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Research Article

The Jacobi Elliptic Equation Method for Solving Fractional Partial Differential Equations

Bin Zheng and Qinghua Feng

School of Science, Shandong University of Technology, Zibo, Shandong 255049, China

Correspondence should be addressed to Qinghua Feng; fqhua@sina.com

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Based on a nonlinear fractional complex transformation, the Jacobi elliptic equation method is extended to seek exact solutions for fractional partial differential equations in the sense of the modified Riemann-Liouville derivative. For demonstrating the validity of this method, we apply it to solve the space fractional coupled Konopelchenko-Dubrovsky (KD) equations and the space-time fractional Fokas equation. As a result, some exact solutions for them including the hyperbolic function solutions, trigonometric function solutions, rational function solutions, and Jacobi elliptic function solutions are successfully found.

1. Introduction

In the nonlinear sciences, it is well known that many nonlinear partial differential equations are widely used to describe the complex phenomena in various fields. The powerful and efficient methods to find analytic solutions and numerical solutions of nonlinear equations have drawn a lot of interest by a diverse group of scientists. Many efficient methods have been presented so far (e.g., see [1–9]). Fractional differential equations are generalizations of classical differential equations of integer order. In recent decades, fractional differential equations have gained much attention as they are widely used to describe various complex phenomena in many fields such as the fluid flow, signal processing, control theory, systems identification, and biology and other areas. Many experts have investigated theoretic problems of fractional differential equations so far, and the concerned fields include the existence and uniqueness of solutions to Cauchy type problems, the methods for explicit and numerical solutions, and the stability of solutions (e.g., see [10–15]). Among these investigations for fractional differential equations, research for seeking analytical or semianalytical solutions of fractional differential equations has been paid an increasing attention. Many analytical or semianalytical methods have been proposed to obtain numerical solutions and exact solutions of fractional differential equations so far. For example, these methods include the (G'/G) method [16–18], the variational iterative method [19–21], and the fractional subequation method [22–26]. Based on these methods, a variety of fractional differential equations have been investigated.

In this paper, we extend the Jacobi elliptic equation method to seek exact solutions for fractional partial differential equations in the sense of modified Riemann-Liouville derivative. Based on a nonlinear fractional complex transformation, certain fractional partial differential equation can be converted into another ordinary differential equation of integer order with respect to one new variable, which can be solved based on the Jacobi elliptic equation. This method belongs to the categories of fractional subequation methods, and with the general solutions of the Jacobi elliptic equation, a series of exact solutions for the fractional partial differential equation can be obtained.

Definition 1. The modified Riemann-Liouville derivative of order α is defined by the following expression [27–30]:

$$D_t^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dt} \\ \times\displaystyle\int_0^t (t-\xi)^{-\alpha}\left(f(\xi)-f(0)\right)d\xi, & 0<\alpha<1, \\ \left(f^{(n)}(t)\right)^{(\alpha-n)}, & n\le\alpha<n+1,\ n\ge 1. \end{cases} \tag{1}$$

Definition 2. The Riemann-Liouville fractional integral of order α on the interval $[0,t]$ is defined by

$$\begin{aligned} I^{\alpha} f(t) &= \frac{1}{\Gamma(1+\alpha)} \int_0^t f(s)(ds)^{\alpha} \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds. \end{aligned} \tag{2}$$

Some important properties for the modified Riemann-Liouville derivative and fractional integral are listed as follows (see [22–27]) (the interval concerned below is always defined by $[0,t]$):

$$D_t^{\alpha} t^{r} = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha}, \tag{3}$$

$$D_t^{\alpha}(f(t)g(t)) = g(t) D_t^{\alpha} f(t) + f(t) D_t^{\alpha} g(t), \tag{4}$$

$$\begin{aligned} D_t^{\alpha} f[g(t)] &= f'_g[g(t)] D_t^{\alpha} g(t) \\ &= D_g^{\alpha} f[g(t)] (g'(t))^{\alpha}, \end{aligned} \tag{5}$$

$$I^{\alpha}(D_t^{\alpha} f(t)) = f(t) - f(0), \tag{6}$$

$$\begin{aligned} I^{\alpha}(g(t) D_t^{\alpha} f(t)) = {} & f(t) g(t) - f(0) g(0) \\ & - I^{\alpha}(f(t) D_t^{\alpha} g(t)). \end{aligned} \tag{7}$$

The rest of this paper is organized as follows. In Section 2, we give the description of the Jacobi elliptic equation method for solving fractional partial differential equations. Then in Section 3 we apply this method to establish exact solutions for the space fractional coupled Konopelchenko-Dubrovsky (KD) equations and the space-time fractional Fokas equation. Some conclusions are presented at the end of the paper.

2. Description of the Jacobi Elliptic Equation Method for Solving Fractional Partial Differential Equations

In this section we give the description of the Jacobi elliptic equation method for solving fractional partial differential equations.

Suppose that a fractional partial differential equation, say in the independent variables $t, x_1, x_2, \ldots, x_n$, is given by

$$\begin{aligned} P\big(&u_1, \ldots, u_k, D_t^{\alpha} u_1, \ldots, D_t^{\alpha} u_k, D_{x_1}^{\beta} u_1, \ldots, D_{x_1}^{\beta} u_k, \ldots, \\ & D_{x_n}^{\gamma} u_1, \ldots, D_{x_n}^{\gamma} u_k, \ldots\big) = 0, \end{aligned} \tag{8}$$

where $u_i = u_i(t, x_1, x_2, \ldots, x_n)$, $i = 1, \ldots, k$, are unknown functions and P is a polynomial in u_i and their various partial derivatives including fractional derivatives.

