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Research Article

Exponential Attractor for Coupled Ginzburg-Landau Equations Describing Bose-Einstein Condensates and Nonlinear Optical Waveguides and Cavities

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The existence of the exponential attractors for coupled Ginzburg-Landau equations describing Bose-Einstein condensates and nonlinear optical waveguides and cavities with periodic initial boundary is obtained by showing Lipschitz continuity and the squeezing property.

1. Introduction

Inertial set was introduced (see [1–5]) in order to overcome some of the theoretical difficulties that are associated with inertial manifolds. An inertial set, by definition, contains the global attractor and attracts all trajectories at a uniform exponential rate. Consequently, it contains the slow transients as well as the global attractor. In the theory of dynamical systems the slow transients correspond to slowly converging stable manifolds that are in some sense close to central manifolds. Numerical simulations of infinite dimensional dynamical systems often capture both slow transients and parts of the attractor. After a large but finite time the state of the system obtained from the numerical calculation may often be at a finite distance from the global attractor but at an infinitesimal distance to the inertial set. In this sense, we propose to call the inertial set an exponential attractor to be consistent with the physical intuition [5].

An exponential attractor is an exponentially attracting compact set with finite fractal dimension that is positively invariant under the forward semiflow. The notion of exponential attractors was introduced by Eden et al. [3] and has been shown to be one of the very important notions in the study of long time behavior of solutions to nonlinear diffusion equations [6]. The easiest way of obtaining an exponential attractor is by taking the intersection of an absorbing set with an inertial manifold.

In the area of hyperbolic evolutionary equations, the existence of exponential attractors has been proved for many equations. In this paper, we will prove the existence of exponential attractor for coupled Ginzburg-Landau equations

$$\begin{gathered} iu_t + \gamma_2 \Delta u + i\gamma u + \left(\sigma_1 + i\sigma_2 |u|^2\right) |u|^2 u + v = 0, \\ iv_t + \gamma_2 \Delta u + (i\Gamma - \chi) v + u = 0, \end{gathered} \tag{1}$$

with the periodic boundary conditions

$$\begin{gathered} u(x,t) = u(x+D,t), \qquad v(x,t) = v(x+D,t), \\ x \in R, \quad t > 0, \end{gathered} \tag{2}$$

and initial value

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in R. \tag{3}$$

Its physical realizations include systems from nonlinear optics and a double-cigar-shaped Bose-Einstein condensate with a negative scattering length. In particular, in the case of the optical systems, u and v are amplitudes of electromagnetic waves in two cores of the system, the evolutional variable t is either time or propagation distance in the dual-core optical fiber, and x is the transverse coordinate in the cavity or the reduced time in the application to the fibers [7].

This paper is organized as follows. In Section 2, we give a description of preliminaries with existence of exponential attractor and the properties of solutions and bounded absorbing sets of (1). In Section 3, the existence of the exponential attractor in V_2 type exponential attractor is proved. In Section 4, we give some conclusions for this paper.

2. Preliminaries

Let V_1, V_2 be two Hilbert spaces, and let V_2 be dense in V_1 and compactly imbedded into V_1. Let $S(t)_{t\geq 0}$ be a continuous map from V_1, V_2 into itself. We study

$$\frac{du}{dt} + Au + g(u) = f(x), \quad t > 0, \ x \in \Omega, \tag{4}$$

$$u(x,0) = u_0(x), \tag{5}$$

Dirichlet problem or periodic boundary problem, (6)

where Ω is a bounded open set in R^n, $\partial\Omega$ is smooth, and A is a positive self-adjoint operator with a compact inverse. Letting $\{w_i\}_{i=1}^{\infty}$ denote the complete set of eigenvectors of A, the corresponding eigenvalues are

$$0 \leq \lambda_1 < \lambda_1 \cdots \lambda_i < \cdots \longrightarrow +\infty. \tag{7}$$

We assume that the nonlinear semigroup $S(t)_{t\geq 0}$ defined in (4)–(6) possesses a compact attractor $\mathbf{B}$ of (V_2, V_1)-type; namely, there exists a compact set $\mathbf{B}$ in V_1, and $\mathbf{B}$ attracts all bounded subsets in V_2 and is invariant under the action of $S(t)_{t\geq 0}$.

Let C be a compact subset of V_2. $S(t)_{t\geq 0}$ leaves the set C invariant and set

$$\mathbf{B} = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} S(t)_{t\geq 0} C}, \tag{8}$$

that is, for $S(t)_{t\geq 0}$ on C, $\mathbf{B}$ is the global attractor.

