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Research Article

H_∞ Loop Shaping Control of Input Saturated Systems with Norm-Bounded Parametric Uncertainty

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This paper proposes a gain-scheduling control design strategy for a class of linear systems with the presence of both input saturation constraints and norm-bounded parametric uncertainty. LMI conditions are derived in order to obtain a gain-scheduled controller that ensures the robust stability and performance of the closed loop system. The main steps to obtain such a controller are given. Differently from other gain-scheduled approaches in the literature, this one focuses on the problem of H_∞ loop shaping control design with input saturation nonlinearity and norm-bounded uncertainty to reduce the effect of the disturbance input on the controlled outputs. Here, the design problem has been formulated in the four-block H_∞ synthesis framework, in which it is possible to describe the parametric uncertainty and the input saturation nonlinearity as perturbations to normalized coprime factors of the shaped plant. As a result, the shaped plant is represented as a linear parameter-varying (LPV) system while the norm-bounded uncertainty and input saturation are incorporated. This procedure yields a linear parameter-varying structure for the controller that ensures the stability of the polytopic LPV shaped plant from the vertex property. Finally, the effectiveness of the method is illustrated through application to a physical system: a VTOL "vertical taking-off landing" helicopter.

1. Introduction

In recent years, input saturation and model uncertainty problems have been extensively studied in the control system literature, where much attention has been focused on the problems of robust stabilization and performance [1–9]. Input saturation is a phenomenon due to inevitable physical limitations of the actuators, such as pumps or compressors that have finite throughput capacity and motors that have a limited range of speed [10]. This saturation can lead to deterioration of the actuator itself or even to the instability of the system. When a system is subject to input saturation, two main issues arise: the guarantee of stability and the containment of performance degradation [7]. To solve this problem, there exist two approaches: two-step and one-step designs [7]. In the first approach, called antiwindup design, a predesigned controller is given without considering the input saturation constraints; usually a standard controller is used. Then, after this controller has been designed, an antiwindup compensator is designed to handle the input saturation nonlinearity. Thus, the antiwindup compensator is designed to ensure that stability is maintained and the performance degradation is contained. Antiwindup compensator research using linear matrix inequalities has been vigorously pursued as can be seen by a significant number of referenced papers [2, 11–14]. This is because LMI techniques offer the advantage of operational simplicity when compared to other approaches. For the second approach, one-step design, the input saturation is directly accounted for in controller design; that is, the controller and an antiwindup compensator are simultaneously computed [15].

Several solutions for input saturation problems have been proposed. However, just few references deal with LPV systems [3, 16–18]. Motivated by this scarcity, recent work focused on employing gain-scheduling controllers designed with an H_∞ approach [19]. This resulted in an advantageous technique using LMI for an H_∞ loop shaping controller design with input saturation, derived from a four-block configuration. Its advantage lies fundamentally in the ease of making trade-offs between performance and robustness to plant uncertainty and the saturation nonlinearities are described as perturbations to normalized coprime factors of the shaped plant.

As gain-scheduling H_∞ loop shaping control can be viewed as robust against nonparametric uncertainties, this controller can also be viewed as a candidate to solve the robust control problem against presence of both nonparametric and parametric uncertainties. Thus, the contribution of this paper, motivated by the results in [19], consists in extending their procedure in order to ensure the robust stability of the closed loop system subject to both the constraint of an input saturation nonlinearity and parametric uncertainties. Here, the parameter uncertainties are assumed to be time-varying but norm-bounded. Sufficient conditions for the existence of the gain scheduled parametric H_∞ loop shaping controller are given in terms of an LMI framework, which also provides the Lyapunov matrix ensuring the stability and robust performance of the LPV controlled system from of the vertex property [20, 21]. The effectiveness of the design method was evaluated using a physical system: a VTOL helicopter [22]. This paper is organized as follows. Section 2 gives some problem statements and preliminaries about the purpose of the paper. In Section 3 a solvability condition for existence of a gain scheduled H_∞ loop shaping controller using the constraint of input saturation and norm-bounded parametric uncertainty is established. In Section 4, an example is considered to evaluate the effectiveness of the proposed method. Finally, Section 5 contains the conclusion.

The notation used in this paper is fairly standard: $\mathbb{R}^{n\times m}$ denotes the set of real $n \times m$ matrices and I_n is the $n \times n$ identity matrix. $M > 0$ (or $M < 0$) means M is symmetric and positive (or negative) definite. $\bullet$ indicates symmetric blocks in the LMIs. Here, the notation

$$G(s) := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{1}$$

is used to denote the transfer function $G(s) = C(sI - A)^{-1}B + D$.

2. Problem Statements and Preliminaries

The exposition in this section closely follows [19], where the loop-shaping design procedure based on H_∞ robust stabilization combined with a four-block configuration is presented. In Figure 1, the block diagram for the H_∞ loop shaping control design problem with input saturation nonlinearity is shown depicted. In this formulation, the input saturation nonlinearity is incorporated through a dynamic weighting function W, and the postcompensator is considered as an identity matrix with proper dimension.