Step 1. Execute a certain nonlinear fractional complex transformation for ξ:

$$u_i(t, x_1, x_2, \ldots, x_n) = U_i(\xi), \quad \xi = \xi(t, x_1, x_2, \ldots, x_n), \tag{9}$$

such that (8) can be turned into the following ordinary differential equation of integer order with respect to the variable ξ:

$$\widetilde{P}\left(U_1, \ldots, U_k, U_1', \ldots, U_k', U_1'', \ldots, U_k'', \ldots\right) = 0. \tag{10}$$

In fact, take $D_t^{\alpha} u_1$; for example, one can suppose a nonlinear fractional complex transformation $\xi = c(t^{\alpha}/\Gamma(1+\alpha))$, and then by using (3) obtain $D_t^{\alpha} u_1 = U_1'(\xi) D_t^{\alpha} \xi = c U_1'(\xi)$.

Step 2. Suppose that the solution of (10) can be expressed by a polynomial in (G'/G) as follows:

$$U_j(\xi) = \sum_{i=0}^{m_j} a_{j,i} \left(\frac{G'}{G}\right)^{i}, \quad j = 1, 2, \ldots, k, \tag{11}$$

where $a_{j,i}$, $i = 0, 1, \ldots, m_j$, $j = 1, 2, \ldots, k$, are constants to be determined later, $a_{j,m} \neq 0$, the positive integer m_j can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (10), and $G = G(\xi)$ satisfies the following Jacobi elliptic equation [31]:

$$\left(G'\right)^2 = e_2 G^4 + e_1 G^2 + e_0, \tag{12}$$

where e_0, e_1, and e_2 are arbitrary constants.

Some general solutions of (12) are listed as follows:

$$G(\xi) = \begin{cases} -\sqrt{e_1}\operatorname{sech}\left(\sqrt{e_1}\xi\right), \\ \quad e_2 = -1,\ e_1 > 0,\ e_0 = 0, \\ -\sqrt{e_1}\operatorname{csch}\left(\sqrt{e_1}\xi\right), \\ \quad e_2 = 1,\ e_1 > 0,\ e_0 = 0, \\ \sqrt{-e_1}\sec\left(\sqrt{-e_1}\xi\right), \\ \quad e_2 = 1,\ e_1 < 0,\ e_0 = 0, \\ \dfrac{1}{\xi + C_0}, \\ \quad e_2 = 1,\ e_1 = 0,\ e_0 = 0, \\ \operatorname{sn}(\xi), \\ \quad e_2 = m^2,\ e_1 = -\left(1+m^2\right),\ e_0 = 0, \\ \operatorname{cn}(\xi), \\ \quad e_2 = -m^2,\ e_1 = 2m^2 - 1,\ e_0 = 1 - m^2, \\ \operatorname{dn}(\xi), \\ \quad e_2 = -1,\ e_1 = 2 - m^2,\ e_0 = m^2 - 1, \\ \operatorname{cs}(\xi), \\ \quad e_2 = 1,\ e_1 = 2 - m^2,\ e_0 = 1 - m^2, \\ \operatorname{sd}(\xi), \\ \quad e_2 = m^2\left(m^2 - 1\right),\ e_1 = 2m^2 - 1,\ e_0 = 1, \\ \operatorname{dc}(\xi), \\ \quad e_2 = 1,\ e_1 = -\left(m^2 + 1\right),\ e_0 = m^2, \end{cases} \tag{13}$$

where C_0 is a constant, $\operatorname{sn}(\xi)$, $\operatorname{cn}(\xi)$, and $\operatorname{dn}(\xi)$ denote the Jacobi elliptic sine function, Jacobi elliptic cosine function, and the Jacobi elliptic function of the third kind, respectively, m is the modulus, and

$$\begin{gathered}\operatorname{cs}(\xi)=\frac{\operatorname{cn}(\xi)}{\operatorname{sn}(\xi)},\qquad \operatorname{sd}(\xi)=\frac{\operatorname{sn}(\xi)}{\operatorname{dn}(\xi)},\qquad \operatorname{dc}(\xi)=\frac{\operatorname{dn}(\xi)}{\operatorname{cn}(\xi)},\\ \operatorname{sc}(\xi)=\frac{1}{\operatorname{cs}(\xi)},\qquad ds(\xi)=\frac{1}{\operatorname{sd}(\xi)},\\ \operatorname{cd}(\xi)=\frac{1}{\operatorname{dc}(\xi)},\qquad \operatorname{nd}(\xi)=\frac{1}{\operatorname{dn}(\xi)},\\ \operatorname{ns}(\xi)=\frac{1}{\operatorname{sn}(\xi)},\qquad \operatorname{nc}(\xi)=\frac{1}{\operatorname{cn}(\xi)}.\end{gathered}\tag{14}$$

Furthermore, one has

$$\left(\frac{G'(\xi)}{G(\xi)}\right)=\begin{cases}-\sqrt{e_1}\tanh\left(\sqrt{e_1}\xi\right),\\ \quad e_2=-1,\ e_1>0,\ e_0=0,\\ -\sqrt{e_1}\coth\left(\sqrt{e_1}\xi\right),\\ \quad e_2=1,\ e_1>0,\ e_0=0,\\ \sqrt{-e_1}\tan\left(\sqrt{-e_1}\xi\right),\\ \quad e_2=1,\ e_1<0,\ e_0=0,\\ -\dfrac{1}{\xi+C_0},\\ \quad e_2=1,\ e_1=0,\ e_0=0,\\ \operatorname{cn}(\xi)\,ds(\xi),\\ \quad e_2=m^2,\ e_1=-\left(1+m^2\right),\ e_0=0,\\ -\operatorname{sn}(\xi)\operatorname{dc}(\xi),\\ \quad e_2=-m^2,\ e_1=2m^2-1,\ e_0=1-m^2,\\ -m^2\operatorname{sn}(\xi)\operatorname{cd}(\xi),\\ \quad e_2=-1,\ e_1=2-m^2,\ e_0=m^2-1,\\ -\dfrac{\operatorname{dc}(\xi)}{\operatorname{sn}(\xi)},\\ \quad e_2=1,\ e_1=2-m^2,\ e_0=1-m^2,\\ \dfrac{\operatorname{cs}(\xi)}{\operatorname{dn}(\xi)},\\ \quad e_2=m^2\left(m^2-1\right),\ e_1=2m^2-1,\ e_0=1,\\ \left(1-m^2\right)\dfrac{\operatorname{sd}(\xi)}{\operatorname{cn}(\xi)},\\ \quad e_2=1,\ e_1=-\left(m^2+1\right),\ e_0=m^2.\end{cases}\tag{15}$$

Other solutions with e_2, e_1, and e_0 taking different values are omitted here for the sake of simplicity.

