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Research Article

Scattering and Bound State Solutions of the Yukawa Potential within the Dirac Equation

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In the presence of spin symmetry case, we obtain bound and scattering states solutions of the Dirac equation for the equal scalar and vector Yukawa potentials for any spin-orbit quantum number κ. The approximate analytical solutions are presented for the bound and scattering states and scattering phase shifts.

1. Introduction

For studying the quantum mechanical systems, it is necessary to pay attention to two points. These two points are to study bound states to take the necessary information about the system under consideration and also solving scattering states for a system under the effect of a potential. Solving both of these problems gives us complete information about a quantum mechanical system under consideration.

The solutions of scattering and/or bound state problem have been investigated for the well-known potentials by applying different methods [1–10]. The analytical scattering state solution of the l-wave Schrödinger Equation for the Eckart potential has been obtained in [11]. The solution of the Schrödinger equation for the modified Morse potential has been studied by Wei and Chen [12]. Rojas and Villalba have found the solutions of the Klein-Gordon equation for one-dimensional Wood-Saxon potential by hypergeometric functions [13]. The exact solutions of scattering state have been studied for the s-wave Schrödinger equation with the Manning-Rosen potential by using standard method [14]. Low momentum scattering states of the Dirac equation have been studied in [15]. Properties of scattering state solutions of the Klein-Gordon equation Coulomb scalar plus vector potential have been studied in [16].

In this work, we have studied bound state and scattering state of the Dirac equation with the Yukawa potential. The Dirac equation describes the particle dynamics in the relativistic quantum mechanics [17, 18]. Thus, solving the Dirac equation is very significant in describing the nuclear shell structure [19, 20]. Also the Yukawa potential has many applications in different areas of physics like high-energy physics [21] and atomic, molecular, and plasma physics [22].

This paper is organized as follows. In Section 2, we briefly introduce the Dirac equation with scalar and vector potential with any spin-orbit quantum number κ. The bound and scattering states of the Yukawa potential within the Dirac equation are presented in Section 3. Finally, our concluding remarks are given in Section 4.

2. Dirac Equation with Scalar and Vector Potential

The Dirac equation with scalar and vector potential ($S(r)$ and $V(r)$) is [$\hbar = c = 1$]

$$[\alpha \cdot p + \beta (M + S(r))] \psi(r) = (E - V(r)) \psi(r), \quad (1)$$

where E is the relativistic energy of the system and $P = -i\nabla$ is the three-dimensional momentum operator. α and β are the usual 4×4 Dirac matrices given as

$$\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad i = 1, 2, 3, \tag{2}$$

where I is the 2×2 unitary matrix and the three 2×2 Pauli matrices σ_i are given as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3}$$

where L is the orbital angular momentum of the spherical nucleons and the total angular momentum operator J and spin-orbit $K = (\sigma \cdot L + 1)$ commute with Dirac Hamiltonian. The eigenvalues of spin-orbit coupling operator are $k = (j + (1/2)) > 0$ and $k = -(j + (1/2)) < 0$ for unaligned spin $j = l - (1/2)$ and the aligned spin $j = l + (1/2)$, respectively. Thus, in the Pauli-Dirac representation,

$$\psi_{nk}(r) = \begin{pmatrix} f_{nk}(r) \\ g_{nk}(r) \end{pmatrix} = \begin{pmatrix} \dfrac{F_{nk}(r)}{r} Y_{jm}^{l}(\theta, \varphi) \\ i \dfrac{G_{nk}(r)}{r} Y_{jm}^{\tilde{l}}(\theta, \varphi) \end{pmatrix}, \tag{4}$$

where $f_{nk}(r)$ is the upper component and $g_{nk}(r)$ is the lower component of the Dirac spinors. $Y_{jm}^{l}(\theta,\varphi)$ and $Y_{jm}^{\tilde{l}}(\theta,\varphi)$ are spin and pseudospin spherical harmonics and m is the projection of the angular momentum on the z-axis. Substituting (4) into (1), one obtains two coupled differential equations for the upper and the lower radial wave functions as follows:

$$\begin{aligned} \left(\frac{d}{dr} + \frac{k}{r}\right) F_{nk}(r) &= \left[M + E_{nk} - \Delta(r)\right] G_{nk}(r), \\ \left(\frac{d}{dr} - \frac{k}{r}\right) G_{nk}(r) &= \left[M - E_{nk} + \Sigma(r)\right] F_{nk}(r), \end{aligned} \tag{5}$$

where

$$\begin{aligned} \Delta(r) &= V(r) - S(r), \\ \Sigma(r) &= V(r) + S(r). \end{aligned} \tag{6}$$