In the absence of input saturation constraints an H_∞ controller K_∞ is designed such that the closed loop system satisfies the condition $\|T_{zw}\|_\infty \leq \gamma$:

$$\left\| \begin{bmatrix} K_\infty \\ I \end{bmatrix} (I - G_s K_\infty)^{-1} \begin{bmatrix} I & G_s \end{bmatrix} \right\|_\infty \leq \frac{1}{\varepsilon_{\max}} = \gamma, \tag{2}$$

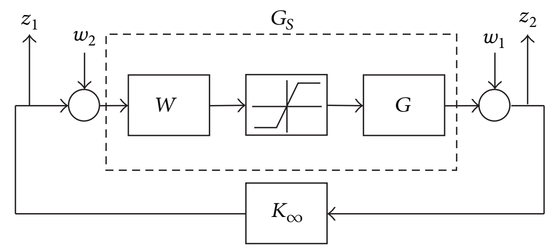

Figure 1: Block diagram for H_∞ loop shaping control using four-block configuration with input saturation.

where $\varepsilon_{\max}$ is the maximum achievable robust stability margin, T_{zw} is the transfer function matrix from the disturbances $[w_1 \;\; w_2]^T$ to the outputs $[z_1 \;\; z_2]^T$, and G_s is the shaped plant. Herein, the designer knows the maximum input $u_{\max,j}$ at the jth channel, that is, the control value that can be used without exceeding the limit of the actuator in channel j. The saturation nonlinearity is defined as follows, in normalized form:

$$\operatorname{sat}_j(u_j) = \begin{cases} -1, & u_j < -1, \\ u_j, & |u_j| \leq 1, \\ +1, & u_j > +1. \end{cases} \tag{3}$$

Normalization always can be achieved by scaling each channel in G and W with the appropriate factor. As shown in [19] the maximum control input in each jth channel can be specified; a corresponding slope α_j for each jth channel is also known. Without losing generality, this slope can be assumed to be less than or equal to 1. Define $\operatorname{sat}_j(u_j) = \theta_j u_j$ with θ_j as

$$\theta_j = \begin{cases} \dfrac{1}{u_j}, & u_j > 1, \\ 1, & |u_j| \leq 1, \\ -\dfrac{1}{u_j}, & u_j < -1, \end{cases} \tag{4}$$

where $j = 1, 2, \ldots, m$ and u_j is constrained by $-u_{\max,j} \leq u_j \leq u_{\max,j}$. It implies that $1/u_{\max,j} \leq \theta_j \leq 1$ $(\alpha_j \leq \theta_j \leq 1)$ and, for the multivariable system with m number of input channels, it can be written as $\operatorname{sat}(u) = \Theta u$ with $\Theta = \operatorname{diag}\{\theta_1, \theta_2, \ldots, \theta_m\}$. It is observed that in the H_∞ loop shaping using four-block configuration the shaped plant G_s can be represented as a polytopic system where the input saturation constraint is allocated between the dynamic weighting function W and the nominal plant G.

For the existence of a gain-scheduled controller, the sufficient conditions proposed by [19] for saturated systems have been elaborated in the following theorem.

Theorem 1 (see [19]). *Consider the nominal plant G and a proper weighting function W, with* $-u_{\max,j} \leq u_j \leq u_{\max,j}$ *being a known saturation bounded for each jth channel where* $j = 1, 2, \ldots, m$. *Then there exists a shaped plant with input* *saturation nonlinearity that can be represented as a polytopic LPV system:*

$$G_s = \left[\begin{array}{c|c} A_s & B_s \\ \hline C_s & 0 \end{array}\right], \tag{5}$$

where (A_s, B_s, C_s) *is quadratically stabilizable and detectable for all* $\Phi = (I - \Theta)$ *in the polytope* $\Omega = \{\sum_{i=1}^{2^m} \lambda_i \xi_i : \lambda_i \geq 0, \sum_{i=1}^{2^m} \lambda_i = 1\}$ *with the vertices in* $[\xi_1, \ldots, \xi_{2^m}]$. *Moreover, there exists a stabilizing LPV controller* K_∞ *such that* $\gamma \geq 1$, *if and only if there exist* $R > 0$, $S > 0$ *that satisfy the following inequalities:*