Step 3. Substituting (11) into (10) and using (12), the left-hand side of (10) is converted into another polynomial in $G^iG'^j$. Collecting all coefficients of the same power and equating them to zero yield a set of algebraic equations for $a_{j,i}$, $i=0,1,\ldots,m$, $j=1,2,\ldots,k$.

Step 4. Solving the equations' system in Step 3, and using the general solutions of (12), we can construct a variety of exact solutions for (8).

3. Application of the Jacobi Elliptic Equation Method to Some Fractional Partial Differential Equations

3.1. Space Fractional Coupled Konopelchenko-Dubrovsky (KD) Equations. Consider the space fractional coupled Konopelchenko-Dubrovsky (KD) equations

$$\begin{aligned}&D_t^{\alpha}u-D_x^{3\beta}u-6buD_x^{\beta}u+\frac{3}{2}a^2u^2D_x^{\beta}u\\&\quad-3D_y^{\gamma}v+3aD_x^{\beta}(uv)=0,\qquad 0<\alpha,\beta,\gamma\leq 1,\\&D_y^{\gamma}u=D_x^{\beta}v,\end{aligned}\tag{16}$$

where $D^{\alpha}(\cdot)$ denotes the modified Riemann-Liouville derivative of order α. Equation (16) is a variation of the coupled Konopelchenko-Dubrovsky (KD) equations of integer order [32].

In the following, we will apply the Jacobi elliptic equation method described in Section 2 to solve (16). To begin with, we suppose $u(x,y,t)=U(\xi)$, $v(x,y,t)=V(\xi)$, where $\xi=(c/\Gamma(1+\alpha))t^{\alpha}+(k/\Gamma(1+\beta))x^{\beta}+(l/\Gamma(1+\gamma))y^{\gamma}+\xi_0$, k,l,c, ξ_0 are all constants with $k,l,c\neq 0$. Then by use of (3) and the first equality of (5) we obtain

$$\begin{aligned}D_t^{\alpha}u&=D_t^{\alpha}U(\xi)=U'(\xi)D_t^{\alpha}\xi=cU'(\xi),\\D_x^{\beta}u&=D_x^{\beta}U(\xi)=U'(\xi)D_x^{\beta}\xi=kU'(\xi),\\D_y^{\gamma}u&=D_y^{\gamma}U(\xi)=U'(\xi)D_y^{\gamma}\xi=lU'(\xi).\end{aligned}\tag{17}$$

Then (16) can be turned into the following form:

$$\begin{gathered}cU'-k^3U'''-6kbUU'+\frac{3}{2}ka^2U^2U'\\-3lV'+3ak(UV)'=0,\\lU'=kV'.\end{gathered}\tag{18}$$

Suppose that the solution of (18) can be expressed by

$$\begin{aligned}U(\xi)&=\sum_{i=0}^{m_1}a_i\left(\frac{G'}{G}\right)^i,\\V(\xi)&=\sum_{i=0}^{m_2}b_i\left(\frac{G'}{G}\right)^i,\end{aligned}\tag{19}$$

where $G=G(\xi)$ satisfies (12). Balancing the order of U''' and U^2U', U' and V' in (18) we have $m_1=m_2=1$. So,

$$\begin{aligned}U(\xi)&=a_0+a_1\left(\frac{G'}{G}\right),\\V(\xi)&=b_0+b_1\left(\frac{G'}{G}\right).\end{aligned}\tag{20}$$

Substituting (20) into (18), using (12), collecting all the terms with the same power of $G^iG'^j$ together, and equating each coefficient to zero yield a set of algebraic equations. Solving these equations yields that

$$
\begin{aligned}
a_0 &= -\frac{2(al-bk)}{a^2k}, \qquad a_1 = \pm\frac{2k}{a},\\
b_0 &= -\frac{cka^2 + 2k^4e_1a^2 + 6bkal - 6b^2k^2 - 3l^2a^2}{a^3k^2}, \qquad (21)\\
b_1 &= \pm\frac{2l}{a}.
\end{aligned}
$$

Substituting the result above into (20) and combining with (15) we can obtain the following exact solutions for (16).

Family 1. When $e_2 = -1$, $e_1 > 0$, $e_0 = 0$, the following hyperbolic function solution can be obtained:

$$
\begin{aligned}
u_1(x,y,t) &= -\frac{2(al-bk)}{a^2k} \pm \frac{2k}{a}\left[-\sqrt{e_1}\tanh\left(\sqrt{e_1}\xi\right)\right],\\
v_1(x,y,t) &= -\frac{cka^2 + 2k^4e_1a^2 + 6bkal - 6b^2k^2 - 3l^2a^2}{a^3k^2}\\
&\quad \pm \frac{2l}{a}\left[-\sqrt{e_1}\tanh\left(\sqrt{e_1}\xi\right)\right],
\end{aligned}
\qquad (22)
$$

where $\xi = (c/\Gamma(1+\alpha))t^{\alpha} + (k/\Gamma(1+\beta))x^{\beta} + (l/\Gamma(1+\gamma))y^{\gamma} + \xi_0$.