Solving (5) leads to a second-order Schrödinger-like differential equation for the upper and the lower components of the Dirac wavefunctions as follows:

$$\begin{aligned} &\left[\frac{d^2}{dr^2} - \frac{k(k-1)}{r^2}\right] G_{nk}(r) \\ &\quad + \left[-\left(M + E_{nk} - \Delta(r)\right)\left(M - E_{nk} + \Sigma(r)\right) \right. \\ &\quad \left. - \frac{(d\Sigma(r)/dr)(d/dr - k/r)}{M - E_{nk} + \Sigma(r)}\right] G_{nk}(r) = 0, \end{aligned} \tag{7a}$$

$$\begin{aligned} &\left[\frac{d^2}{dr^2} - \frac{k(k+1)}{r^2}\right] F_{nk}(r) \\ &\quad + \left[-\left(M + E_{nk} - \Delta(r)\right)\left(M - E_{nk} + \Sigma(r)\right) \right. \\ &\quad \left. + \frac{(d\Delta(r)/dr)(d/dr + k/r)}{M + E_{nk} - \Delta(r)}\right] F_{nk}(r) = 0, \end{aligned} \tag{7b}$$

where $k(k-1) = \tilde{l}(\tilde{l}+1)$ and $k(k+1) = l(l+1)$.

When scalar potential $S(r)$ is equal to the vector potential $V(r)$, (7b) becomes

$$\begin{aligned} &\left[\frac{d^2}{dr^2} - \frac{k(k+1)}{r^2} - 2\left(E_{nk} + M\right) V(r)\right] F_{nk}(r) \\ &\quad = \left[M^2 - E_{nk}^2\right] F_{nk}(r) \end{aligned} \tag{8}$$

and with (5), we have

$$G_{nk}(r) = \frac{1}{M + E_{nk}} \left[\frac{d}{dr} + \frac{k}{r}\right] F_{nk}(r). \tag{9}$$

These are two equations with equal scalar and vector potential that we have used in this work.

3. Dirac Equation with the Yukawa Potential

According to (8), we have

$$\begin{aligned} &\left[\frac{d^2}{dr^2} - \frac{k(k+1)}{r^2} - 2\left(E_{nk} + M\right) V(r) \right. \\ &\quad \left. + E_{nk}^2 - M^2\right] F_{nk}(r) = 0. \end{aligned} \tag{10}$$

The Yukawa potential is

$$V(r) = -\frac{A}{r} e^{-\alpha r}, \tag{11}$$

where α is the screening parameter and A is the strength of the potential [21, 23].

Instead of the centrifugal term in (10), the following approximation has been used in recent years:

$$\frac{1}{r^2} \approx 4\alpha^2 \frac{e^{-2\alpha r}}{\left(1 - e^{-2\alpha r}\right)^2}; \tag{12}$$

that is, it has a good accuracy for small values of the potential parameter [24, 25].

Substituting (11) and (12) into (10), we obtain the following form of the wave equation:

$$\begin{aligned} &\left[\frac{d^2}{dr^2} - 4\alpha^2 k(k+1) \frac{e^{-2\alpha r}}{\left(1 - e^{-2\alpha r}\right)^2} + 4A\alpha\left(E_{nk} + M\right) \right. \\ &\quad \left. \times \frac{e^{-2\alpha r}}{1 - e^{-2\alpha r}} + E_{nk}^2 - M^2\right] F_{nk}(r) = 0. \end{aligned} \tag{13}$$