$$\begin{gathered}
\left[\begin{array}{c|c} N_R & 0 \\ \hline 0 & I \end{array}\right]^T \left(\begin{array}{ccc|c} A_{s_i}R + RA_{s_i}^T & R & 0 \; C_s^T & 0 \;\; B_s \\ & & & 0 \;\; 0 \\ \cdot & & -\gamma I & I \;\; 0 \\ \hline \cdot & & \cdot & -\gamma I \end{array}\right) \left[\begin{array}{c|c} N_R & 0 \\ \hline 0 & I \end{array}\right] < 0, \quad i = 1, \ldots, 2^m, \\
\left[\begin{array}{c|c} N_S & 0 \\ \hline 0 & I \end{array}\right]^T \left(\begin{array}{ccc|c} A_{s_i}^T S + SA_{s_i} & S & 0 \; B_s & 0 \;\; C_s^T \\ & & & 0 \;\; I \\ \cdot & & -\gamma I & 0 \;\; 0 \\ \hline \cdot & & \cdot & -\gamma I \end{array}\right) \left[\begin{array}{c|c} N_S & 0 \\ \hline 0 & I \end{array}\right] < 0, \quad i = 1, \ldots, 2^m, \\
\begin{pmatrix} R & I \\ I & S \end{pmatrix} \geq 0,
\end{gathered} \tag{6}$$

where N_R *and* N_S *denote bases of the null spaces of* $\begin{bmatrix} B_s^T & I & 0 \end{bmatrix}$ *and* $[C_s \;\; I \;\; 0]$, *respectively.*

The proof of this theorem and additional details can be found in [19]. Conditions (6) are numerically tractable and solvable. From these sufficient conditions an H_∞ gain-scheduled controller can be synthesized.

In Theorem 1, the solvability conditions are defined by two positive definite matrices $R > 0$ and $S > 0$. If there exist feasible solutions, the gain-scheduled controller matrices in form

$$K_\infty(\Phi) = \left[\begin{array}{c|c} A_K(\Phi) & B_K(\Phi) \\ \hline C_K(\Phi) & D_K(\Phi) \end{array}\right] := \sum_{i}^{2^m} \lambda_i \left[\begin{array}{c|c} A_{K_i} & B_{K_i} \\ \hline C_{K_i} & D_{K_i} \end{array}\right] \tag{7}$$

are then calculated following the method described in [23]. The main steps to the understanding of the gain-scheduling control design will be described. Firstly, consider the closed loop system

$$\left[\begin{array}{c} \dot{x} \\ \dot{x}_K \\ \hline z_1 \\ z_2 \end{array}\right] = \left[\begin{array}{cc|cc} A_s + B_s D_K C_s & B_s C_K & B_s D_K & B_s \\ B_K C_s & A_K & B_K & 0 \\ \hline D_K C_s & C_K & D_K & 0 \\ C_s & 0 & I & 0 \end{array}\right] \left[\begin{array}{c} x \\ x_K \\ \hline w_1 \\ w_2 \end{array}\right] \tag{8}$$

which can be written by the state-space equations

$$\Sigma_0 := \begin{cases} \dot{x}_{\text{cl}} = A_{\text{cl}}(\Phi) x_{\text{cl}} + B_{\text{cl}}(\Phi) w \\ z = C_{\text{cl}}(\Phi) x_{\text{cl}} + D_{\text{cl}}(\Phi) w \end{cases} \tag{9}$$

with x_K being the controller state, $x_{\text{cl}} = \begin{bmatrix} x^T & x_K^T \end{bmatrix}^T$, $w = \begin{bmatrix} w_1^T & w_2^T \end{bmatrix}^T$, and $z = \begin{bmatrix} z_1^T & z_2^T \end{bmatrix}^T$. Here, A_{cl}, B_{cl}, C_{cl}, and D_{cl} are the affine functions of $K_\infty(\Phi)$. In a second step, find the Lyapunov matrix P from of the following linear equation:

$$P \begin{pmatrix} R & I \\ M^T & 0 \end{pmatrix} = \begin{pmatrix} I & S \\ 0 & N^T \end{pmatrix}. \tag{10}$$

P is a unique solution, where M and N are full row rank matrices with $MN^T = I - RS$ [23]. Now, using the well known *Bounded Real Lemma* [24] that ensures the internal stability and the H_∞ norm constraint, determine the gain-scheduled controller if and only if there exists a $P > 0$ such that

$$\begin{pmatrix} A_{\text{cl}}^T(\xi_i) P + PA_{\text{cl}}(\xi_i) & PB_{\text{cl}}(\xi_i) & C_{\text{cl}}^T(\xi_i) \\ \cdot & -\gamma I & D_{\text{cl}}^T(\xi_i) \\ \cdot & \cdot & -\gamma I \end{pmatrix} < 0, \quad i = 1, \ldots, 2^m. \tag{11}$$

Applying the procedure described in [19, 20, 23] for the LMIs conditions above, we obtain the gain-scheduled H_∞ loop shaping controller that ensures the robust stability of the closed loop system subject to constraint of input saturation nonlinearity. Now, some important results that are required to establish the main results of this paper will be described.