Definition 1. A compact set M is called an exponential attractors for $S(t)_{t\geq 0}, C$ if

(i) $\mathbf{B} \subseteq M \subseteq C$;

(ii) $S(t)M \subseteq M$, for every $t \geq 0$;

(iii) M has finite fractal dimension $d_F < \infty$;

(iv) There exist constants c_1 and c_2 such that

$$\operatorname{dist}_{v_2}(S(t)C, M) \leq c_1 \exp(-c_2 t), \quad \forall t > 0, \tag{9}$$

where

$$\operatorname{dist}_{v_2}(A, D) = \sup_{x\in A} \inf_{y\in D} \|x - y\|_{V_2}. \tag{10}$$

Definition 2. If there exists a bounded function $l(t)$ independent u and v such that

$$\|S(t)u - S(t)v\|_{V_2} \leq l(t)\|u - v\|_{V_2}, \tag{11}$$

for every $u, v \in C$, then we say $S(t)$ is Lipschitz continuous in C and $l(t)$ is Lipschitz constant for $S(t)$ in C.

Definition 3. A continuous semigroup $S(t)_{t\geq 0}$ is said to satisfy the squeezing property on C if there exists $t_* > 0$ such that $S(t_*)$ satisfies the following.

For every $\delta \in (0, (1/8))$, there exists an orthogonal projection P_{N_0} of rank equal to N_0 such that for every u and v in C if

$$\left\|P_{N_0}(S(t_*)u - S(t_*)v)\right\|_{v_2} \leq \left\|Q_{N_0}(S(t_*)u - S(t_*)v)\right\|_{v_2} \tag{12}$$

holds, then we also have

$$\left\|S(t_*)u - S(t_*)v\right\|_{v_2} \leq \delta\|u - v\|_{v_2}, \tag{13}$$

where $Q_{N_0} = I - P_{N_0}$.

Theorem 4 (see [3]). *Suppose (4)–(6) satisfy the following conditions.*

(1) *There exist nonlinear semigroup $S(t)_{t\geq 0}$ and a compact attractor $\mathbf{B}$.*

(2) *There exists a compact set $\mathbf{C}$ in V_2 which is positively invariant for $S(t)_{t\geq 0}$.*

(3) *$S(t)_{t\geq 0}$ is Lipschitz continuous and is squeezing in C.*

Then (4)–(6) admit an exponential attractor M in V_2 for $S(t)_{t\geq 0}$ and

$$M = \bigcup_{0\leq t\leq t_*} S(t) M_*, \tag{14}$$

where

$$M_* = \mathbf{B} \bigcup \left(\bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} S(t_*)^j \left(E^{(k)}\right) \right). \tag{15}$$

Moreover,

$$\begin{gathered} d_F(M) \leq 1 + CN_0, \\ \operatorname{dist}_{V_2}(S(t)B, M) \leq C_0 \exp(-C_1 t), \end{gathered} \tag{16}$$

where N_0, $E(k)$ are defined as in [4], C, C_0, C_1 are the constants independent of B, and t_* is a positive constant.

Proposition 5. *There exists $t_0(B_0)$ such that*

$$B^* = \overline{\bigcup_{0\leq t\leq t_0} S(t) B_0} \tag{17}$$

is a compact positively invariant set in V_1 and is absorbing set for all bounded subsets in V_2, where B_0 is a closed absorbing set in V_2 for $S(t)_{t\geq 0}$.

Proposition 6. *Let B_0, B_1 be bounded and closed absorbing sets for (4)–(6) in (V_2, V_1), respectively. Then there exists a compact attractor A^* of (V_2, V_1)-type. For the proof of Proposition 5 and Proposition 6, we refer the reader to [5].*

Denoting by $|\cdot|_{L^p}$ the norm in $L^p(0, L)$, $1 \leq p \leq \infty$, for simplicity, we denote by $|\cdot|_0$ and $|\cdot|_\infty$ the norm in the case

$p = 2$ and $p = \infty$, respectively. Suppose that $H = L^2(0, L)$, $E_i = H^i(0, L) \times H^i(0, L)$ $(i = 1, 2)$, where $H^i(0, L)$ is a Hilbert space for the scalar product

$$((\cdot,\cdot))_{H^i} = (\cdot,\cdot) + \sum_{j=1}^{i} \left(D^j \cdot, D^j \cdot\right), \quad D = \frac{\partial}{\partial x}. \tag{18}$$

The norm of E_i is defined by $\|(u, v)\|_{E_i}^2 = \|u\|_{H^i}^2 + \|v\|_{H^i}^2$.

We now establish some time-uniform a priori estimates on (u, v) in E_1 and E_2, respectively.

Lemma 7. *Assume that $(u_0, v_0) \in E_1$; then*

$$\|(u, v)\|_{E_1}^2 \le c\|(u_0, v_0)\|_{E_1}^2 e^{-\delta_1 t} + c_1. \tag{19}$$

Thus there exists $t_1 = t_1(R) > 0$ such that

$$\|(u, v)\|_{E_1}^2 \le c_2, \qquad t \ge t_1, \tag{20}$$

whenever $\|(u_0, v_0)\|_{E_1} \le R$.

Lemma 8. *Assume that $(u_0, v_0) \in E_2$; then*

$$\|(u, v)\|_{E_2}^2 \le c\|(u_0, v_0)\|_{E_2}^2 e^{-\delta_2 t} + c_3. \tag{21}$$

Thus there exists $t_2 = t_2(R) > 0$ such that

$$\|(u, v)\|_{E_2}^2 \le c_4, \quad t \ge t_2, \tag{22}$$

whenever $\|(u_0, v_0)\|_{E_2} \le R$.