In Figures 1 and 2, the solitary wave solutions $u_1(x,y,t)$, $v_1(x,y,t)$ in (22) with some special parameters are demonstrated.

Family 2. When $e_2 = 1$, $e_1 > 0$, $e_0 = 0$,

$$
\begin{aligned}
u_2(x,y,t) &= -\frac{2(al-bk)}{a^2k} \pm \frac{2k}{a}\left[-\sqrt{e_1}\coth\left(\sqrt{e_1}\xi\right)\right],\\
v_2(x,y,t) &= -\frac{cka^2 + 2k^4e_1a^2 + 6bkal - 6b^2k^2 - 3l^2a^2}{a^3k^2}\\
&\quad \pm \frac{2l}{a}\left[-\sqrt{e_1}\coth\left(\sqrt{e_1}\xi\right)\right],
\end{aligned}
\qquad (23)
$$

where $\xi = (c/\Gamma(1+\alpha))t^{\alpha} + (k/\Gamma(1+\beta))x^{\beta} + (l/\Gamma(1+\gamma))y^{\gamma} + \xi_0$.

Family 3. When $e_2 = 1$, $e_1 < 0$, $e_0 = 0$, the following trigonometric function solution can be obtained:

$$
\begin{aligned}
u_3(x,y,t) &= -\frac{2(al-bk)}{a^2k} \pm \frac{2k}{a}\sqrt{-e_1}\tan\left(\sqrt{-e_1}\xi\right),\\
v_3(x,y,t) &= -\frac{cka^2 + 2k^4e_1a^2 + 6bkal - 6b^2k^2 - 3l^2a^2}{a^3k^2}\\
&\quad \pm \frac{2l}{a}\sqrt{-e_1}\tan\left(\sqrt{-e_1}\xi\right),
\end{aligned}
\qquad (24)
$$

where $\xi = (c/\Gamma(1+\alpha))t^{\alpha} + (k/\Gamma(1+\beta))x^{\beta} + (l/\Gamma(1+\gamma))y^{\gamma} + \xi_0$.

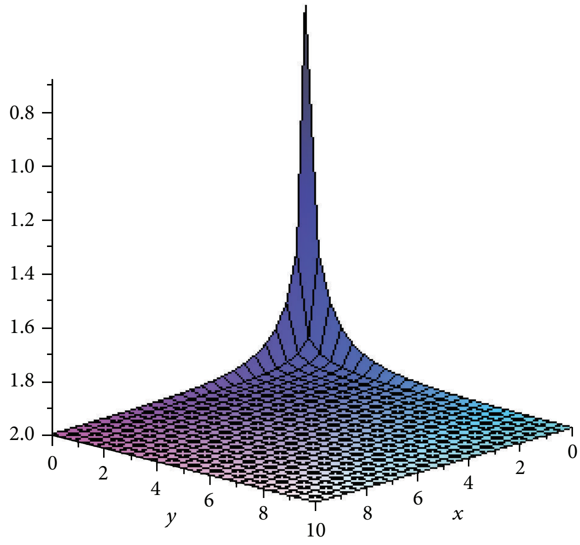

FIGURE 1: The solitary wave solution u_1 with $e_1 = 1$, $a = b = c = k = 1 = 1$, $t = 0.1$, and $\alpha = 1/2$.

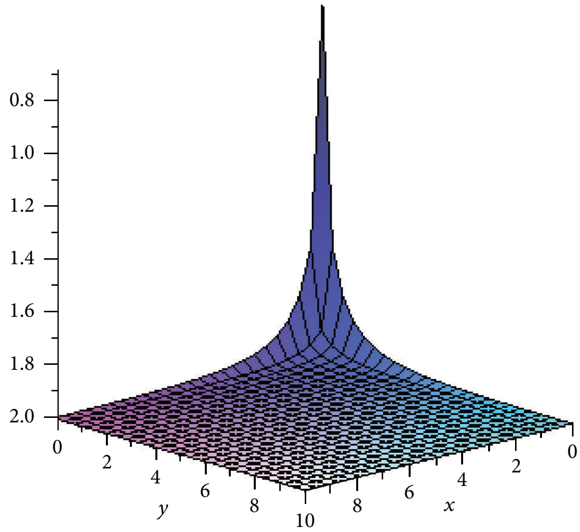

FIGURE 2: The solitary wave solution v_1 with $e_1 = 1$, $a = b = c = k = 1 = 1$, $t = 0.1$, and $\alpha = 1/2$.

In Figures 3 and 4, the periodic wave solutions $u_3(x,y,t)$, $v_3(x,y,t)$ in (24) with some special parameters are demonstrated.

Family 4. When $e_2 = 1$, $e_1 = 0$, $e_0 = 0$, the following rational function solution can be obtained:

$$
\begin{aligned}
u_4(x,y,t) &= -\frac{2(al-bk)}{a^2k} \pm \frac{2k}{a}\left(-\frac{1}{\xi + C_0}\right),\\
v_4(x,y,t) &= -\frac{cka^2 + 6bkal - 6b^2k^2 - 3l^2a^2}{a^3k^2}\\
&\quad \pm \frac{2l}{a}\left(-\frac{1}{\xi + C_0}\right),
\end{aligned}
\qquad (25)
$$

where $\xi = (c/\Gamma(1+\alpha))t^{\alpha} + (k/\Gamma(1+\beta))x^{\beta} + (l/\Gamma(1+\gamma))y^{\gamma} + \xi_0$.