3. Main Results

Consider the plant G described by state-space models in the form

$$G(\Delta) := \begin{cases} \dot{x}(t) = [A + \Delta A] x(t) + Bv(t) \\ y(t) = Cx(t) + Dv(t), \end{cases} \tag{12}$$

where $x \in \mathbb{R}^n$ is the state vector, $v \in \mathbb{R}^m$ the input vector, and $y \in \mathbb{R}^p$ the output vector. Herein, A, B, C, and D are known constant matrices that describe the nominal system and ΔA is a matrix function representing the time-varying parametric uncertainty. The parametric uncertainty is considered in form $\Delta A = F\Delta(t)E$, where F and E are known constant matrices with appropriate dimensions and $\Delta(t)$ is an unknown matrix with Lebesgue measurable elements such that $\|\Delta(t)\|_2 \leq 1$

[25]. Now, consider that the dynamic weighting function W can be defined as

$$W := \begin{cases} \dot{x}_w(t) = A_w x_w(t) + B_w p(t) \\ u(t) = C_w x_w(t) + D_w p(t), \end{cases} \tag{13}$$

where $x_w \in \mathbb{R}^{n_w}$ is the state vector, $p \in \mathbb{R}^{m_w}$ the input vector, and $u \in \mathbb{R}^{p_w}$ the output vector of the dynamic weighting function. Following the same procedure described by [19] the state-space model of the H_∞ loop shaping framework (Figure 1) in the presence of input saturation and uncertainties in the A matrix can be written as

$$\begin{aligned} \dot{x}(t) &= [A + \Delta A]\, x(t) + B\Theta C_w x_w(t) + B\Theta D_w p(t), \\ \dot{x}_w(t) &= A_w x_w(t) + B_w p(t), \\ y(t) &= Cx(t) + D\Theta C_w x_w + D\Theta D_w p(t). \end{aligned} \tag{14}$$

Following a few mathematical manipulations we can write the shaped plant $G_{s\Delta}(\Phi)$ as

$$\begin{gathered} \begin{bmatrix} \dot{x} \\ \dot{x}_w \\ \hline y \end{bmatrix} = \left[\begin{array}{cc|c} A + \Delta A & B\Theta C_w & B\Theta D_w \\ 0 & A_w & B_w \\ \hline C & D\Theta C_w & D\Theta D_w \end{array}\right] \begin{bmatrix} x \\ x_w \\ \hline p \end{bmatrix}, \\ G_{s\Delta}(\Phi) \Longrightarrow \begin{bmatrix} \dot{x}_s \\ \hline y \end{bmatrix} = \left[\begin{array}{c|c} \overbrace{A_s + \Delta A_s}^{A_{D_s}} & B_s \\ \hline C_s & D_s \end{array}\right] \begin{bmatrix} \dot{x}_s \\ \hline p \end{bmatrix}. \end{gathered} \tag{15}$$

Thus, the state-space matrices can be written in the form

$$\begin{aligned} A_{D_s} &= \underbrace{\begin{bmatrix} A & B\Theta C_w \\ 0 & A_w \end{bmatrix}}_{A_s} + \underbrace{\begin{bmatrix} F\Delta(t)E & 0 \\ 0 & 0 \end{bmatrix}}_{\Delta A_s}; \\ A_s &= \begin{bmatrix} A & BC_w \\ 0 & A_w \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} (I - \Theta) \begin{bmatrix} 0 & -C_w \end{bmatrix}; \\ B_s &= \begin{bmatrix} BD_w \\ B_w \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} (I - \Theta)(-D_w); \\ C_s &= \begin{bmatrix} C & DC_w \end{bmatrix} + D(I - \Theta)\begin{bmatrix} 0 & -C_w \end{bmatrix}; \\ D_s &= D\Theta D_w = DD_w + D(I - \Theta)(-D_w). \end{aligned} \tag{16}$$

Making $\Phi = (I - \Theta)$ and substituting (16) and emphasizing that $\Phi = \operatorname{diag}(\phi_1, \ldots, \phi_m)$ with $0 \le \phi_j \le (1 - \alpha_j)$ we obtain

$$\begin{aligned} A_s &= \widetilde{A} + \overline{B}\Phi\overline{C}, \\ B_s &= \widetilde{B} + \overline{B}\Phi\overline{D}, \\ C_s &= \widetilde{C} + D\Phi\overline{C}, \\ D_s &= \widetilde{D} + D\Phi\overline{D}. \end{aligned} \tag{17}$$

For technical simplification, it is considered that G and W are strictly proper; that is, $D = 0$ and $D_w = 0$ in (16) such that overall closed loop system is given by

$$\Sigma_1 := \begin{cases} \dot{x}_{\mathrm{cl}} = A_{\mathrm{cl}_\Delta}(\Phi)\, x_{\mathrm{cl}} + B_{\mathrm{cl}}(\Phi)\, w \\ z = C_{\mathrm{cl}}(\Phi)\, x_{\mathrm{cl}} + D_{\mathrm{cl}} w, \end{cases} \tag{18}$$

where $x_{\mathrm{cl}} = \begin{bmatrix} x^T & x_K^T \end{bmatrix}^T$ and

$$\begin{aligned} A_{\mathrm{cl}_\Delta}(\Phi) &= A_{\mathrm{cl}}(\Phi) + \underbrace{\begin{bmatrix} F \\ 0 \end{bmatrix}}_{\widetilde{F}} \Delta \underbrace{\begin{bmatrix} E & 0 \end{bmatrix}}_{\widetilde{E}} \text{ with} \\ A_{\mathrm{cl}}(\Phi) &= \begin{bmatrix} A_s & B_s C_K \\ B_K C_s & A_K \end{bmatrix}; \\ B_{\mathrm{cl}}(\Phi) &= \begin{bmatrix} 0 & B_s \\ B_K & 0 \end{bmatrix}; \\ C_{\mathrm{cl}}(\Phi) &= \begin{bmatrix} 0 & C_K \\ C_s & 0 \end{bmatrix}; \\ D_{\mathrm{cl}} &= \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}. \end{aligned} \tag{19}$$

