Research Article

Polyharmonic Multiquadric Particular Solutions for Reissner/Mindlin Plate

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Analytical particular solutions of the polyharmonic multiquadrics are derived for both the Reissner and Mindlin thick-plate models in a unified formulation. In the derivation, the three coupled second-order partial differential equations are converted into a product operator of biharmonic and Helmholtz operators using the Hörmander operator decomposition technique. Then a method is introduced to eliminate the Helmholtz operator, which enables the utilization of the polyharmonic multiquadrics. Then, the analytical particular solutions of displacements, shear forces, and bending or twisting moments corresponding to the polyharmonic multiquadrics are all explicitly derived. Numerical examples are carried out to validate these particular solutions. The results obtained by the present method are more accurate than those by the traditional multiquadrics and splines.

1. Introduction

Boundary-type numerical methods have been emerged as a popular research field because only boundary discretizations are required when they are applied to solve homogeneous problems. These methods include the boundary element method (BEM) [1], method of fundamental solutions (MFS) [2, 3], and Trefftz methods (TM) [4, 5]. When they are applied for solving an inhomogeneous partial differential equation, the dual reciprocity method (DRM) can be used [6, 7].

In the early development of DRM, the ad hoc radial function, $1 + r$, was exclusively used. In order to improve the accuracy of the computation, Golberg and Chen [8] and Karur and Ramachandran [9] applied the theory of radial basis functions (RBFs) to the DRM. Among these RBFs, Hardy's multiquadrics (MQ) [10] and Duchon's augmented polyharmonic spline (APS) [11] are the most popular ones. For example, Golberg [12, 13], Chen [14], and Karur and Ramachandran [9] demonstrated the superiority of the APS over the ad hoc radial function. Then, Golberg et al. [15] further improved the accuracy of the approximated particular solution by utilizing the MQ. Recently, Tsai [16] generalized the MQ to the polyharmonic multiquadrics (PMQ) and showed further accuracy improvement of the PMQ over

the traditional MQ. In this study, the accuracy improvement of the PMQ over the MQ and APS is demonstrated for the Reissner and Mindlin thick-plate models.

When the MQ is adopted in a DRM procedure, the applicability depends on the availability of the analytical particular solution of the basis function associated with the partial differential operator of a given problem. Golberg et al. [15] derived the analytical particular solutions for the Laplace equation. Samaan and Rashed [17, 18] and Tsai and Hsu [19] found the analytical particular solutions, respectively, for the two- and three-dimensional elasticity problems, which can be converted to a biharmonic equation using the Galerkin-Papkovich vector [20]. Basically, the applicability of MQ is limited to the harmonic and biharmonic operators [21] before Tsai [16] derived the analytical particular solutions of MQ associated with the polyharmonic operators, which are denoted as the PMQ in the following.

When applying the DRM for solving the Mindlin thick-plate model [22], Wen et al. [23] and Tsai and Wu [24] found the analytical particular solutions for the ad hoc RBF and the APS, respectively. In addition, the analytical particular solutions for the Chebyshev polynomials [25] and the APS [26] were derived for the Reissner thick-plate model [27, 28]. Basically, a product operator of biharmonic and Helmholtz
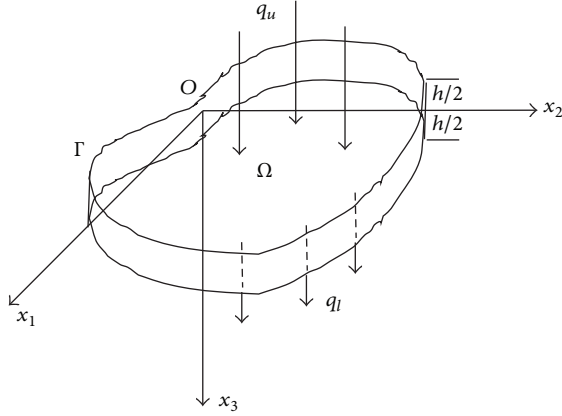
FIGURE 1: Configuration of a thick plate.

operators resulted when applying the Hörmander operator decomposition technique [29] to the coupled PDEs of the Reissner and Mindlin thick-plate models as demonstrated in the previous studies [24–26]. This seems to infer that the accuracy improvement of the PMQ over the APS is not possible for the thick-plate models.

In this study, a method is introduced to eliminate the Helmholtz operator, which makes the application of the PMQ for the thick-plate models possible. Numerical experiments are carried out to demonstrate the superior accuracy of the PMQ over the traditional MQ and the first few orders of the APS.

This paper is organized as follows: the unified Reissner/Mindlin plate model is introduced in Section 2. Then, a review of the PMQ is given in Section 3. And the particular solutions of the PMQ associated with the Reissner/Mindlin plate model are derived in Section 4. Section 5 gives the formulation of MFS-DRM. Finally, some numerical experiments are carried out to validate the particular solutions in Section 6 and the conclusions are drawn in Section 7.

2. Reissner/Mindlin Plate Model

In the following, indices i, j, k are in the range $\{1, 2, 3\}$ and indices α, β, γ are in the range $\{1, 2\}$. As described in Figure 1, we consider a plate of uniform thickness h with its middle plane being a domain Ω with boundary Γ in the $x_1 - x_2$ plane and thickness coordinate x_3. The plate is subjected to a normal loading with intensity $q(x_1, x_2)$. Then, the equations of equilibrium are given by

$$\frac{\partial Q_\alpha}{\partial x_\alpha} + q = 0,$$
$$\frac{\partial \Upsilon_{\alpha\beta}}{\partial x_\beta} - Q_\alpha = 0, \tag{1}$$

where Q_α is the transverse shear force and $\Upsilon_{\alpha\beta}$ is the bending or twisting moment.