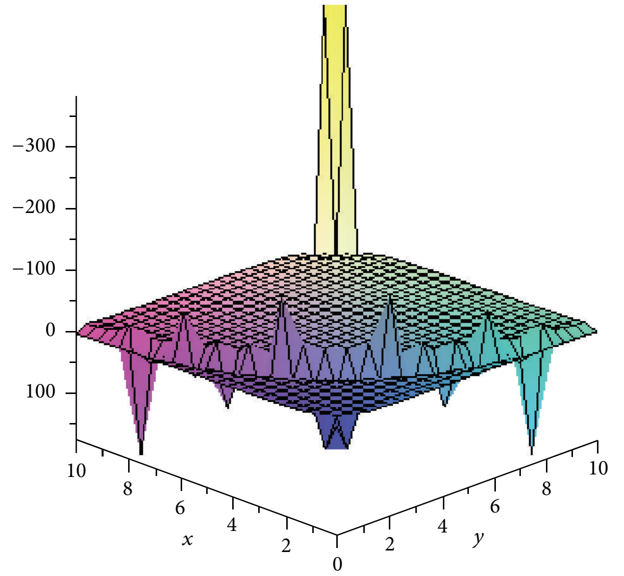

FIGURE 3: The periodic wave solution u_3 with $e_1 = -1$, $a = b = c = k = 1 = 1$, $t = 0.5$, and $\alpha = 1/2$.

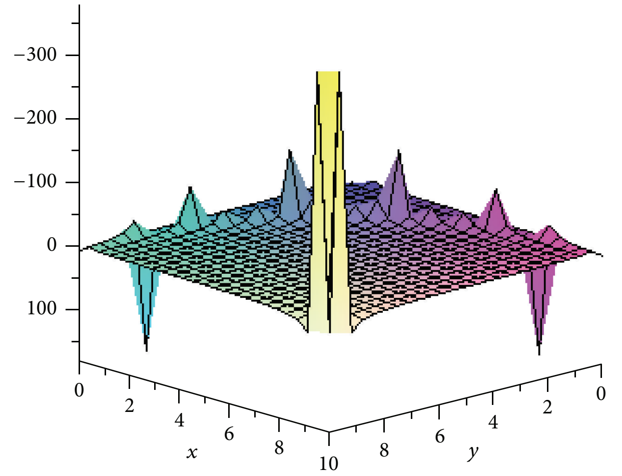

FIGURE 4: The periodic wave solution v_3 with $e_1 = -1$, $a = b = c = k = 1 = 1$, $t = 0.5$, and $\alpha = 1/2$.

Family 5. When $e_2 = m^2$, $e_1 = -(1+m^2)$, $e_0 = 1$, the following Jacobi elliptic function solution can be obtained:

$$u_5(x,y,t) = -\frac{2(al-bk)}{a^2k} \pm \frac{2k}{a}\operatorname{cn}(\xi)\, ds(\xi),$$

$$v_5(x,y,t) = -\frac{cka^2 - 2k^4(1+m^2)a^2 + 6bkal - 6b^2k^2 - 3l^2a^2}{a^3k^2} \pm \frac{2l}{a}\operatorname{cn}(\xi)\, ds(\xi), \tag{26}$$

where $\xi = (c/\Gamma(1+\alpha))t^{\alpha} + (k/\Gamma(1+\beta))x^{\beta} + (l/\Gamma(1+\gamma))y^{\gamma} + \xi_0$.

Family 6. When $e_2 = -m^2$, $e_1 = 2m^2 - 1$, $e_0 = 1 - m^2$,

$$u_6(x,y,t) = -\frac{2(al-bk)}{a^2k} \pm \frac{2k}{a}\left[-\operatorname{sn}(\xi)\operatorname{dc}(\xi)\right],$$

$$v_6(x,y,t) = -\frac{cka^2 + 2k^4(2m^2-1)a^2 + 6bkal - 6b^2k^2 - 3l^2a^2}{a^3k^2} \pm \frac{2l}{a}\left[-\operatorname{sn}(\xi)\operatorname{dc}(\xi)\right], \tag{27}$$

where $\xi = (c/\Gamma(1+\alpha))t^{\alpha} + (k/\Gamma(1+\beta))x^{\beta} + (l/\Gamma(1+\gamma))y^{\gamma} + \xi_0$.

Family 7. When $e_2 = -1$, $e_1 = 2 - m^2$, $e_0 = m^2 - 1$,

$$u_7(x,y,t) = -\frac{2(al-bk)}{a^2k} \pm \frac{2k}{a}\left[-m^2\operatorname{sn}(\xi)\operatorname{cd}(\xi)\right],$$

$$v_7(x,y,t) = -\frac{cka^2 + 2k^4(2-m^2)a^2 + 6bkal - 6b^2k^2 - 3l^2a^2}{a^3k^2} \pm \frac{2l}{a}\left[-m^2\operatorname{sn}(\xi)\operatorname{cd}(\xi)\right], \tag{28}$$

where $\xi = (c/\Gamma(1+\alpha))t^{\alpha} + (k/\Gamma(1+\beta))x^{\beta} + (l/\Gamma(1+\gamma))y^{\gamma} + \xi_0$.

Family 8. When $e_2 = 1$, $e_1 = 2 - m^2$, $e_0 = 1 - m^2$,

$$u_8(x,y,t) = -\frac{2(al-bk)}{a^2k} \pm \frac{2k}{a}\left[-\frac{\operatorname{dc}(\xi)}{\operatorname{sn}(\xi)}\right],$$

$$v_8(x,y,t) = -\frac{cka^2 + 2k^4(2-m^2)a^2 + 6bkal - 6b^2k^2 - 3l^2a^2}{a^3k^2} \pm \frac{2l}{a}\left[-\frac{\operatorname{dc}(\xi)}{\operatorname{sn}(\xi)}\right], \tag{29}$$

where $\xi = (c/\Gamma(1+\alpha))t^{\alpha} + (k/\Gamma(1+\beta))x^{\beta} + (l/\Gamma(1+\gamma))y^{\gamma} + \xi_0$.