3.1. Bound State Solutions. For the bound state, by taking the following variable ($z \to 1$ for $r \to 0$ and $z \to 0$ for $r \to \infty$)

$$z = e^{-2\alpha r} \tag{14}$$

one can obtain (13) in the following form:

$$\begin{aligned}
&\left[z(1-z)\frac{d^2}{dz^2} + (1-z)\frac{d}{dz} + \frac{1}{z(1-z)} \right.\\
&\quad\times\left\{ -k(k+1)z + \frac{A}{\alpha}(E_{nk}+M)\left(z-z^2\right) \right.\\
&\quad\left.\left. + \frac{E_{nk}^2 - M^2}{4\alpha^2}(1-z)^2 \right\} \right] F_{nk}(z) = 0.
\end{aligned} \tag{15}$$

Taking the form of the wave function

$$F_{nk}(z) = z^{\mu}(1-z)^{\nu} f_{nk}(z) \tag{16}$$

and substituting this equation into (15), one gets a hypergeometric-type equation as follows [26]:

$$\begin{aligned}
&z(1-z) f_{nk}''(z) + \left(2\mu + 1 - (2\mu+2\nu+1)z\right) f_{nk}'(z)\\
&\quad + \left[\mu(\mu-1)\frac{(1-z)}{z} - 2\mu\nu \right.\\
&\qquad + \nu(\nu-1)\frac{z}{1-z} + \mu\frac{(1-z)}{z} - \nu\\
&\qquad - k(k+1)\frac{1}{(1-z)} + \frac{A}{\alpha}(E_{nk}+M)\\
&\qquad \left. + \frac{\left(E_{nk}^2 - M^2\right)}{4\alpha^2}\frac{(1-z)}{z} \right] f_{nk}(z) = 0,
\end{aligned} \tag{17}$$

where

$$\mu = i\sqrt{\frac{\left(E_{nk}^2 - M^2\right)}{4\alpha^2}}, \qquad \nu(\nu-1) = k(k+1) \quad (\nu > 0). \tag{18}$$

Comparing (17) with the hypergeometric equation of the form [26]

$$z(1-z) f_{nk}''(z) + [c - (a+b+1)z] f_{nk}'(z) - ab f_{nk}(z) = 0 \tag{19}$$

we can obtain the wavefunction as the hypergeometric function:

$$f_{nk}(z) = {}_2F_1\left(a_{nk}, b_{nk}, c_{nk}; z\right), \tag{20}$$

where

$$\begin{aligned}
a_{nk} &= \mu + \nu + \sqrt{\frac{A}{\alpha}(E_{nk}+M) - \frac{\left(E_{nk}^2 - M^2\right)}{4\alpha^2}},\\
b_{nk} &= \mu + \nu - \sqrt{\frac{A}{\alpha}(E_{nk}+M) - \frac{\left(E_{nk}^2 - M^2\right)}{4\alpha^2}},\\
c_{nk} &= 2\mu + 1.
\end{aligned} \tag{21}$$

Then, with (14) and (16), we have the upper-spinor component of wavefunction for the Dirac equation with the Yukawa potential as follows:

$$\begin{aligned}
F_{nk}(r) &= C_{nk} e^{-2\alpha\mu r}\left(1 - e^{-2\alpha r}\right)^{\nu}\\
&\quad\times {}_2F_1\left(a_{nk}, b_{nk}, c_{nk}; e^{-2\alpha r}\right),
\end{aligned} \tag{22}$$

where C_{nk} is the normalization constant. With (9), the lower-spinor component can be obtained as

$$\begin{aligned}
G_{nk}(r) &= \frac{C_{nk}}{M + E_{nk}}\left[\frac{d}{dr} + \frac{k}{r}\right] e^{-2\alpha\mu r}\left(1 - e^{-2\alpha r}\right)^{\nu}\\
&\quad\times {}_2F_1\left(a_{nk}, b_{nk}, c_{nk}; e^{-2\alpha r}\right).
\end{aligned} \tag{23}$$