In the Reissner [27, 28] and Mindlin [22] thick-plate theory, u_α represents an average slope angle and u_3 is the lateral deflection of the plate in the middle surface. According to the theories, the constitutive equations are given by

$$Q_\alpha = \frac{D(1-\nu)\lambda^2}{2}\left(u_\alpha + \frac{\partial u_3}{\partial x_\alpha}\right),$$
$$\Upsilon_{\alpha\beta} = D\frac{1-\nu}{2}\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha} + \frac{2\nu}{1-\nu}\frac{\partial u_\gamma}{\partial x_\gamma}\delta_{\alpha\beta}\right) \tag{2}$$
$$+ \frac{\varepsilon\nu}{(1-\nu)\lambda^2}q\delta_{\alpha\beta}$$

with

$$\lambda = \frac{\pi}{h},$$
$$\varepsilon = 0 \tag{3}$$

for the Mindlin plate model and

$$\lambda = \frac{\sqrt{10}}{h},$$
$$\varepsilon = 1 \tag{4}$$

for the Reissner plate model. Combining (1)~(2) results in the following governing equation:

$$\widetilde{L}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = -\widetilde{E}q \tag{5}$$

with

$$\widetilde{L} = \begin{pmatrix} \dfrac{D(1-\nu)}{2}\left(\nabla^2 - \lambda^2\right) + \dfrac{D(1+\nu)}{2}\dfrac{\partial^2}{\partial x_1{}^2} & \dfrac{D(1+\nu)}{2}\dfrac{\partial^2}{\partial x_1 \partial x_2} & -\dfrac{D\lambda^2(1-\nu)}{2}\dfrac{\partial}{\partial x_1} \\ \dfrac{D(1+\nu)}{2}\dfrac{\partial^2}{\partial x_1 \partial x_2} & \dfrac{D(1-\nu)}{2}\left(\nabla^2 - \lambda^2\right) + \dfrac{D(1+\nu)}{2}\dfrac{\partial^2}{\partial x_2{}^2} & -\dfrac{D\lambda^2(1-\nu)}{2}\dfrac{\partial}{\partial x_2} \\ \dfrac{D\lambda^2(1-\nu)}{2}\dfrac{\partial}{\partial x_1} & -\dfrac{D\lambda^2(1-\nu)}{2}\dfrac{\partial}{\partial x_2} & \dfrac{D\lambda^2(1-\nu)}{2}\nabla^2 \end{pmatrix},$$

$$\widetilde{E} = \begin{pmatrix} \dfrac{\varepsilon\nu}{(1-\nu)\lambda^2}\dfrac{\partial}{\partial x_1} \\ \dfrac{\varepsilon\nu}{(1-\nu)\lambda^2}\dfrac{\partial}{\partial x_2} \\ 1 \end{pmatrix}, \tag{6}$$

where D and v are Young's modulus and Poisson ratio, respectively.

Equation (5) is a coupled system of three second-order PDEs for the three unknown functions u_j, and, therefore, three boundary conditions are required as

$$\mathbf{B}_1 \cdot \mathbf{u} = H_1 (x_1, x_2),$$

$$\mathbf{B}_2 \cdot \mathbf{u} = H_2 (x_1, x_2), \qquad (7)$$

$$\mathbf{B}_3 \cdot \mathbf{u} = H_3 (x_1, x_2),$$

where $H_i(x_1, x_2)$ are given boundary data.

3. Polyharmonic Multiquadrics

Before deriving the particular solutions of PMQ associated with the Reissner/Mindlin plate model, we give a short review on the PMQ previously derived by the method of undetermined coefficients together with the Laurent series in [16]. Considering the Lth order PMQ ψ_L, it is governed by the following hierarchical relation:

$$\nabla^2 \psi_L = \psi_{L-1}, \qquad (8)$$

$$\psi_0 = \sqrt{c^2 + r^2}, \qquad (9)$$

$$\left(\nabla^2\right)^i \psi_L = 0 \quad \text{as } r \longrightarrow 0 \text{ for } 0 \leq i \leq L - 1. \qquad (10)$$

Here, (10) is introduced so that the PMQ are hierarchically unique and c is a shape parameter to tune the numerical accuracy [10].

Then, by using the method of undetermined coefficients, we can find the solution of (8)~(10) as

$$\frac{\psi_L}{c^{2L+1}} = \sum_{j=0}^{L} A_{L,j} R^{2j} \sqrt{1 + R^2}$$

$$+ \sum_{j=0}^{L-1} B_{L,j} R^{2j} \ln\left(\frac{1 + \sqrt{1 + R^2}}{2}\right) + \sum_{j=0}^{L-1} C_{L,j} R^{2j} \qquad (11)$$

with

$$R = \frac{r}{c} \qquad (12)$$

for any positive integer L. And, the unknown $C_{L,i}$ in (11) can be solved by using the uniqueness condition (10) as

$$C_{L,i} = -\left(\sum_{j=0}^{i} P_{L,i,j} + \sum_{j=0}^{i-1} Q_{L,i,j}\right) \quad \text{for } 0 \leq i \leq L - 1 \qquad (13)$$

with

$$P_{L,i,j} = \frac{(-1)^{i-j-1} (2i - 2j - 3)!!}{(i - j)! 2^{i-j}} A_{L,j},$$

$$Q_{L,i,j} = \frac{(-1)^{i-j+1} (2i - 2j - 1)!!}{(i - j)! 2^{i-j+1} (i - j)} B_{L,j}. \qquad (14)$$

In (14), the double factorial is defined as

$$n!!$$

$$= \begin{cases} n \cdot (n - 2) \cdots 5 \cdot 3 \cdot 1 & n \text{ is an odd positive integer} \\ n \cdot (n - 2) \cdots 6 \cdot 4 \cdot 2 & n \text{ is an even positive integer} \\ 1 & n = 0, -1 \\ \dfrac{(-1)^{(n+1)/2}}{(-n - 2)!!} & n \leq -3 \text{ is an odd negative integer.} \end{cases} \qquad (15)$$

The PMQ defined in (11) can be expanded into the Maclaurin series as

$$\frac{\psi_L}{c^{2L+1}} = \sum_{i=L}^{\infty} \left(\sum_{j=0}^{L} P_{L,i,j} + \sum_{j=0}^{L-1} Q_{L,i,j}\right) R^{2i}. \qquad (16)$$

Observing (16), we can conclude that the proposed PMQ are infinitely differentiable since they only consist of even-coefficient polynomials [16, 30]. Then, substituting (16) into (8) and (9) and performing some mathematical manipulations can result in

$$\sum_{j=0}^{L} \frac{P_{L,i+L,j} 4^L (i + L)!^2}{i!^2} + \sum_{j=0}^{L-1} \frac{Q_{L,i+L,j} 4^L (i + L)!^2}{i!^2}$$

$$= \frac{(-1)^{i-1} (2i - 3)!!}{i! 2^i} \quad \text{for all nonnegative integer } i, \qquad (17)$$

which can be used for solving the unknown coefficients $A_{L,j}$ and $B_{L,j}$ in (11) and (16). In practice, we simply enforce i equal to $0, 1, \ldots, 2L$ in (17) which results in $2L + 1$ linear equations for solving the $2L + 1$ unknown coefficients $A_{L,j}$ and $B_{L,j}$ of the PMQ. After $A_{L,j}$ and $B_{L,j}$ are obtained, (13) can be used to have $C_{L,j}$. Then, the PMQ ψ_L can be computed simply by (11). For the cases when r is very small or equal to zero, one can alternatively use the Maclaurin series (16).