Family 9. When $e_2 = m^2(m^2 - 1)$, $e_1 = 2m^2 - 1$, $e_0 = 1$,

$$u_9(x,y,t) = -\frac{2(al-bk)}{a^2k} \pm \frac{2k}{a}\frac{\operatorname{cs}(\xi)}{\operatorname{dn}(\xi)},$$

$$v_9(x,y,t) = -\frac{cka^2 + 2k^4(2m^2-1)a^2 + 6bkal - 6b^2k^2 - 3l^2a^2}{a^3k^2} \pm \frac{2l}{a}\frac{\operatorname{cs}(\xi)}{\operatorname{dn}(\xi)}, \tag{30}$$

where $\xi = (c/\Gamma(1+\alpha))t^{\alpha} + (k/\Gamma(1+\beta))x^{\beta} + (l/\Gamma(1+\gamma))y^{\gamma} + \xi_0$.

Family 10. When $e_2 = 1, e_1 = -(m^2+1), e_0 = m^2$,

$$u_{10}(x,y,t) = -\frac{2(al-bk)}{a^2k} \pm \frac{2k}{a}\left[\left(1-m^2\right)\frac{\mathrm{sd}(\xi)}{\mathrm{cn}(\xi)}\right],$$
$$v_{10}(x,y,t) = -\frac{cka^2 - 2k^4\left(1+m^2\right)a^2 + 6bkal - 6b^2k^2 - 3l^2a^2}{a^3k^2} \pm \frac{2l}{a}\left[\left(1-m^2\right)\frac{\mathrm{sd}(\xi)}{\mathrm{cn}(\xi)}\right], \quad (31)$$

where $\xi = (c/\Gamma(1+\alpha))t^{\alpha} + (k/\Gamma(1+\beta))x^{\beta} + (l/\Gamma(1+\gamma))y^{\gamma} + \xi_0$.

Remark 3. We note that the exact solutions established in (22)–(31) are new exact solutions to the space fractional coupled Konopelchenko-Dubrovsky (KD) equations.

3.2. Space-Time Fractional Fokas Equation.

Consider the space-time fractional Fokas equation [33, 34]

$$4\frac{\partial^{2\alpha}q}{\partial t^{\alpha}\partial x_1^{\alpha}} - \frac{\partial^{4\alpha}q}{\partial x_1^{3\alpha}\partial x_2^{\alpha}} + \frac{\partial^{4\alpha}q}{\partial x_2^{3\alpha}\partial x_1^{\alpha}} + 12\frac{\partial^{\alpha}q}{\partial x_1^{\alpha}}\frac{\partial^{\alpha}q}{\partial x_2^{\alpha}} + 12q\frac{\partial^{2\alpha}q}{\partial x_1^{\alpha}\partial x_2^{\alpha}} - 6\frac{\partial^{2\alpha}q}{\partial y_1^{\alpha}\partial y_2^{\alpha}} = 0, \quad 0 < \alpha \le 1. \quad (32)$$

In [33, 34], the authors solved (32) by use of the Riccati equation method and a fractional subequation method, respectively. Based on the two methods, some exact solutions for it were obtained. Now we will apply the Jacobi elliptic equation method described in Section 2 to solve (32).

Suppose $q(t,x_1,x_2,y_1,y_2) = U(\xi)$, where $\xi = (k_1x_1^{\alpha}/\Gamma(1+\alpha)) + (k_2x_2^{\alpha}/\Gamma(1+\alpha)) + (l_1y_1^{\alpha}/\Gamma(1+\alpha)) + (l_2y_2^{\alpha}/\Gamma(1+\alpha)) + (ct^{\alpha}/\Gamma(1+\alpha)) + \xi_0$, $k_1, k_2, l_1, l_2, c, \xi_0$ are all constants with $k_1, k_2, l_1, l_2, c \neq 0$. Then by use of (3) and the first equality in (5), (32) can be turned into the following form:

$$4ck_1U'' - k_1^3k_2U^{(4)} + k_2^3k_1U^{(4)} + 12k_1k_2U'^2 + 12k_1k_2UU'' - 6l_1l_2U'' = 0. \quad (33)$$

Suppose that the solution of (33) can be expressed by

$$U(\xi) = \sum_{i=0}^{n} a_i\left(\frac{G'}{G}\right)^i, \quad (34)$$

where $G = G(\xi)$ satisfies (12). By balancing the order between the highest order derivative term and nonlinear term in (33), we can obtain $n = 2$. So we have

$$U(\xi) = a_0 + a_1\left(\frac{G'}{G}\right) + a_2\left(\frac{G'}{G}\right)^2. \quad (35)$$

Substituting (35) into (33), using (12), collecting all the terms with the same power of $G^iG'^j$ together, and equating each coefficient to zero yield a set of algebraic equations. Solving these equations yields that

$$a_0 = -\frac{4k_1^{3\alpha}k_2^{\alpha}e_1 + 2c^{\alpha}k_1^{\alpha} - 3l_1^{\alpha}l_2^{\alpha} - 4k_2^{3\alpha}k_1^{\alpha}e_1}{6k_1^{\alpha}k_2^{\alpha}}, \quad a_1 = 0, \quad a_2 = k_1^{2\alpha} - k_2^{2\alpha}. \quad (36)$$

Substituting the result above into (35) and combining with (15) we can obtain the following exact solutions to (32).

