_I\xi e^{cT^\alpha/\Gamma(\alpha+1)}}{\Gamma\left(\alpha\right)}\right. \\
&\left.\cdot\frac{\left(1 + T^\alpha\right)}{\alpha[\lambda]^\alpha}\right\}.
\end{aligned} \tag{79}$$

There exists a sufficiently large λ such that $\widehat{\mu}$ is very small and using (71) and (75) we can derive $\lim_{k\to\infty}\|e_k\|_{\lambda,\alpha} = 0$. This completes the proof of Theorem 18. □

5. Examples

In this section, we give two examples to illustrate our results above.

Example 1. Let $\alpha = 1/2$, $L_f = 1/8$, $L_I = 1/12$, $m = 2$, $n = 1$, and $T = 0.2$. We consider the following nonlinear fractional multiple time-delays differential equations

$$\begin{aligned} {}^cD_{0+}^{1/2}x(t) &= \frac{1}{8}\sin(5x(t)) + \frac{1}{8}\frac{x^2(t-1)}{1+x^2(t-1)} \\ &\quad + \frac{1}{8}\cos\left(x\left(t-\frac{1}{2}\right)\right) \\ &\quad + \frac{1}{12}\sin\left(3x\left(t-\frac{1}{3}\right)\right)u(t), \\ &\qquad t\in[0,0.2], \\ x(0) &= 1, \quad t\in[-1,0] \end{aligned} \tag{80}$$

and the inequalities

$$\begin{aligned} &\left|{}^cD_t^{1/2}z(t) - f\left(t, z(t)\, z(t-1), z\left(t-\frac{1}{2}\right)\right)\right. \\ &\quad \left.+ I\left(t, z\left(t-\frac{1}{3}\right)\right)u(t)\right| \le \epsilon, \\ &\left|{}^cD_t^{1/2}z(t) - f\left(t, z(t)\, z(t-1), z\left(t-\frac{1}{2}\right)\right)\right. \\ &\quad \left.+ I\left(t, z\left(t-\frac{1}{3}\right)\right)u(t)\right| \le \epsilon E_{1/2}\left(t^{1/2}\right). \end{aligned} \tag{81}$$

Define $f(t, x(t), x(t-1), x(t-1/2)) = (1/8)\sin(5x(t)) + (1/8)(x^2(t-1)/(1+x^2(t-1))) + (1/8)\cos(x(t-1/2))$ and $I(t, x(t-1/3)) = (1/12)\sin(3x(t-1/3))$ and set $b_u = 1$, $\lambda = 4.2$. Thus $c = 0.5417$; then $c(1+T^\alpha)/\Gamma(\alpha+1)[\lambda]^\alpha = 0.4423 < 1$. Now all the assumptions in Theorems 9, 12, and 15 are satisfied, problem (80) has a unique solution, and the first equation in (80) is Ulam-Hyers stable with $|z(t) - x(t)| \le c\epsilon$, $t \in [-1, 0.2]$ and Ulam-Hyers-Mittag-Leffler stable with

$$|z(t) - x(t)| \le c_{E_{1/2}}\epsilon E_{1/2}\left(t^{1/2}\right), \quad t\in[-1,0.2], \tag{82}$$

where $c = 26.9701$, $c_{E_{1/2}} = 1.3447$.

Example 2. Let $\alpha = 1/2$, $L_f = 1/8$, $L_I = 1/12$, $m = 2$, $n = 1$, and $T = 0.2$. We consider the following nonlinear fractional multiple time-delays differential equations

$$\begin{aligned} {}^cD_{0+}^{1/2}x_k(t) &= \frac{1}{8}\sin(5x_k(t)) + \frac{1}{8}\frac{x_k^2(t-1)}{1+x_k^2(t-1)} \\ &\quad + \frac{1}{8}\cos\left(x_k\left(t-\frac{1}{2}\right)\right) \\ &\quad + \frac{1}{12}\sin\left(3x_k\left(t-\frac{1}{3}\right)\right)u_k(t), \\ &\qquad t\in[0,0.2], \\ x(0) &= 1, \quad t\in[-1,0], \\ y_k(t) &= 2x_k(t) + \frac{3}{2}\sin x_k(t) + 2\int_0^t u_k(s)\,ds, \\ &\qquad t\in[0,0.2] \end{aligned} \tag{83}$$