For the purpose of a later derivation of the particular solutions associated with the Reissner/Mindlin plate model, we need to find the formulas when the radial differential operator $(d/rdr)^M$ is applied onto (11) and (16). This can be done for arbitrary nonnegative integer M since the proposed

PMQ are infinitely differentiable. Then, the required formulas for (11) and (16) can be derived, respectively, as

$$\left(\frac{d}{rdr}\right)^M \frac{\psi_L}{c^{2L-2M+1}} = \sum_{j=0}^{L} A_{L,j} V_{M,j}(R)$$
$$+ \sum_{j=0}^{L-1} B_{L,j} W_{M,j}(R) + \sum_{j=0}^{L-1} C_{L,j} U_{M,j}(R),$$

$$\left(\frac{d}{rdr}\right)^M \frac{\psi_L}{c^{2L-2M+1}} = \sum_{i=L}^{\infty} \left(\sum_{j=0}^{L} P_{L,i,j} + \sum_{j=0}^{L-1} Q_{L,i,j}\right) U_{M,i}(R) \tag{18}$$

with

$$U_{M,j}(R)$$

$$\equiv \left(\frac{d}{RdR}\right)^M R^{2j}$$

$$= \begin{cases} \dfrac{(2j)!!}{(2j-2M)!!} R^{2j-2M}, & \text{for } j \geq M \geq 0, \\ 0, & \text{for } M > j \geq 0, \\ \dfrac{(-1)^M (-2j+2M-2)!!}{(-2j-2)!!} R^{2j-2M}, & \text{for } j < 0, \end{cases} \tag{19}$$

$$V_{M,j}(R)$$

$$\equiv \left(\frac{d}{RdR}\right)^M \left(R^{2j}\sqrt{1+R^2}\right) \tag{20}$$

$$= \sum_{k=0}^{M} \frac{k!(M-k)!}{M!} U_{M-k,j}(R) \left(\frac{d}{RdR}\right)^k \sqrt{1+R^2},$$

$$W_{M,j}(R)$$

$$\equiv \left(\frac{d}{RdR}\right)^M \left(R^{2j}\ln\left(\frac{1+\sqrt{1+R^2}}{2}\right)\right)$$

$$= \sum_{k=0}^{M} \frac{k!(M-k)!}{M!} U_{M-k,j}(R) \left(\frac{d}{RdR}\right)^k \ln\left(\frac{1+\sqrt{1+R^2}}{2}\right). \tag{21}$$

In deriving (18), we have used

$$c^{2M}\left(\frac{d}{rdr}\right)^M = \left(\frac{d}{RdR}\right)^M. \tag{22}$$

And, in deriving (20) and (21), we have used the product rule of differentiation as follows:

$$\left(\frac{d}{RdR}\right)^M (fg)$$

$$= \sum_{k=0}^{M} \frac{k!(M-k)!}{M!} \left(\left(\frac{d}{RdR}\right)^{M-k} f\right)\left(\left(\frac{d}{RdR}\right)^k g\right) \tag{23}$$

with f and g being two arbitrary radial functions. Finally, additional formulas are required to complete (20) and (21), respectively, as follows:

$$\left(\frac{d}{RdR}\right)^k \sqrt{1+R^2} = \frac{(-1)^{k-1}(2k-3)!!}{\sqrt{1+R^2}^{2k-1}}, \tag{24}$$

$$\left(\frac{d}{RdR}\right)^k \ln\left(\frac{1+\sqrt{1+R^2}}{2}\right)$$

$$= \begin{cases} \dfrac{1}{R^2} - \dfrac{1}{R^2\sqrt{1+R^2}}, & \text{for } k=1 \\ U_{k-1,-1}(R) \\ \quad - \displaystyle\sum_{l=0}^{k-1} \dfrac{l!(k-1-l)!}{(k-1)!} U_{k-1-l,-1}(R)\left(\dfrac{d}{RdR}\right)^l \dfrac{1}{\sqrt{1+R^2}}, \\ \hspace{5cm} \text{for } k>1 \end{cases} \tag{25}$$

with

$$\left(\frac{d}{RdR}\right)^l \frac{1}{\sqrt{1+R^2}} = \frac{(-1)^l (2l-1)!!}{\sqrt{1+R^2}^{2l+1}}. \tag{26}$$

The proposed formulas can be implemented into a floating-point subroutine for evaluating $(d/rdr)^M \psi_L(r)$ with arbitrary nonnegative integers M and L. Therefore, the subroutine can be invoked when implementing numerical methods. In addition, the source codes can be obtained from the author via e-mails.

4. Analytical Particular Solution of Plate Model

Analytical particular solutions are required when the DRM is applied for solving a Reissner/Mindlin plate problem under an arbitrary loading. In this section, the analytical particular solutions of the PMQ associated with the Reissner/Mindlin plate model will be derived, which are governed by

$$\tilde{L}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = -\tilde{E}\psi_L(r), \tag{27}$$

where

$$r = \sqrt{x_1^2 + x_2^2}. \tag{28}$$

Formally, the particular solutions $u_j(x_1, x_2)$, governed by (27), should be obtainable by using the Hörmander

operator decomposition technology [29], which begins with the definition of the adjoint operator $\widetilde{L}^{\text{adj}}$ as

$$\widetilde{L}^{\text{adj}}\widetilde{L} = \widetilde{L}\widetilde{L}^{\text{adj}} = \det\left(\widetilde{L}\right)\widetilde{I}, \qquad (29)$$

where $\det(\widetilde{L})$ is the determinant of $\widetilde{L}$ and $\widetilde{I}$ is the three-by-three identity matrix. Here, both $\det(\widetilde{L})$ and $\widetilde{L}^{\text{adj}}$ can be obtained by linear algebraic manipulations, respectively, defined as