This system is quadratically stable with $\|T_{zw}\|_\infty \le \gamma$ [26]. From the generalized plant in four-block configuration considering $D_K(\Phi) = 0$, synthesis conditions are now presented for the design of the gain-scheduled parametric H_∞ loop shaping controller subject to both constraint of input saturation and parametric uncertainties.

Theorem 2. *There exists a stabilizing gain-scheduled controller $K_\infty(\Phi)$ subject to both constraint of input saturation with $-u_{\max,j} \le u_j \le u_{\max,j}$ and parametric uncertainties such that $\gamma \ge 1$ if for some $\mu > 0$ there exist $Y > 0$, $X > 0$, satisfying the following inequalities:*

$$\begin{gathered} \begin{bmatrix} \Xi_1 & \Xi_2 & \Xi_3 & \Xi_4 \\ \cdot & -\gamma I & D_{cl}{}^T & 0 \\ \cdot & \cdot & -\gamma I & 0 \\ \cdot & \cdot & \cdot & -\mu I \end{bmatrix} < 0 \quad \text{for } i = 1, \ldots, 2^m, \\ \begin{bmatrix} Y & I \\ I & X \end{bmatrix} > 0, \end{gathered} \tag{20}$$

where

$$\Xi_1 = \begin{bmatrix} A_{s_i}Y + B_sH_i + YA_{s_i}^T + H_i^TB_s^T & \cdots & \cdots & A_{s_i} + M_i^T \\ M_i + A_{s_i}^T & \cdots & \cdots & XA_{s_i} + C_s^TL_i^T + L_iC_s + A_{s_i}^TX \end{bmatrix},$$
$$\Xi_2 = \begin{bmatrix} 0 & B_s \\ L_i & XB_s \end{bmatrix},$$
$$\Xi_3 = \begin{bmatrix} H_i^T & YC_s^T \\ 0 & C_s^T \end{bmatrix},$$
$$\Xi_4 = \begin{bmatrix} YE^T & \mu F \\ E^T & \mu XF \end{bmatrix}. \tag{21}$$

Moreover, a suitable gain-scheduled controller when Φ varies in the polytope is described as

$$C_{K_i} = H_iV,$$
$$B_{K_i} = \left(U^T\right)^{-1}L_i,$$
$$A_{K_i} = \left(U^T\right)^{-1}\left(M_i - XA_{s_i}Y - XB_sH_i - U^TB_{K_i}C_sY\right)V^{-1}, \tag{22}$$

where U and V are arbitrary nonsingular matrices satisfying $VU^T = I - XY$ with $i = 1, \ldots, 2^m$.

Proof. According to [21], a closed loop system $(A_{\mathrm{cl}_i}, B_{\mathrm{cl}_i}, C_{\mathrm{cl}_i}, D_{\mathrm{cl}_i})$ is stable with $\|T_{zw}\|_\infty \le \gamma$ if and only if there exists a $P > 0$ such that

$$\begin{bmatrix} A_{\mathrm{cl}_i}^TP + PA_{\mathrm{cl}_i} & PB_{\mathrm{cl}_i} & C_{\mathrm{cl}_i}^T \\ \cdot & -\gamma I & D_{\mathrm{cl}_i}^T \\ \cdot & \cdot & -\gamma I \end{bmatrix} < 0, \quad i = 1, \ldots, 2^m. \tag{23}$$

For norm-bounded parametric uncertainty, an important lemma will be used to prove the main results in this paper.

Lemma 3 (see [27]). *Consider $Q = Q^T$, G and W matrices with appropriate dimensions*

$$Q + G\Delta(t)W + W^T\Delta(t)^TG^T < 0 \quad \text{for } \|\Delta(t)\|_2 \le 1 \tag{24}$$

if only if there exists $\mu > 0$ such that

$$Q + \mu GG^T + \mu^{-1}W^TW < 0. \tag{25}$$

The lemma above and Schur complement imply that there exists a Lyapunov matrix $P = P^T > 0$ such that

$$\begin{bmatrix} A_{\mathrm{cl}_i}^TP + PA_{\mathrm{cl}_i} & PB_{\mathrm{cl}_i} & C_{\mathrm{cl}_i}^T & \widetilde{E}^T & \mu P\widetilde{F} \\ \cdot & -\gamma I & D_{\mathrm{cl}}^T & 0 & 0 \\ \cdot & \cdot & -\gamma I & 0 & 0 \\ \cdot & \cdot & \cdot & -\mu I & 0 \\ \cdot & \cdot & \cdot & \cdot & -\mu I \end{bmatrix} < 0, \tag{26}$$