Therefore,

$$\begin{aligned}
&G_{nk}(r)\\
&\quad = \frac{C_{nk}}{M + E_{nk}}\left[k\frac{e^{-2\alpha\mu r}}{r}\left(1 - e^{-2\alpha r}\right)^{\nu} \right.\\
&\qquad \times {}_2F_1\left(a_{nk}, b_{nk}, c_{nk}; e^{-2\alpha r}\right)\\
&\qquad - 2\alpha\mu e^{-2\alpha\mu r}\left(1 - e^{-2\alpha r}\right)^{\nu}\\
&\qquad \times {}_2F_1\left(a_{nk}, b_{nk}, c_{nk}; e^{-2\alpha r}\right)\\
&\qquad + 2\alpha\nu e^{-2\alpha(\mu+1)r}\left(1 - e^{-2\alpha r}\right)^{\nu-1}\\
&\qquad \times {}_2F_1\left(a_{nk}, b_{nk}, c_{nk}; e^{-2\alpha r}\right)\\
&\qquad + \left(\frac{a_{nk}b_{nk}}{c_{nk}}\right) e^{-2\alpha\mu r}\left(1 - e^{-2\alpha r}\right)^{\nu}\\
&\qquad \left. \times {}_2F_1\left(a_{nk}+1, b_{nk}+1, c_{nk}+1; e^{-2\alpha r}\right) \right].
\end{aligned} \tag{24}$$

Finally, the spinor wave function under the condition of equal scalar and vector potentials with (4), (7b), (22), and (24) becomes

$$\begin{aligned}
\Psi(r) &= \begin{pmatrix} \dfrac{F_{nk}(r)}{r} Y_{jm}^{l}(\theta, \varphi) \\ i\dfrac{G_{nk}(r)}{r} Y_{jm}^{\tilde{l}}(\theta, \varphi) \end{pmatrix}\\
&= \frac{C_{nk}}{r}\begin{pmatrix} 1 \\ \dfrac{-i}{M + E_{nk}}(\sigma \cdot \hat{r})\,\mathfrak{R}(r) \end{pmatrix} e^{-2\alpha\mu r}\left(1 - e^{-2\alpha r}\right)^{\nu}\\
&\quad \times Y_{jm}^{l}(\theta, \varphi)\, {}_2F_1\left(a_{nk}, b_{nk}, c_{nk}; e^{-2\alpha r}\right),\\
\mathfrak{R}(r) &= \frac{k}{r} - 2\alpha\mu + 2\alpha\nu e^{-2\alpha r}\left(1 - e^{-2\alpha r}\right)^{-1}\\
&\quad + \left(\frac{a_{nk}b_{nk}}{c_{nk}}\right)\frac{{}_2F_1\left(a_{nk}+1, b_{nk}+1, c_{nk}+1; e^{-2\alpha r}\right)}{{}_2F_1\left(a_{nk}, b_{nk}, c_{nk}; e^{-2\alpha r}\right)}.
\end{aligned} \tag{25}$$

By considering the finiteness of the solution, the quantum condition is given by

$$\mu + \nu - \sqrt{\frac{A}{\alpha}\left(E_{nk}+M\right) - \frac{\left(E_{nk}^2 - M^2\right)}{4\alpha^2}} = -n, \quad n = 0, 1, 2, 3. \tag{26}$$

It is the energy eigenvalue equation of bound state for the upper component of the Dirac equation with the Yukawa potential.

3.2. Scattering State. Now, we turn to solve (10) for scattering state. For this purpose, we use a new variable as follows:

$$y = 1 - e^{-2\alpha r}, \qquad y = 1 - z, \qquad (r \longrightarrow \infty, \quad y \longrightarrow 1, \quad r \longrightarrow 0, \quad y \longrightarrow 0) \tag{27}$$

and obtain

$$\left[y\left(1-y\right)\frac{d^2}{dy^2} - y\frac{d}{dy} - k\left(k+1\right)\frac{1}{y} + \frac{A}{\alpha}\left(E_{nk}+M\right) + \frac{y}{4\alpha^2\left(1-y\right)}\left(E_{nk}^2 - M^2\right)\right]F_{nk}\left(y\right) = 0. \tag{28}$$

Considering the boundary condition of the scattering state, we take the following trial wavefunction:

$$F_{nk}\left(y\right) = y^{\eta}\left(1-y\right)^{\beta} f_{nk}\left(y\right). \tag{29}$$