$$\det\left(\widetilde{L}\right) = \frac{D^3\lambda^2(1-\nu)^2}{4}\left(\nabla^2 - \lambda^2\right)\nabla^2\nabla^2, \qquad (30)$$

$$\widetilde{L}^{\text{adj}} = \begin{pmatrix} 2\nabla^4 - \left[(1+\nu)\nabla^2 + (1-\nu)\lambda^2\right]\dfrac{\partial^2}{\partial x_1{}^2} & -\left[(1+\nu)\nabla^2 + (1-\nu)\lambda^2\right]\dfrac{\partial^2}{\partial x_1\partial x_2} & (1-\nu)\dfrac{\partial}{\partial x_1}\left(\nabla^2 - \lambda^2\right) \\[2.5ex] -\left[(1+\nu)\nabla^2 + (1-\nu)\lambda^2\right]\dfrac{\partial^2}{\partial x_1\partial x_2} & 2\nabla^4 - \left[(1+\nu)\nabla^2 + (1-\nu)\lambda^2\right]\dfrac{\partial^2}{\partial x_2{}^2} & (1-\nu)\dfrac{\partial}{\partial x_2}\left(\nabla^2 - \lambda^2\right) \\[2.5ex] -(1-\nu)\dfrac{\partial}{\partial x_1}\left(\nabla^2 - \lambda^2\right) & -(1-\nu)\dfrac{\partial}{\partial x_2}\left(\nabla^2 - \lambda^2\right) & \dfrac{\left(\nabla^2 - \lambda^2\right)\left[2\nabla^2 - (1-\nu)\lambda^2\right]}{\lambda^2} \end{pmatrix}. \qquad (31)$$

Then, the particular solutions can be found by assuming

$$\begin{pmatrix} u_1(x_1, x_2) \\ u_2(x_1, x_2) \\ u_3(x_1, x_2) \end{pmatrix} = \widetilde{L}^{\text{adj}}\widetilde{E}F(r), \qquad (32)$$

where $F(r)$ is an unknown function to be determined. In (32), the required operator $\widetilde{L}^{\text{adj}}\widetilde{E}$ can be obtained as

$$\widetilde{L}^{\text{adj}}\widetilde{E}$$

$$= \begin{pmatrix} \dfrac{D^2(1-\nu)}{4}\dfrac{\partial}{\partial x_1}\left(\nabla^2 - \lambda^2\right)\left[\varepsilon\nu\nabla^2 + (1-\nu)\lambda^2\right] \\[2ex] \dfrac{D^2(1-\nu)}{4}\dfrac{\partial}{\partial x_2}\left(\nabla^2 - \lambda^2\right)\left[\varepsilon\nu\nabla^2 + (1-\nu)\lambda^2\right] \\[2ex] \dfrac{D^2(1-\nu)}{4}\left(\nabla^2 - \lambda^2\right)\left[(2-\varepsilon\nu)\nabla^2 - (1-\nu)\lambda^2\right] \end{pmatrix}. \qquad (33)$$

After substituting (32) into (27) and using (29) and (30), the resulted equation indicates that the following particular solution is required:

$$-\frac{D^3\lambda^2(1-\nu)^2}{4}\left(\nabla^2 - \lambda^2\right)\nabla^2\nabla^2 F = \psi_L. \qquad (34)$$

However, the analytical particular solution of the above equation is not known due to the existence of the Helmholtz operator.

In this study, we find that the Helmholtz operator can be eliminated by rewriting (33) as

$$\widetilde{L}^{\text{adj}}\widetilde{E} = \frac{D^3\lambda^2(1-\nu)^2}{4}\left(\nabla^2 - \lambda^2\right)\widetilde{E}_1, \qquad (35)$$

where the operator $\widetilde{E}_1$ is defined by

$$\widetilde{E}_1 = \frac{1}{D\lambda^2(1-\nu)}\begin{pmatrix} \dfrac{\partial}{\partial x_1}\left[\varepsilon\nu\nabla^2 + (1-\nu)\lambda^2\right] \\[2ex] \dfrac{\partial}{\partial x_2}\left[\varepsilon\nu\nabla^2 + (1-\nu)\lambda^2\right] \\[2ex] (2-\varepsilon\nu)\nabla^2 - (1-\nu)\lambda^2 \end{pmatrix}. \qquad (36)$$

Then, (29) and (35) can be combined to have

$$\widetilde{L}\widetilde{L}^{\text{adj}}\widetilde{E} = -\frac{D^3\lambda^2(1-\nu)^2}{4}\left(\nabla^2 - \lambda^2\right)\nabla^2\nabla^2\widetilde{E}$$

$$\Longrightarrow \widetilde{L}\left\{\frac{D^3\lambda^2(1-\nu)^2}{4}\left(\nabla^2 - \lambda^2\right)\right\}\widetilde{E}_1$$

$$= -\frac{D^3\lambda^2(1-\nu)^2}{4}\left(\nabla^2 - \lambda^2\right)\nabla^2\nabla^2\widetilde{E}$$

$$\Longrightarrow \widetilde{L}\widetilde{E}_1 = -\nabla^2\nabla^2\widetilde{E}. \qquad (37)$$

The above equations have suggested us to modify the Hörmander operator decomposition technology by assuming

$$\begin{pmatrix} u_1(x_1, x_2) \\ u_2(x_1, x_2) \\ u_3(x_1, x_2) \end{pmatrix} = \widetilde{E}_1 F_1(r), \qquad (38)$$

where $F_1(r)$ is an unknown function to be determined. Substituting (38) into (27) and using (37) result in

$$\nabla^2\nabla^2 F_1(r) = \psi_L. \qquad (39)$$

By using the PMQ definitions (8) and (9), we can have the particular solution of (39) as

$$F_1(r) = \psi_{L+2}. \qquad (40)$$

Substituting (40) into (38) and using (36) can result in the desired analytical particular solutions of displacements as

$$u_\alpha(x_1, x_2)$$

$$= \frac{-x_\alpha}{D\lambda^2(1-\nu)} \qquad (41)$$

$$\cdot\left(\varepsilon\nu\left(\frac{d}{r\,dr}\right)\psi_{L+1} + (1-\nu)\lambda^2\left(\frac{d}{r\,dr}\right)\psi_{L+2}\right),$$