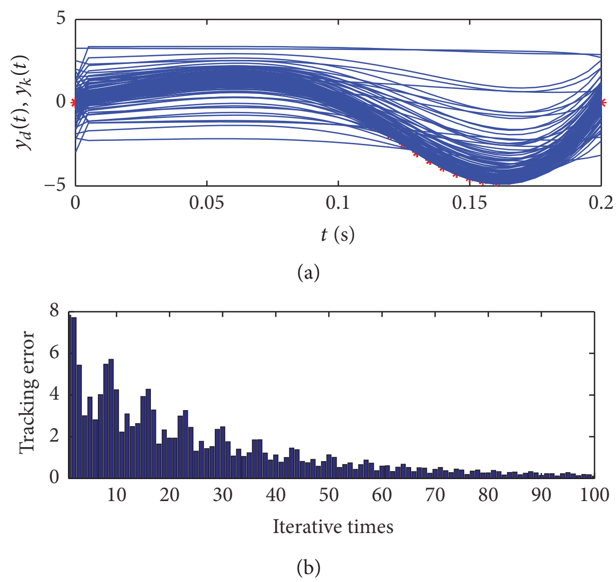

Figure 1: The system output $y_k(t)$ (blue line), the desired trajectory $y_d(t)$ (red line), and the tracking error for (83).

and the desired reference trajectory $y_d(t) = 10\sin(10\pi t)(1 + \cos(10\pi t)) + 2$, $t \in [0, 0.2]$.

Consider problem (83) and the open-loop PID-type ILC updating law with initial state learning:

$$\begin{aligned} x_{k+1}(0) &= x_k(0) + \frac{1}{5}e_k(0), \\ u_{k+1}(t) &= u_k(t) + \frac{1}{10}e_k(t) + \frac{1}{2}\dot{e}_k(t) + \frac{1}{5}\int_0^t e_k(s)\,ds. \end{aligned} \tag{84}$$

Obviously, $r_0 = 0.9 < 1$ and $r_1 = 0.048 < 1$. All the conditions of Theorem 17 are satisfied.

Next, consider problem (83) and the close-loop PID-type ILC updating law with initial state learning:

$$\begin{aligned} x_{k+1}(0) &= x_k(0) + \frac{3}{2}e_{k+1}(0), \\ u_{k+1}(t) &= u_k(t) + \frac{3}{2}e_{k+1}(t) + \frac{5}{2}\dot{e}_{k+1}(t) \\ &\quad + 6\int_0^t e_{k+1}(s)\,ds. \end{aligned} \tag{85}$$

Thus $\overline{r}_0 = 1.75 > 1$ and $r_2 = 2.76 > 1$. All the conditions of Theorem 18 are satisfied.

Numerical simulation diagram for the open-loop control is shown in Figure 1(a).

Figure 1(b) shows the supremum norm of the tracking error in each iteration and the 100th error is 0.0979.

6. Conclusions

This paper is twofold: in the first part we show the existence and uniqueness result and present Ulam-Hyers and Ulam-Hyers-Mittag-Leffler stability results for fractional nonlinear multiple time-delays systems. In the second part we apply PID-type learning updating laws to obtain sequences of tracking trajectory to approximate a given trajectory. In the future, we will study the related topic of noninstantaneous impulsive systems [48].

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The first and second author's work was partially supported by National Natural Science Foundation of China (11661016), Training Object of High Level and Innovative Talents of Guizhou Province ((2016)4006), Unite Foundation of Guizhou Province ([2015]7640), and Graduate ZDKC([2015]003). The third author acknowledges the support by National Natural Science Foundation of China (11671339). The authors are grateful to Associate Professor Dr. Qian Chen for her discussion, careful reading of the manuscript, and valuable comments.