where

$$Q = \begin{bmatrix} A_{\mathrm{cl}_i}^TP + PA_{\mathrm{cl}_i} & PB_{\mathrm{cl}_i} & C_{\mathrm{cl}_i}^T \\ \cdot & -\gamma I & D_{\mathrm{cl}} \\ \cdot & \cdot & -\gamma I \end{bmatrix},$$
$$G = \begin{bmatrix} PF \\ 0 \\ 0 \end{bmatrix},$$
$$W = \begin{bmatrix} E & 0 & 0 \end{bmatrix}. \tag{27}$$

Now, partition the Lyapunov matrix P in accordance with the structure of the matrix A_{cl} and matrices given by X, U, Y, and V [24]; we obtain

$$P = \begin{bmatrix} X & U^T \\ U & \widehat{X} \end{bmatrix},$$
$$P^{-1} = \begin{bmatrix} Y & V^T \\ V & \widehat{Y} \end{bmatrix} \tag{28}$$

with $X = X^T$, $Y = Y^T$, and $VU^T = I - XY$. To eliminate the nonlinear terms in (26) it is necessary to consider the transformation matrix Ψ given by

$$\Psi = \begin{bmatrix} Y & I \\ V & 0 \end{bmatrix}. \tag{29}$$

Thus it is possible to apply the following congruence transformation on P for the inequality $P > 0$:

$$P > 0 \Longrightarrow \Psi^TP\Psi > 0,$$
$$\begin{bmatrix} Y & V^T \\ I & 0 \end{bmatrix}\begin{bmatrix} X & U^T \\ U & \widehat{X} \end{bmatrix}\begin{bmatrix} Y & I \\ V & 0 \end{bmatrix} = \begin{bmatrix} Y & I \\ I & X \end{bmatrix} > 0. \tag{30}$$

Now, premultiplying and postmultiplying (26) by $\operatorname{diag}\{\Psi^T \; I \; I \; I \; I\}$ and its transpose, respectively, yield

$$\begin{bmatrix} \Gamma_1 & \Psi^T P B_{\mathrm{cl}_i} & \Psi^T C_{\mathrm{cl}_i}^T & \Psi^T \widetilde{E}^T & \mu \Psi^T P \widetilde{F} \\ \cdot & -\gamma I & D_{\mathrm{cl}}^T & 0 & 0 \\ \cdot & \cdot & -\gamma I & 0 & 0 \\ \cdot & \cdot & \cdot & -\mu I & 0 \\ \cdot & \cdot & \cdot & \cdot & -\mu I \end{bmatrix} < 0 \quad (31)$$

with $\Gamma_1 = \Psi^T (A_{\mathrm{cl}_i}^T P + P A_{\mathrm{cl}_i}) \Psi$. Solving the terms $C_{\mathrm{cl}_i} \Psi$, $B_{\mathrm{cl}_i}^T P \Psi$, and Γ_1 we obtain a gain-scheduled controller $K_\infty(\Phi)$ that ensures the robust stability of the closed loop system subject to the input saturation constraint and norm-bounded parametric uncertainty:

$$\begin{aligned} C_{K_i} &= H_i V, \\ B_{K_i} &= \left(U^T\right)^{-1} L_i, \\ A_{K_i} &= \left(U^T\right)^{-1} \left(M_i - X A_{s_i} Y - X B_s H_i - U^T B_{K_i} C_s Y\right) V^{-1}, \end{aligned} \quad (32)$$

where the optimization variables are γ, X, Y, H_i, L_i, and M_i with $VU^T = I - XY$ and $i = 1, \ldots, 2^m$. □

It can be noted that (20) cannot be solved directly via an LMI software, due to the product of μ with X. To overcome this difficulty, it is important to set an a priori value for μ and afterward use an LMI solver. Thus it is necessary to tune the parameter μ until a feasible solution is found.

4. Numerical Example

This section describes a realistic design of a gain scheduled H_∞ loop shaping controller considering both input saturation constraint and parametric uncertainties for a VTOL helicopter. The synthesis procedure described in this section was implemented using the following software suite: Matlab 7.5.0, SeDuMi [28], and Yalmip [29].