$$u_3(x_1, x_2) = \frac{-(2-\varepsilon\nu)\psi_{L+1} + (1-\nu)\lambda^2\psi_{L+2}}{D\lambda^2(1-\nu)}, \qquad (42)$$

where we have used

$$\nabla^2 F_1 = \nabla^2 \psi_{L+2} = \psi_{L+1}. \tag{43}$$

In addition, the required formulas of $d\psi_{L+1}/rdr$ and $d\psi_{L+2}/rdr$ have been provided in the previous section. Then, the corresponding shear forces and bending/twisting moments can be obtained by using the definitions

$$Q_\alpha (x_1, x_2) = \frac{D(1-\nu)\lambda^2}{2} \left(u_\alpha^M + \frac{\partial u_3^M}{\partial x_\alpha} \right), \tag{44}$$

$$\Upsilon_{\alpha\beta}(x_1, x_2) = \frac{D(1-\nu)}{2} \left(\frac{\partial u_\alpha^M}{\partial x_\beta} + \frac{\partial u_\beta^M}{\partial x_\alpha} + \frac{2\nu}{1-\nu} \frac{\partial u_\gamma^M}{\partial x_\gamma} \delta_{\alpha\beta} \right)$$
$$+ \frac{\delta_{\alpha\beta}\varepsilon\nu\sqrt{r^2+c^2}}{\lambda^2(1-\nu)}. \tag{45}$$

Equations (41), (42), (44), and (45) are sufficient to obtain the analytical particular solutions of shear forces and bending/twisting moments, respectively, as follows:

$$Q_\alpha (x_1, x_2) = -x_\alpha \left(\frac{d}{rdr} \right) \psi_{L+1}, \tag{46}$$

$$\Upsilon_{\alpha\beta}(x_1, x_2)$$
$$= -\delta_{\alpha\beta} \left\{ \frac{\left(\varepsilon(\nu^2 - \nu)\sqrt{r^2+c^2} + \nu(1-\nu)\lambda^2 F_2(r) \right)}{\lambda^2(1-\nu)} \right.$$
$$+ \frac{1}{\lambda^2} \left(\varepsilon\nu \left(\frac{d}{rdr} \right) \psi_{L+1} \right.$$
$$\left. + (1-\nu)\lambda^2 \left(\frac{d}{rdr} \right) \psi_{L+2} \right) \right\}$$
$$- \frac{x_\alpha x_\beta}{\lambda^2} \left(\varepsilon\nu \left(\frac{d}{rdr} \right)^2 \psi_{L+1} + (1-\nu)\lambda^2 \left(\frac{d}{rdr} \right)^2 \psi_{L+2} \right), \tag{47}$$

where the additional terms $(d/rdr)^2\psi_{L+1}$ and $(d/rdr)^2\psi_{L+2}$ have been given in the previous section.

This completes the derivation of the analytical particular solutions of the PMQ associated with the Reissner/Mindlin plate model.

5. MFS-DRM Formulation

Now, we are in a position to review the application of the MFS-DRM procedure [24, 26] for solving the well-posed thick-plate problem governed by (5) and (7). First of all, we need the principle of superposition as
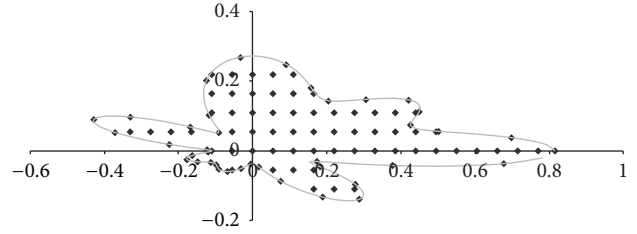
$$u_j = u_j^h + u_j^P, \tag{48}$$

Figure 2: Field points of the DRM.

where the particular solution u_j^P satisfies

$$\tilde{L} \begin{pmatrix} u_1^P \\ u_2^P \\ u_3^P \end{pmatrix} = -\tilde{E}q \tag{49}$$

without specifying any boundary condition. In addition, the homogeneous solution u_j^h satisfies

$$\tilde{L} \begin{pmatrix} u_1^h \\ u_2^h \\ u_3^h \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{50}$$

with modified boundary conditions

$$\mathbf{B}_1 \cdot \mathbf{u}^h = H_1(x_1, x_2) - \mathbf{B}_1 \cdot \mathbf{u}^P$$
$$\mathbf{B}_2 \cdot \mathbf{u}^h = H_2(x_1, x_2) - \mathbf{B}_2 \cdot \mathbf{u}^P \tag{51}$$
$$\mathbf{B}_3 \cdot \mathbf{u}^h = H_3(x_1, x_2) - \mathbf{B}_3 \cdot \mathbf{u}^P,$$

where $\mathbf{u}^h = (u_1^h, u_2^h, u_3^h)$ and $\mathbf{u}^P = (u_1^P, u_2^P, u_3^P)$.

In order to apply the DRM for solving the particular solution u_j^P, the loading intensity needs to be first approximated by the PMQ as follows:

$$q(x_1, x_2) \cong \sum_{n=1}^{N} a_n \psi_L(r_n), \tag{52}$$

where r_n is the distance between (x_1, x_2) and the nth DRM field point, (ξ_{n1}, ξ_{n2}), as depicted in Figure 2. In order to solve the unknown coefficient a_n, (52) should be collocated on the N DRM field points as

$$q(\xi_{l1}, \xi_{l2}) = \sum_{n=1}^{N} a_n \psi_L(r_{ln}) \tag{53}$$

for $l = 1, 2, \ldots, N$ and r_{ln} is the distance between the lth and nth DRM field points. Then, the corresponding particular solution can be approximated by

$$u_j^P \cong \sum_{n=1}^{N} a_n u_j(r_n) \tag{54}$$

with the particular solutions $u_j(r)$ given by (41) and (42). Also, the particular solutions of shear force and bending or twisting moment can be approximated, respectively, as

$$Q_\alpha{}^P \cong \sum_{n=1}^{N} a_n Q_\alpha(r_n), \tag{55}$$

$$\Upsilon_{\alpha\beta}{}^P \cong \sum_{n=1}^{N} \alpha_n \Upsilon_{\alpha\beta}(r_n). \tag{56}$$

In (55) and (56), the particular solutions Q_α and $\Upsilon_{\alpha\beta}$ are given in (46) and (47), respectively.