Family 1. When $e_2 = -1, e_1 > 0, e_0 = 0$,

$$q_1(t,x_1,x_2,y_1,y_2) = -\frac{4k_1^{3\alpha}k_2^{\alpha}e_1 + 2c^{\alpha}k_1^{\alpha} - 3l_1^{\alpha}l_2^{\alpha} - 4k_2^{3\alpha}k_1^{\alpha}e_1}{6k_1^{\alpha}k_2^{\alpha}} + e_1\left(k_1^{2\alpha} - k_2^{2\alpha}\right)\tanh^2\left(\sqrt{e_1}\xi\right), \quad (37)$$

where $\xi = k_1x_1^{\alpha}/\Gamma(1+\alpha) + k_2x_2^{\alpha}/\Gamma(1+\alpha) + l_1y_1^{\alpha}/\Gamma(1+\alpha) + l_2y_2^{\alpha}/\Gamma(1+\alpha) + ct^{\alpha}/\Gamma(1+\alpha) + \xi_0$.

Family 2. When $e_2 = 1, e_1 > 0, e_0 = 0$,

$$q_2(t,x_1,x_2,y_1,y_2) = -\frac{4k_1^{3\alpha}k_2^{\alpha}e_1 + 2c^{\alpha}k_1^{\alpha} - 3l_1^{\alpha}l_2^{\alpha} - 4k_2^{3\alpha}k_1^{\alpha}e_1}{6k_1^{\alpha}k_2^{\alpha}} + e_1\left(k_1^{2\alpha} - k_2^{2\alpha}\right)\coth^2\left(\sqrt{e_1}\xi\right), \quad (38)$$

where $\xi = k_1x_1^{\alpha}/\Gamma(1+\alpha) + k_2x_2^{\alpha}/\Gamma(1+\alpha) + l_1y_1^{\alpha}/\Gamma(1+\alpha) + l_2y_2^{\alpha}/\Gamma(1+\alpha) + ct^{\alpha}/\Gamma(1+\alpha) + \xi_0$.

Family 3. When $e_2 = 1, e_1 < 0, e_0 = 0$,

$$q_3(t,x_1,x_2,y_1,y_2) = -\frac{4k_1^{3\alpha}k_2^{\alpha}e_1 + 2c^{\alpha}k_1^{\alpha} - 3l_1^{\alpha}l_2^{\alpha} - 4k_2^{3\alpha}k_1^{\alpha}e_1}{6k_1^{\alpha}k_2^{\alpha}} - e_1\left(k_1^{2\alpha} - k_2^{2\alpha}\right)\tan^2\left(\sqrt{-e_1}\xi\right), \quad (39)$$

where $\xi = k_1x_1^{\alpha}/\Gamma(1+\alpha) + k_2x_2^{\alpha}/\Gamma(1+\alpha) + l_1y_1^{\alpha}/\Gamma(1+\alpha) + l_2y_2^{\alpha}/\Gamma(1+\alpha) + ct^{\alpha}/\Gamma(1+\alpha) + \xi_0$.

Family 4. When $e_2 = 1, e_1 = 0, e_0 = 0$,

$$q_4(t,x_1,x_2,y_1,y_2) = -\frac{2c^{\alpha}k_1^{\alpha} - 3l_1^{\alpha}l_2^{\alpha}}{6k_1^{\alpha}k_2^{\alpha}} + \left(k_1^{2\alpha} - k_2^{2\alpha}\right)\frac{1}{\left(\xi + C_0\right)^2}, \quad (40)$$

where $\xi = k_1 x_1^{\alpha}/\Gamma(1+\alpha) + k_2 x_2^{\alpha}/\Gamma(1+\alpha) + l_1 y_1^{\alpha}/\Gamma(1+\alpha) + l_2 y_2^{\alpha}/\Gamma(1+\alpha) + ct^{\alpha}/\Gamma(1+\alpha) + \xi_0$.

Family 5. When $e_2 = m^2$, $e_1 = -(1+m^2)$, $e_0 = 0$,

$$\begin{aligned} q_5(t, x_1, x_2, y_1, y_2) &= -\frac{-4k_1^{3\alpha}k_2^{\alpha}\left(1+m^2\right) + 2c^{\alpha}k_1^{\alpha} - 3l_1^{\alpha}l_2^{\alpha} + 4k_2^{3\alpha}k_1^{\alpha}\left(1+m^2\right)}{6k_1^{\alpha}k_2^{\alpha}} \\ &\quad + \left(k_1^{2\alpha} - k_2^{2\alpha}\right)\left[\mathrm{cn}(\xi)\, ds(\xi)\right]^2, \end{aligned} \tag{41}$$

where $\xi = k_1 x_1^{\alpha}/\Gamma(1+\alpha) + k_2 x_2^{\alpha}/\Gamma(1+\alpha) + l_1 y_1^{\alpha}/\Gamma(1+\alpha) + l_2 y_2^{\alpha}/\Gamma(1+\alpha) + ct^{\alpha}/\Gamma(1+\alpha) + \xi_0$.

Family 6. When $e_2 = -m^2$, $e_1 = 2m^2 - 1$, $e_0 = 1 - m^2$,

$$\begin{aligned} q_6(t, x_1, x_2, y_1, y_2) &= -\frac{4k_1^{3\alpha}k_2^{\alpha}\left(2m^2-1\right) + 2c^{\alpha}k_1^{\alpha} - 3l_1^{\alpha}l_2^{\alpha} - 4k_2^{3\alpha}k_1^{\alpha}\left(2m^2-1\right)}{6k_1^{\alpha}k_2^{\alpha}} \\ &\quad + \left(k_1^{2\alpha} - k_2^{2\alpha}\right)\left[\mathrm{sn}(\xi)\,\mathrm{dc}(\xi)\right]^2, \end{aligned} \tag{42}$$

where $\xi = k_1 x_1^{\alpha}/\Gamma(1+\alpha) + k_2 x_2^{\alpha}/\Gamma(1+\alpha) + l_1 y_1^{\alpha}/\Gamma(1+\alpha) + l_2 y_2^{\alpha}/\Gamma(1+\alpha) + ct^{\alpha}/\Gamma(1+\alpha) + \xi_0$.