4.1. VTOL Helicopter. This example was inspired by [22] where a robust controller was designed in order to keep the closed loop system stable when subject to airspeed changes. Here, the model is slightly modified to address the present problem, being described by the following states equations:

$$G(\Delta) := \begin{cases} \dot{x}(t) = [A + \Delta A]\, x(t) + B v(t) \\ y(t) = C x(t) + D v(t) \end{cases} \quad (33)$$

with

$$\begin{aligned} A &= \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.010 & 0.0024 & -4.0208 \\ 0.1002 & \rho_1 & -0.707 & \rho_2 \\ 0 & 0 & 1 & 0 \end{bmatrix}; \\ B &= \begin{bmatrix} 0.4422 & 0.1761 \\ 3.5446 & -7.592 \\ -5.52 & 4.49 \\ 0 & 0 \end{bmatrix}; \\ C &= \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}; \\ D &= 0, \end{aligned} \quad (34)$$

where the states variables are horizontal velocity, x_1, vertical velocity, x_2, pitch rate, x_3, and pitch angle, x_4; the saturating input variables are collective pitch control, v_1, and longitudinal cyclic pitch control, v_2. Moreover, the following bound parameters are assumed: $\rho_1 = 0.3681 + \Delta\rho_1$ with $|\Delta\rho_1| \le 0.05$ and $\rho_2 = 1.42 + \Delta\rho_2$ with $|\Delta\rho_2| \le 0.01$. Now, assuming the parametric uncertainty $\Delta A = F\Delta(t)E$, we can represent the uncertainty using the following matrices:

$$\begin{aligned} F &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.05 & 0.01 \\ 0 & 0 \end{bmatrix}, \\ E &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &\text{for } \|\Delta(t)\|_2 \le 1. \end{aligned} \quad (35)$$

The following specifications are chosen for the controller to be designed: (1) reference signal tracking and disturbance rejection with error not exceeding 10%(2) zero steady error for the step input; (3) stability of the controlled system for input saturation limits of $u_{\max/\min} = \pm 2.5$. Typically, a dynamic weighting function W is used to enforce the performance specifications in the frequency domain. This weighting function W must be selected in order to obtain high gain at low frequencies, roll-off rates of approximately 20 dB/decade at the desired bandwidth(s), and higher rates at high frequencies [30]. To satisfy these requirements, we chose

$$W = \begin{bmatrix} \dfrac{12.5s + 7}{0.01s^2 + 4s + 0.8} & 0 \\ 0 & \dfrac{12.5s + 2}{0.01s^2 + 2s} \end{bmatrix}. \quad (36)$$

From this choice the shaped plant has been defined. The frequency response of the shaped plant in comparison to nominal plant is presented in Figure 2.

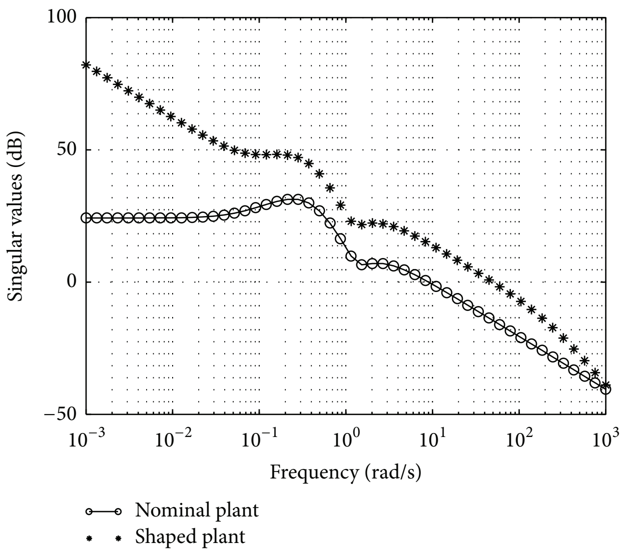

Figure 2: Singular values of the shaped and nominal plants.

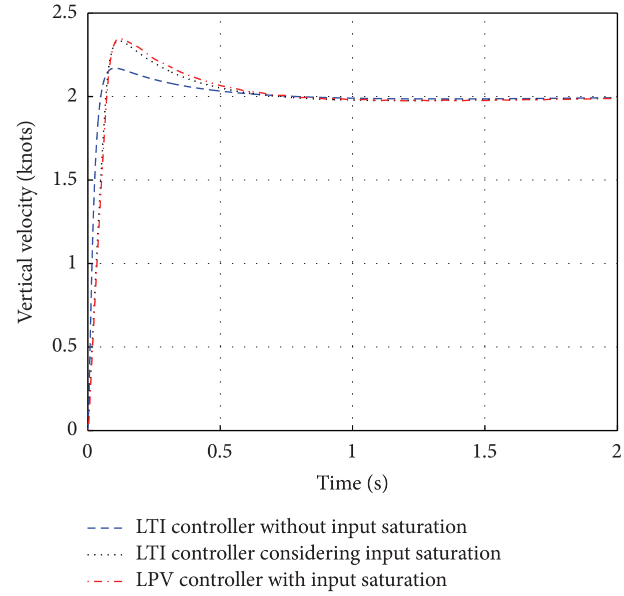

Figure 3: Step response for the controllers designed considering input saturation and parametric uncertainty.