$$\tilde{L} \begin{pmatrix} u_{11}{}^* & u_{12}{}^* & u_{13}{}^* \\ u_{21}{}^* & u_{22}{}^* & u_{23}{}^* \\ u_{31}{}^* & u_{31}{}^* & u_{33}{}^* \end{pmatrix} = - \begin{pmatrix} \delta(x_1 - s_1, x_2 - s_2) & 0 & 0 \\ 0 & \delta(x_1 - s_1, x_2 - s_2) & 0 \\ 0 & 0 & \delta(x_1 - s_1, x_2 - s_2) \end{pmatrix} \tag{58}$$

and (s_{m1}, s_{m2}) are M source points outside the plate domain as described in Figure 3. In (57), the $3M$ unknowns b_m^k can be solved by collocating the modified boundary conditions (51) on M boundary points. More details of the MFS and the explicit fundamental solutions can be found in [24, 26]. After both the particular and homogeneous solutions are solved, the principle of superposition (48) can be applied for obtaining the desired solution.

6. Numerical Results

In order to validate the derived particular solutions, three numerical cases are considered. Typically, $D = \lambda = 2$ and $\nu = 0.3$ are selected in these numerical studies. Furthermore, the MFS is always arranged to be sufficient and accurate such that the numerical errors are majorly controlled by the DRM. The root mean square errors (RMSEs) of the numerical solutions are defined as

$$\sqrt{\frac{\sum_{i=1}^{3} \sum_{l=1}^{\bar{L}} \left(u^{\text{NUM}}{}_{i,l} - u^{\text{ANA}}{}_{i,l}\right)^2}{3\bar{L}}}, \tag{59}$$

where $u^{\text{NUM}}{}_{i,l}$ is the numerical solution obtained by the MFS-DRM at the lth point of the $\bar{L}$ considered positions and $u^{\text{ANA}}{}_{i,l}$ is the corresponding analytical solution of the problem. Here, $\bar{L}$ will be chosen to be sufficiently large to ensure the utilization of the RMSEs.

In our numerical experiments, we consider the following Reissner/Mindlin plate problem:

$$\tilde{L} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = -D\tilde{E} \{\sin x_1 (x_2 + \sinh x_2 + x_2 \sinh x_2)\}. \tag{60}$$

After the particular solutions are approximated by the DRM, the modified boundary conditions (51) become well defined. Therefore, it is ready to use the MFS for solving the homogeneous solution. Formally, the homogeneous solution can be approximated by

$$u_j{}^h \cong \sum_{k=1}^{3} \sum_{m=1}^{M} b_m^k u_{jk}{}^* (x_1, x_2; s_{m1}, s_{m2}), \tag{57}$$

where $u_{jk}{}^*(x_1, x_2; s_1, s_2)$ are the fundamental solutions defined by

This problem has been considered for both the Mindlin [24] and the Reissner [26] cases and their analytical particular solutions can be unified as

$$u_1 = \Big(\cos x_1 \Big\{ 48 x_2 \big[\lambda^2 (\nu - 1) + \varepsilon \nu \big]$$

$$- 6 \big[(1 + x_2)^2 \lambda^2 (\nu - 1)$$

$$+ (1 + 2x_2 (2 + x_2)) \varepsilon \nu \big] \cosh x_2$$

$$+ \big[(9 + x_2 (9 + 2x_2 (3 + x_2))) \lambda^2 (\nu - 1)$$

$$+ 12 (1 + x_2) \varepsilon \nu \big] \sinh x_2 \Big\} \Big)$$

$$\cdot \big(48 \lambda^2 (1 - \nu) \big)^{-1},$$

$$u_2 = \Big(\sin x_1 \Big\{ 48 \big[\lambda^2 - \big(\varepsilon + \lambda^2 \big) \nu \big]$$

$$+ \big[(3 + x_2 (3 - 2x_2 (3 + x_2)))$$

$$\cdot \lambda^2 (\nu - 1) + 12 (1 + x_2) \varepsilon \nu \big] \cosh x_2$$

$$+ 3 \big[-\lambda^2 (\nu - 1)$$

$$+ 2 (2x_2 (2 + x_2) - 1) \varepsilon \nu \big] \sinh x_2 \Big\} \Big)$$

$$\cdot \big(48 \lambda^2 (\nu - 1) \big)^{-1},$$

$$u_3 = \Big(\sin x_1 \Big\{ 48 \big[2 + \lambda^2 (1 - \nu) - \varepsilon \nu \big] x_2$$

$$+ 6 \big[\lambda^2 (\nu - 1) (1 + x_2)^2$$
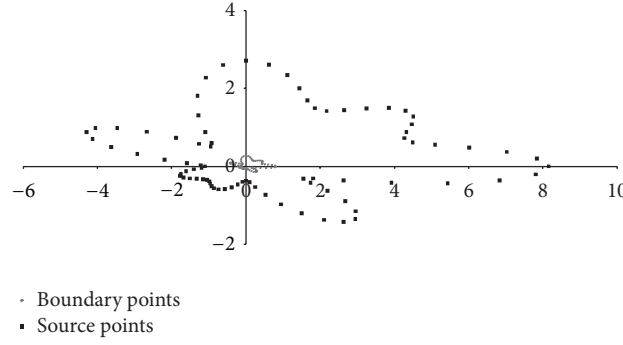
- Boundary points
- Source points

FIGURE 3: Boundary and source points of the MFS.

TABLE 1: The RMSEs for Example 1 (Div for divergence).