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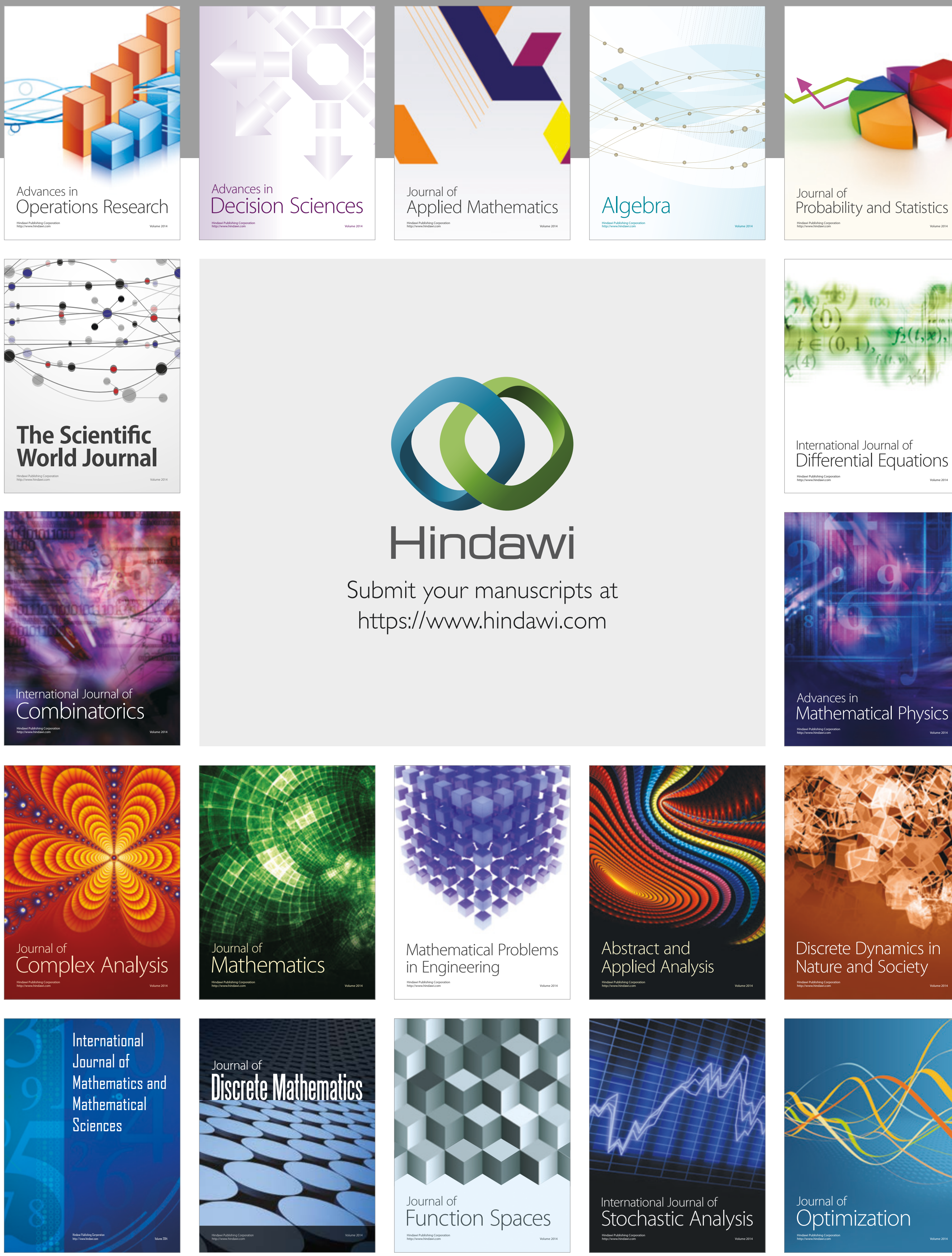
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