And inserting this equation into (28), we obtain the following equation:

$$\begin{aligned} &y\left(1-y\right) f_{nk}''\left(y\right) + \left[2\eta - \left(2\eta + 2\beta + 1\right)y\right] f_{nk}'\left(y\right) \\ &\quad + \left[\eta\left(\eta-1\right)\frac{\left(1-y\right)}{y} - 2\beta\eta + \beta\left(\beta-1\right)\frac{y}{\left(1-y\right)} - \eta \right. \\ &\quad + \beta\frac{y}{\left(1-y\right)} - k\left(k+1\right)\frac{1}{y} + \frac{A}{\alpha}\left(E_{nk}+M\right) \\ &\quad \left. + \frac{1}{4\alpha^2}\left(E_{nk}^2 - M^2\right)\frac{y}{\left(1-y\right)}\right] f_{nk}\left(y\right) = 0, \end{aligned} \tag{30}$$

where

$$\beta = i\sqrt{\frac{\left(E_{nk}^2 - M^2\right)}{4\alpha^2}}, \qquad \eta\left(\eta - 1\right) = k\left(k+1\right) \quad \left(\eta > 0\right). \tag{31}$$

This equation is a hypergeometric type. Thus, (29) is as follows: (N_{nk} is the normalization constant)

$$F_{nk}\left(y\right) = N_{nk} y^{\eta}\left(1-y\right)^{\beta} {}_2F_1\left(a, b, c; y\right) \tag{32}$$

that is, the upper component of wavefunction. The lower component is

$$\begin{aligned} G_{nk}\left(r\right) = \frac{N_{nk}}{M + E_{nk}}\left[\frac{d}{dr} + \frac{k}{r}\right] e^{-2\alpha\beta r}\left(1 - e^{-2\alpha r}\right)^{\eta} \\ \times {}_2F_1\left(a_{nk}, b_{nk}, c_{nk}; 1 - e^{-2\alpha r}\right). \end{aligned} \tag{33}$$

And therefore, total wavefunction of scattering state with (4), (7b), (32), and (33) is

$$\begin{aligned} \psi\left(r\right) &= \begin{pmatrix} \dfrac{F_{nk}\left(r\right)}{r} Y_{jm}^{l}\left(\theta, \varphi\right) \\ i\dfrac{G_{nk}\left(r\right)}{r} Y_{jm}^{\tilde{l}}\left(\theta, \varphi\right) \end{pmatrix} \\ &= \frac{N_{nk}}{r}\begin{pmatrix} 1 \\ \dfrac{-i}{E_{nk}+M}\left(\sigma \cdot \hat{r}\right)\mathfrak{R}\left(r\right) \end{pmatrix} \\ &\quad \times e^{-2\alpha\beta r}\left(1 - e^{-2\alpha r}\right)^{\eta} Y_{jm}^{l}\left(\theta, \varphi\right) \\ &\quad \times {}_2F_1\left(a_{nk}, b_{nk}, c_{nk}; 1 - e^{-2\alpha r}\right), \end{aligned} \tag{34}$$

where

$$\begin{aligned} \mathfrak{R}\left(r\right) &= \frac{k}{r} - 2\alpha\beta + 2\alpha\eta e^{-2\alpha r}\left(1 - e^{-2\alpha r}\right)^{-1} + \left(\frac{a_{nk}b_{nk}}{c_{nk}}\right) \\ &\quad \times \frac{{}_2F_1\left(a_{nk}+1, b_{nk}+1, c_{nk}+1; 1 - e^{-2\alpha r}\right)}{{}_2F_1\left(a_{nk}, b_{nk}, c_{nk}; 1 - e^{-2\alpha r}\right)}, \\ a_{nk} &= \eta + \beta + \sqrt{\frac{A}{\alpha}\left(E_{nk}+M\right) - \frac{E_{nk}^2 - M^2}{4\alpha^2}}, \\ b_{nk} &= \eta + \beta - \sqrt{\frac{A}{\alpha}\left(E_{nk}+M\right) - \frac{E_{nk}^2 - M^2}{4\alpha^2}}, \\ c_{nk} &= 2\eta. \end{aligned} \tag{35}$$

According to (26), we obtain the following form of energy eigenvalue equation for scattering states:

$$\eta + \beta - \sqrt{\frac{A}{\alpha}\left(E_{nk}+M\right) - \frac{E_{nk}^2 - M^2}{4\alpha^2}} = -n. \tag{36}$$