	$N = 6 \times 6$	$N = 8 \times 8$	$N = 10 \times 10$	$N = 12 \times 12$
Chebyshev	$2.10E - 06$	$8.62E - 09$	$2.18E - 09$	$3.81E - 10$
MQ	$6.10E - 06$	$4.30E - 07$	$4.30E - 07$	$2.74E - 08$
PMQ ($L = 2$)	$6.04E - 06$	$2.17E - 07$	$2.05E - 08$	$5.94E - 09$
PMQ ($L = 4$)	$4.51E - 06$	$1.20E - 07$	$5.12E - 09$	$1.66E - 09$
APS (order = 1)	$2.22E - 04$	$9.38E - 05$	$4.96E - 05$	$2.85E - 05$
APS (order = 3)	$3.71E - 05$	$9.08E - 06$	$3.02E - 06$	$1.49E - 06$
APS (order = 5)	$4.88E - 06$	$7.35E - 07$	$1.90E - 07$	$5.90E - 08$
APS (order = 7)	Div	$8.14E - 08$	$1.27E - 08$	$3.20E - 09$
APS (order = 9)	Div	Div	$2.44E - 09$	$4.20E - 10$
APS (order = 11)	Div	Div	Div	$3.81E - 10$

$$+ (\varepsilon \nu - 2) (1 + 2x_2 (2 + x_2)) \big] \cosh x_2$$

$$- \Big[12 (\varepsilon \nu - 2) (1 + x_2)$$

$$+ \lambda^2 (\nu - 1) (9 + x_2 (9 + 2x_2 (3 + x_2))) \Big]$$

$$\cdot \sinh x_2 \Big\} \Big)$$

$$\cdot \left(48\lambda^2 (1 - \nu) \right)^{-1}.$$

$$(61)$$

Example 1. First, we consider a Reissner plate defined in $[-1, 1]^2$ subjected to clamped boundary conditions, which are set according to the analytical solutions in (61).

The particular solutions of the present problem are solved by the DRM based on the APS [26], Chebyshev polynomials [25], and PMQ. Table 1 gives the RMSEs for the solutions obtained by the DRMs with different bases and N. Basically, the solutions obtained by the Chebyshev polynomials are the most accurate and the application is straightforward if the arbitrary loadings are given by functions. However, when

the loadings are measured on some scattered points, a two-stage method [31] is required which is relatively more difficult compared to the direct application of either APS or PMQ. In the table, the superior accuracy of the MQ over the first few orders of the APS can be observed. In addition, the accuracy improvement for the numerical solutions obtained by the high-order PMQ over the traditional MQ can also be observed. Overall, the best accuracy of the high-order PMQ is close to that of the high-order APS while the former RBF does not need augmented polynomials. In practice, it can be cumbersome to implement the particular solutions of the high-order augmented polynomials and thus the superiority of the PMQ over the APS becomes more relevant.

For both MQ and PMQ, the optimal shape parameters c are searched as demonstrated in Figure 4. Basically, the numerical accuracy improves with increasing c value. However, as c is continuously made larger, the interpolation matrix becomes ill-conditioned [32]. Then, a typical error contour is given in Figure 5 for the solution obtained by the MFS-DRM based on the MQ. In the figure, the larger errors are on the boundary as expected.

Finally, the RMSE comparison for different numbers of nodes is given in Figure 6, in which higher resolution gives better accuracy. Overall, these results should have validated the polyharmonic multiquadric particular solutions of $u_i(r)$.

TABLE 2: The RMSEs for Example 2.

	$N = 6 \times 6$	$N = 8 \times 8$	$N = 10 \times 10$	$N = 12 \times 12$
Chebyshev	$5.36E - 06$	$7.98E - 08$	$4.43E - 08$	$1.07E - 08$
MQ	$5.51E - 05$	$1.03E - 05$	$1.42E - 06$	$6.88E - 07$
PMQ ($L = 2$)	$4.97E - 05$	$5.07E - 06$	$5.51E - 07$	$1.15E - 07$
PMQ ($L = 4$)	$3.12E - 05$	$3.30E - 06$	$1.64E - 07$	$4.21E - 08$
APS (order = 3)	$1.85E - 03$	$4.77E - 04$	$1.64E - 04$	$6.88E - 05$
APS (order = 5)	$1.11E - 04$	$1.13E - 05$	$2.27E - 06$	$6.49E - 07$
APS (order = 7)	Div	$4.20E - 06$	$7.06E - 07$	$1.63E - 07$
APS (order = 9)	Div	Div	$5.66E - 08$	$1.20E - 08$
APS (order = 11)	Div	Div	Div	$1.08E - 08$

(a)

(b)
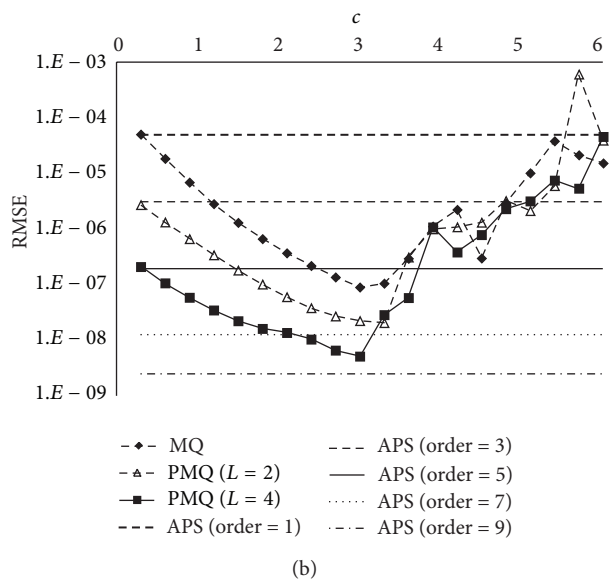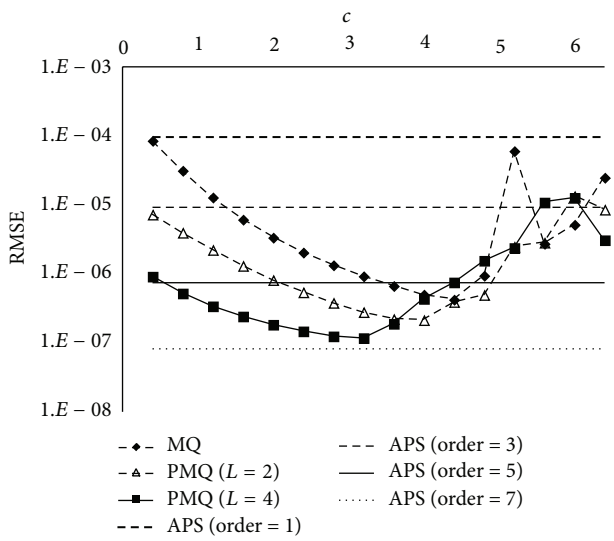
FIGURE 4: Optimal shape parameter searches of Example 1 for (a) $N = 8 \times 8$ and (b) $N = 10 \times 10$.
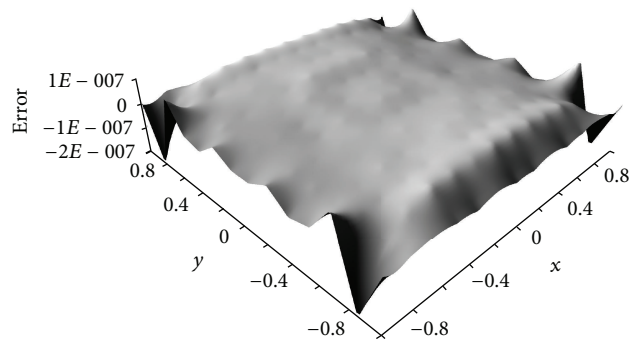

FIGURE 5: Error contour for Example 1 with $N = 10 \times 10$ and $c = 3$.