Family 7. When $e_2 = -1$, $e_1 = 2 - m^2$, $e_0 = m^2 - 1$,

$$\begin{aligned} q_7(t, x_1, x_2, y_1, y_2) &= -\frac{4k_1^{3\alpha}k_2^{\alpha}\left(2-m^2\right) + 2c^{\alpha}k_1^{\alpha} - 3l_1^{\alpha}l_2^{\alpha} - 4k_2^{3\alpha}k_1^{\alpha}\left(2-m^2\right)}{6k_1^{\alpha}k_2^{\alpha}} \\ &\quad + \left(k_1^{2\alpha} - k_2^{2\alpha}\right)\left[m^2\mathrm{sn}(\xi)\,\mathrm{cd}(\xi)\right]^2, \end{aligned} \tag{43}$$

where $\xi = k_1 x_1^{\alpha}/\Gamma(1+\alpha) + k_2 x_2^{\alpha}/\Gamma(1+\alpha) + l_1 y_1^{\alpha}/\Gamma(1+\alpha) + l_2 y_2^{\alpha}/\Gamma(1+\alpha) + ct^{\alpha}/\Gamma(1+\alpha) + \xi_0$.

Family 8. When $e_2 = 1$, $e_1 = 2 - m^2$, $e_0 = 1 - m^2$,

$$\begin{aligned} q_8(t, x_1, x_2, y_1, y_2) &= -\frac{4k_1^{3\alpha}k_2^{\alpha}\left(2-m^2\right) + 2c^{\alpha}k_1^{\alpha} - 3l_1^{\alpha}l_2^{\alpha} - 4k_2^{3\alpha}k_1^{\alpha}\left(2-m^2\right)}{6k_1^{\alpha}k_2^{\alpha}} \\ &\quad + \left(k_1^{2\alpha} - k_2^{2\alpha}\right)\left[\frac{\mathrm{dc}(\xi)}{\mathrm{sn}(\xi)}\right]^2, \end{aligned} \tag{44}$$

where $\xi = k_1 x_1^{\alpha}/\Gamma(1+\alpha) + k_2 x_2^{\alpha}/\Gamma(1+\alpha) + l_1 y_1^{\alpha}/\Gamma(1+\alpha) + l_2 y_2^{\alpha}/\Gamma(1+\alpha) + ct^{\alpha}/\Gamma(1+\alpha) + \xi_0$.

Family 9. When $e_2 = m^2(m^2 - 1)$, $e_1 = 2m^2 - 1$, $e_0 = 1$,

$$\begin{aligned} q_9(t, x_1, x_2, y_1, y_2) &= -\frac{4k_1^{3\alpha}k_2^{\alpha}\left(2m^2-1\right) + 2c^{\alpha}k_1^{\alpha} - 3l_1^{\alpha}l_2^{\alpha} - 4k_2^{3\alpha}k_1^{\alpha}\left(2m^2-1\right)}{6k_1^{\alpha}k_2^{\alpha}} \\ &\quad + \left(k_1^{2\alpha} - k_2^{2\alpha}\right)\left[\frac{\mathrm{cs}(\xi)}{\mathrm{dn}(\xi)}\right]^2, \end{aligned} \tag{45}$$

where $\xi = k_1 x_1^{\alpha}/\Gamma(1+\alpha) + k_2 x_2^{\alpha}/\Gamma(1+\alpha) + l_1 y_1^{\alpha}/\Gamma(1+\alpha) + l_2 y_2^{\alpha}/\Gamma(1+\alpha) + ct^{\alpha}/\Gamma(1+\alpha) + \xi_0$.

Family 10. When $e_2 = 1$, $e_1 = -(m^2 + 1)$, $e_0 = m^2$,

$$\begin{aligned} q_{10}(t, x_1, x_2, y_1, y_2) &= -\frac{-4k_1^{3\alpha}k_2^{\alpha}\left(1+m^2\right) + 2c^{\alpha}k_1^{\alpha} - 3l_1^{\alpha}l_2^{\alpha} + 4k_2^{3\alpha}k_1^{\alpha}\left(1+m^2\right)}{6k_1^{\alpha}k_2^{\alpha}} \\ &\quad + \left(k_1^{2\alpha} - k_2^{2\alpha}\right)\left[\left(1-m^2\right)\frac{\mathrm{sd}(\xi)}{\mathrm{cn}(\xi)}\right]^2, \end{aligned} \tag{46}$$

where $\xi = k_1 x_1^{\alpha}/\Gamma(1+\alpha) + k_2 x_2^{\alpha}/\Gamma(1+\alpha) + l_1 y_1^{\alpha}/\Gamma(1+\alpha) + l_2 y_2^{\alpha}/\Gamma(1+\alpha) + ct^{\alpha}/\Gamma(1+\alpha) + \xi_0$.

Remark 4. If we put $e_1 = -\sigma$, $C_0 = \omega$, then the solutions (37)–(40) reduce to the results established in [33, (22)]. On the other hand, as a different subequation was used here from that in [34], one can see that our results are essentially different from those in [34]. Furthermore, we note that the Jacobi elliptic function solutions denoted in (41)–(46) are new exact solution to the space-time fractional Fokas equation so far to the best of our knowledge.

4. Conclusions

Based on nonlinear fractional complex transformation, we have extended the Jacobi elliptic equation method to seek exact solutions for fractional partial differential equations in the sense of modified Riemann-Liouville derivative. By use of this method, the space fractional coupled Konopelchenko-Dubrovsky (KD) equations and the space-time fractional Fokas equation are solved successfully. With the aid of the mathematical software, a series of exact solutions including not only hyperbolic function solutions, trigonometric function solutions, and rational function solutions but also Jacobi elliptic function solutions for the two equations have been found. Being concise and powerful, this method can be applied to solve many other fractional partial differential equations.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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