Now, by finding the asymptotic form of (32) for large r, we try to obtain the scattering phase shifts. For this purpose, we use the following property of the hypergeometric function:

$$\begin{aligned} {}_2F_1\left(a, b, c; x\right) &= \frac{\Gamma\left(c\right)\Gamma\left(c-a-b\right)}{\Gamma\left(c-a\right)\Gamma\left(c-b\right)} \\ &\quad \times {}_2F_1\left(a, b; a+b-c+1; 1-x\right) \\ &\quad + \left(1-x\right)^{c-a-b}\frac{\Gamma\left(c\right)\Gamma\left(a+b-c\right)}{\Gamma\left(a\right)\Gamma\left(b\right)} \\ &\quad \times {}_2F_1\left(c-a, c-b; c-a-b+1; 1-x\right), \end{aligned} \tag{37}$$

$${}_2F_1\left(a, b, c, 0\right) = 1. \tag{38}$$

Therefore, we obtain

$$\begin{aligned} &{}_2F_1\left(a,b,c;1-e^{-2\alpha r}\right) \\ &\quad = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \\ &\qquad \times {}_2F_1\left(a,b;a+b-c+1;e^{-2\alpha r}\right) \\ &\qquad + \left(e^{-2\alpha r}\right)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \\ &\qquad \times {}_2F_1\left(c-a,c-b;c-a-b+1;e^{-2\alpha r}\right). \end{aligned} \tag{39}$$

By using this definition

$$\beta = ik_1, \qquad k_1 = \sqrt{\frac{E_{nk}^2-M^2}{4\alpha^2}} \tag{40}$$

the upper component of wavefunction is as follows:

$$\begin{aligned} F_{nk}(r) &= N_{nk}\left(1-e^{-2\alpha r}\right)^{\eta}e^{-2ik_1\alpha r} \\ &\quad \times {}_2F_1\left(a_{nk},b_{nk},c_{nk};1-e^{-2\alpha r}\right). \end{aligned} \tag{41}$$

Now, with (38) and (39) in the limit $r \to \infty$ we have

$$\begin{aligned} F_{\substack{nk\\ r\to\infty}}(r) \longrightarrow\ & N_{\substack{nk\\ r\to\infty}}\left(1-e^{-2\alpha r}\right)^{\eta}e^{-2ik_1\alpha r} \\ &\times \left\{\frac{\Gamma(c_{nk})\Gamma(c_{nk}-a_{nk}-b_{nk})}{\Gamma(c_{nk}-a_{nk})\Gamma(c_{nk}-b_{nk})}\right. \\ &\quad \left. + \frac{\Gamma(c_{nk})\Gamma(a_{nk}+b_{nk}-c_{nk})}{\Gamma(a_{nk})\Gamma(b_{nk})}e^{4ik_1\alpha r}\right\}. \end{aligned} \tag{42}$$

Then

$$\begin{aligned} F_{nk}(r\longrightarrow\infty) \longrightarrow\ & N_{nk}\Gamma(c_{nk}) \\ &\times \left|\frac{\Gamma(c_{nk}-a_{nk}-b_{nk})}{\Gamma(c_{nk}-a_{nk})\Gamma(c_{nk}-b_{nk})}\right| \\ &\times \left(e^{-i(2k_1\alpha r-\delta)}+e^{i(2k_1\alpha r-\delta)}\right) \\ =\ & 2N_{nk}\Gamma(c_{nk})\left|\frac{\Gamma(c_{nk}-a_{nk}-b_{nk})}{\Gamma(c_{nk}-a_{nk})\Gamma(c_{nk}-b_{nk})}\right| \\ &\times \cos(2k_1\alpha r-\delta) \\ =\ & 2N_{nk}\Gamma(c_{nk})\left|\frac{\Gamma(c_{nk}-a_{nk}-b_{nk})}{\Gamma(c_{nk}-a_{nk})\Gamma(c_{nk}-b_{nk})}\right| \\ &\times \sin\left(\delta-2k_1\alpha r+\frac{\pi}{2}\right). \end{aligned} \tag{43}$$