Example 2. Then, we consider a similar Mindlin plate problem by changing the boundary conditions to the free boundary condition on one side. Table 2 addresses the RMSEs for the numerical results in this example. The results are very similar except that the errors are slightly larger due to the numerical approximations on the free boundary condition.

In addition, the optimal shape parameter searches and the error comparison for different numbers of nodes are given in Figures 7 and 8, respectively. The results are similar to the previous example. Therefore, the polyharmonic multiquadric particular solutions of $Q_\alpha(r)$ and $\Upsilon_{\alpha\beta}(r)$ are also validated.

7. Discussions

In this paper, analytical particular solutions of the polyharmonic multiquadrics associated with Reissner and Mindlin thick plates in bending are derived. In the derivations, the coupled system of the three second-order governing equations is transformed into a single sixth-order product operator of biharmonic and Helmholtz operators. Then, a method is introduced to eliminate the Helmholtz operator, which enables the use of the polyharmonic multiquadric. Numerical results are performed to validate the derived particular solutions and to demonstrate the superiority of the polyharmonic multiquadrics over the traditional multiquadrics and the augmented polynomial splines.
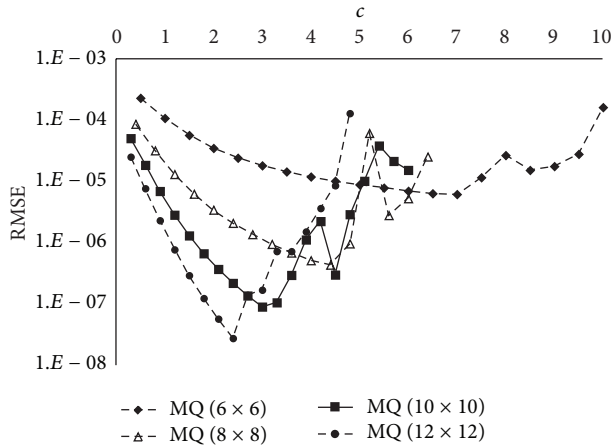
FIGURE 6: RMSE comparisons of Example 1 for different numbers of nodes.
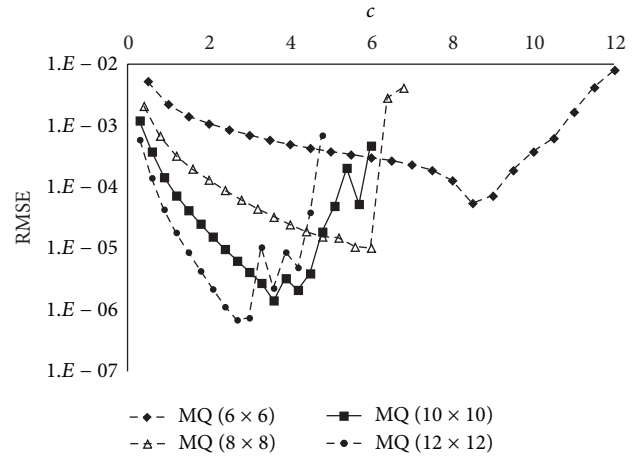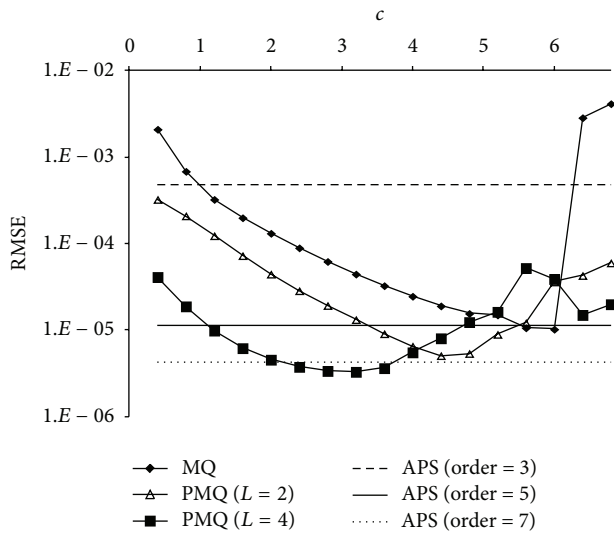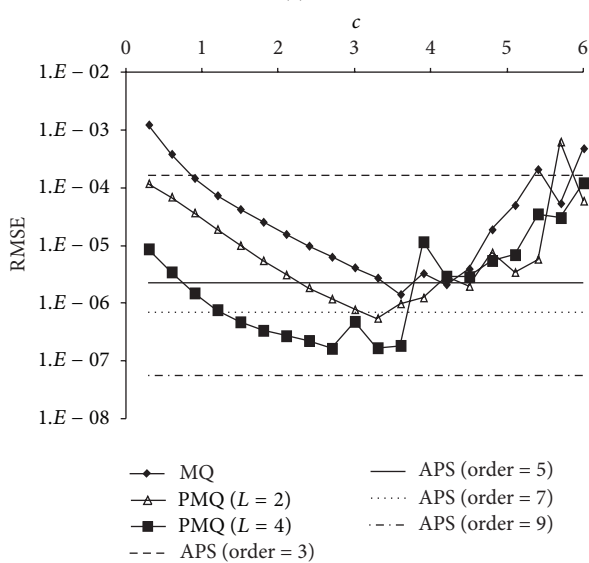

FIGURE 8: RMSE comparisons of Example 2 for different numbers of nodes.

(a)

(b)

FIGURE 7: Optimal shape parameter searches of Example 2 for (a) $N = 8 \times 8$ and (b) $N = 10 \times 10$.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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