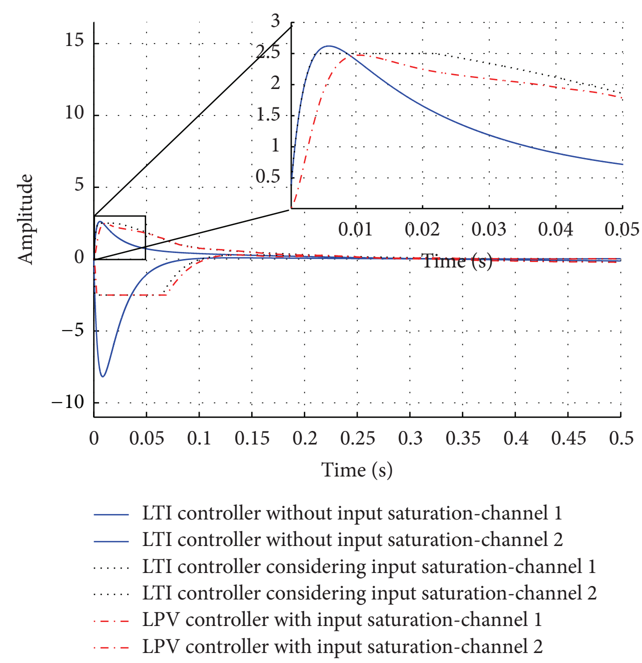

Figure 4: Control input in both channels for the controllers designed considering input saturation and parametric uncertainty.

To demonstrate the effectiveness of the proposed method, two different designs will be discussed. The first consists in the design of an H_∞ loop shaping controller for the VTOL helicopter considering only norm-bounded parametric uncertainty; this design will be used as a reference for performance evaluation. The second consists in the proposed LPV approach considering both the input saturation constraint and norm-bounded parametric uncertainty. In this design, the same dynamic weighting function and $\mu = 52000$ have been considered. Suitable values of μ can be found by a grid search, in this example in the interval $(0, 10^5]$. Furthermore, two varying parameters (ϕ_1, ϕ_2) that indicate the level of actuator's saturation are defined. Herein, it is assumed that ϕ_1 varies in the range between 0 and 0.4 and ϕ_2 varies between 0 and 0.2, representing a polytope with four vertices and input saturation limits around $u_{\max/\min} = \pm 2.5$. For both designs the maximum robust stability margin $\varepsilon_{\max} > 0.25$ $(\gamma < 4)$ was achieved. In the absence of input saturation, that is, considering only norm-bounded parametric uncertainty, the H_∞ loop shaping controller was a robust stability margin of 0.60 while the gain-scheduled H_∞ controller was a margin of 0.40.

Figures 3 and 4 illustrate the performance of the two controllers designed. They satisfied the requirements of reference signal tracking, zero steady error for the step input, and guarantee of robust stability of the controlled system for input saturation of $u_{\max/\min} = \pm 2.5$. It can be noted that another simulation for better evaluation of the proposed method is incorporated. It consists basically in the performance of the H_∞ loop shaping controller designed considering a saturation block. Taking into account the control input in both channels obtained with the two controllers designed, it is observed that gain scheduled H_∞ loop shaping has a better performance than the H_∞ loop shaping controller incorporating the saturation block.

Next, the guaranteed simultaneous gain/phase margins [31, 32] were used to evaluate the controlled system's robustness. These margins were defined as

$$-20\log_{10}\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\,\text{dB} \le GM \le 20\log_{10}\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\,\text{dB}$$
$$-\sin^{-1}\varepsilon \text{ degrees} \le PM \le \sin^{-1}\varepsilon \text{ degrees}. \tag{37}$$

The H_∞ loop shaping controller considering only norm-bounded parametric uncertainty has a gain margin of ± 6.02 dB; its phase margin is 36.86°. The gain-scheduled H_∞ loop shaping controller has a gain margin of ± 3.67 dB and phase margin of 23.57°. The guaranteed simultaneous gain/phase margins for the gain-scheduled controller are less than for the LTI controller without input saturation considering only parametric uncertainties. This conservative margin occurs because in (22) there is a style set of X, Y being used for all polytope vertices. From these results, it is concluded that the proposed method is an interesting alternative for systems subject to input saturation and parametric uncertainties.

5. Conclusions

New sufficient LMI conditions for the synthesis of gain scheduled H_∞ loop shaping controllers considering both input saturation constraints and norm-bounded parametric have been presented. The methodology addresses the design problem in the four-block H_∞ synthesis framework, in which it is possible, through a linear parameter-varying structure, to describe the parametric uncertainty and the input saturation nonlinearity as perturbations to normalized coprime factors of the shaped plant. Moreover, it is observed that the synthesis is quite simple, but in practical terms some adjustments are necessary to avoid high gains and an inappropriate limited control input must be considered. One difficulty found in this methodology is the exhaustive attempt to find the parameter μ, which is capable of providing good performance and a feasible solution. Future work should use computational techniques in order to reduce this exhaustive attempt to find μ. Finally, performance and robustness analysis illustrate that the controllers obtained can be an advantageous strategy for a class of linear systems with the presence of both constraint of input saturation and parametric uncertainty.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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