The general boundary condition of the scattering state wave function on the "$(k_1/2\pi)$ scale" is

$$\varphi(r) = 2\sin\left(k_1 r-\frac{\pi}{2}l+\delta_l\right). \tag{44}$$

Therefore, we have

$$\begin{aligned} &F_{nk}(r\longrightarrow\infty) \\ &\quad\longrightarrow 2N_{nk}\Gamma(c_{nk}) \\ &\qquad\times\left|\frac{\Gamma(c_{nk}-a_{nk}-b_{nk})}{\Gamma(c_{nk}-a_{nk})\Gamma(c_{nk}-b_{nk})}\right| \\ &\qquad\times\sin\left(2\alpha k_1 r+\frac{\pi}{2}\right. \\ &\qquad\qquad\left.+\arg\left(\frac{\Gamma(c_{nk}-a_{nk}-b_{nk})}{\Gamma(c_{nk}-a_{nk})\Gamma(c_{nk}-b_{nk})}\right)\right). \end{aligned} \tag{45}$$

Thus, we obtain the scattering phase shifts as follows:

$$\begin{aligned} \delta_{l,nk} &= \frac{\pi}{2}(l+1)+\arg\Gamma(c_{nk}-a_{nk}-b_{nk}) \\ &\quad -\arg\Gamma(c_{nk}-a_{nk})-\arg\Gamma(c_{nk}-b_{nk}). \end{aligned} \tag{46}$$

Then we have the scattering phase shifts of the upper component of wavefunction as

$$\begin{aligned} \delta_{l,nk} &= \frac{\pi}{2}(l+1)+\arg\Gamma(2\beta) \\ &\quad -\arg\Gamma\left(\eta+\beta-\sqrt{2\eta^2+\beta^2+4\eta\beta+\frac{A}{\alpha}(E_{nk}+M)}\right) \\ &\quad -\arg\Gamma\left(\eta+\beta+\sqrt{2\eta^2+\beta^2+4\eta\beta+\frac{A}{\alpha}(E_{nk}+M)}\right). \end{aligned} \tag{47}$$

For the lower component, with (34), we have

$$\begin{aligned} G_{\substack{nk\\ r\to\infty}}(r) \longrightarrow\ & \frac{N_{nk}}{E_{\substack{nk\\ r\to\infty}}+M}\mathfrak{R}(r)\left(1-e^{-2\alpha r}\right)^{\eta} \\ &\times e^{-2ik_1\alpha r}\left\{\frac{\Gamma(c_{nk})\Gamma(c_{nk}-a_{nk}-b_{nk})}{\Gamma(c_{nk}-a_{nk})\Gamma(c_{nk}-b_{nk})}\right. \\ &\quad+\frac{\Gamma(c_{nk})\Gamma(a_{nk}+b_{nk}-c_{nk})}{\Gamma(a_{nk})\Gamma(b_{nk})} \\ &\quad\left.\times e^{4ik_1\alpha r}\right\} \end{aligned}$$

Table 1: Energy eigenvalues of the Yukawa potential for different values of n and k (in $\hbar = c = 1$ unit) for $M = 0.5 f\,\text{m}^{-1}$, $A = 0.1$, and $\alpha = 0.01$.

n	$l, k > 0$	$l, j = l - 1/2$	ν	$E_{k>0}$	$l, k < 0$	$l, j = l + 1/2$	ν	$E_{k<0}$
0	1, 1	$0p_{1/2}$	2	−0.4996006388	0, −1	$0s_{1/2}$	1	−0.4999001896
0	2, 2	$0d_{3/2}$	3	−0.4991009891	1, −2	$0p_{3/2}$	2	−0.4996006388
0	3, 3	$0f_{5/2}$	4	−0.4984006400	2, −3	$0d_{5/2}$	3	−0.4991009891
0	4, 4	$0g_{7/2}$	5	−0.4974987420	3, −4	$0f_{7/2}$	4	−0.4984006400
1	1, 1	$1p_{1/2}$	2	−0.4991009891	0, −1	$1s_{1/2}$	1	−0.4996006388
1	2, 2	$1d_{3/2}$	3	−0.4984006400	1, −2	$1p_{3/2}$	2	−0.4991009891
1	3, 3	$1f_{5/2}$	4	−0.4974987420	2, −3	$1d_{5/2}$	3	−0.4984006400
1	4, 4	$1g_{7/2}$	5	−0.4963942080	3, −4	$1f_{7/2}$	4	−0.4974987420
1	1, 1	$2p_{1/2}$	2	−0.4984006400	0, −1	$2s_{1/2}$	1	−0.4991009891
2	2, 2	$2d_{3/2}$	3	−0.4974987420	1, −2	$2p_{3/2}$	2	−0.4984006400
2	3, 3	$2f_{5/2}$	4	−0.4963942080	2, −3	$2d_{5/2}$	3	−0.4974987420
2	4, 4	$2g_{7/2}$	5	−0.4950856720	3, −4	$2f_{7/2}$	4	−0.4963942080

$$
\begin{aligned}
&\lim_{r\to\infty} \mathfrak{R}(r) \\
&\quad = \lim_{r\to\infty} \left(\frac{k}{r} - 2\alpha\beta + 2\alpha\eta e^{-2\alpha r} \right. \\
&\qquad \times \left(1 - e^{-2\alpha r}\right)^{-1} + \left(\frac{a_{nk} b_{nk}}{c_{nk}}\right) \\
&\qquad \left. \times \frac{{}_2F_1\left(a_{nk}+1, b_{nk}+1, c_{nk}+1;\ 1 - e^{-2\alpha r}\right)}{{}_2F_1\left(a_{nk}, b_{nk}, c_{nk};\ 1 - e^{-2\alpha r}\right)} \right) \\
&\quad = -2\alpha\beta + \frac{a_{nk} b_{nk}}{c_{nk}}.
\end{aligned} \tag{48}
$$

Then we have

$$
\begin{aligned}
G_{nk}(r \longrightarrow \infty) &\longrightarrow H_{nk} \Gamma(c_{nk}) \\
&\quad \times \left| \frac{\Gamma(c_{nk} - a_{nk} - b_{nk})}{\Gamma(c_{nk} - a_{nk}) \Gamma(c_{nk} - b_{nk})} \right| \\
&\quad \times \left(e^{-i(2k_1 \alpha r - \delta)} + e^{i(2k_1 \alpha r - \delta)} \right),
\end{aligned} \tag{49}
$$

where

$$
H_{nk} = \frac{N_{nk}}{E_{nk} + M} \left(-2\alpha\beta + \frac{a_{nk} b_{nk}}{c_{nk}} \right). \tag{50}
$$

Thus with (43), (44), (45), and (49), we have the following:

$$
\begin{aligned}
&G_{nk}(r \longrightarrow \infty) \\
&\quad \longrightarrow 2H_{nk} \Gamma(c_{nk}) \\
&\qquad \times \left| \frac{\Gamma(c_{nk} - a_{nk} - b_{nk})}{\Gamma(c_{nk} - a_{nk}) \Gamma(c_{nk} - b_{nk})} \right| \\
&\qquad \times \sin\left(2\alpha k_1 r + \frac{\pi}{2} \right. \\
&\qquad\qquad \left. + \arg\left(\frac{\Gamma(c_{nk} - a_{nk} - b_{nk})}{\Gamma(c_{nk} - a_{nk}) \Gamma(c_{nk} - b_{nk})} \right) \right).
\end{aligned} \tag{51}
$$

And finally we have

$$
\begin{aligned}
\delta_{l,nk} = \frac{\pi}{2}(l+1) &+ \arg\Gamma(c_{nk} - a_{nk} - b_{nk}) \\
&- \arg\Gamma(c_{nk} - a_{nk}) - \arg\Gamma(c_{nk} - b_{nk}).
\end{aligned} \tag{52}
$$

Therefore, the scattering phase shifts of two components are equal but in a constant coefficient.

4. Conclusions

We have studied the Dirac equation with the Yukawa potential and have obtained bound and scattering states of this problem. The energy eigenvalues, eigenstates, and scattering phase shifts have been presented. The numerical results of the energy eigenvalues of this work have been compared with the ones obtained in the literature in Table 1.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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