Theorem 9. *Assume that all the parameters of* (1) *are positive. For (u_0, v_0) given in E_i $(i = 1, 2)$, there exists a unique solution*

$$(u, v) \in L^\infty(R_+, E_i). \tag{23}$$

And also

$$(u, v) \in \mathscr{C}(R_+, E_1), \quad \forall (u_0, v_0) \in E_1. \tag{24}$$

Furthermore, the solution operator of the system is a continuous semigroup $S(t)$ on E_1 which possesses bounded absorbing sets $B_i \subset E_i$, for $i = 1, 2$.

Thus, we observe that Lemmas 7 and 8 show that there exists constant k depending only on the data that the balls

$$B_1 = \left\{(u, v) \in E_1, \|u\|_{H_1} + \|v\|_{H_1} \le k\right\},$$
$$B_2 = \left\{(u, v) \in E_2, \|u\|_{H_2} + \|v\|_{H_2} \le k\right\} \tag{25}$$

are bounded absorbing sets for $S(t)$ in E_1 and E_2, respectively:

Let

$$V_1 = E_1, \qquad V_2 = E_2, \qquad B = \overline{\bigcup_{t\ge0} S(t) B_2}, \tag{26}$$

then B is a compact invariant subset in V_2; we know that semigroup $S(t)$ defined by problem (31)–(34) possesses a V_2-type compact attractor. According to Theorem 4, we need only to show the Lipschitz continuity and the squeezing property of the dynamical system $S(t)$ in B, respectively. That is what we proceed to do in the following sections.

3. Exponential Attractor in V_2 for Problem (1)-(2)

In this section, we show the existence of the exponential attractor in V_2 for problem (1)-(2). In order to prove the Lipschitz continuity and the squeezing property, we need to extend Hölder inequality

$$\int_\Omega \left|u(x) u_2(x) \cdots u_k(x)\right| d_x \le \prod_{j=1}^{k} \left\|u_j\right\|_{L^{p_j}}, \tag{27}$$

where $\sum_{j=1}^{k}(1/p_j) = 1$, $p_j > 1$ and Gagliardo-Nirenberg (G-N) inequality

$$\left\|\nabla^j u\right\|_p \le c\left\|\nabla^m u\right\|_r^a \|u\|_q^{1-a}, \tag{28}$$

where

$$\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + \frac{1-a}{q},$$
$$1 \le q, \quad r \le \infty, \quad 0 \le j < m, \quad \frac{j}{m} \le a \le 1, \tag{29}$$

and the Young's inequality

$$ab \le \frac{\varepsilon}{p} a^p + \frac{1}{q} \varepsilon^{(-q/b)} b^q, \quad a, b, \varepsilon > 0, \ 1 < p,$$
$$q < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{30}$$

Theorem 10. *Assume $w_1(t) = (u_1(t), v_1(t))$, and $w_2(t) = (u_2(t), v_2(t))$ are two solutions of problem* (1)-(2) *with initial values $w_{10} = (u_{10}, v_{10})$, $w_{20} = (u_{20}, v_{20}) \in B = H^2 \times H^2$; then one has*

$$\left\|w_1(t) - w_2(t)\right\|_{V_2} \le \exp(2C_0 t) \left\|w_{10} - w_{20}\right\|_{V_2}. \tag{31}$$

Proof. Letting $h(t) = u_1(t) - u_2(t)$, $g(t) = v_1(t) - v_2(t)$, from (1)-(2), we have

$$ih_t + \gamma_2 \Delta h + i\gamma h + f(u_1, u_2) + g = 0, \tag{32}$$

$$ig_t + \gamma_2 \Delta g + (i\Gamma - \chi) g + h = 0, \tag{33}$$

with periodic initial value

$$h(x, t) = h(x + D, t), \qquad g(x, t) = g(x + D, t),$$
$$x \in R, \quad t > 0, \tag{34}$$

$$h(x, 0) = u_{10}(x) - u_{20}(x), \qquad g(x, 0) = v_{10}(x) - v_{20}(x),$$
$$x \in R, \tag{35}$$

where

$$f(u_1, u_2) = \sigma_1 \left(|u_1|^2 u_1 - |u_2|^2 u_2\right) + i\sigma_2 \left(|u_1|^4 u_1 - |u_2|^4 u_2\right). \tag{36}$$

Taking $\phi_1(u) = |u|^2$ and $\phi_2(u) = |u|^4$, then we get

$$\phi_1'(\xi)\, h = |u_1|^2 - |u_2|^2, \tag{37}$$

$$\phi_2'(\eta)\, h = |u_1|^4 - |u_2|^4. \tag{38}$$

Substituting (37) and (38) into (36), we get

$$\begin{aligned} f(u_1, u_2) &= \sigma_1 \left(|u_1|^2 u_1 - |u_1|^2 u_2 + |u_1|^2 u_2 - |u_2|^2 u_2\right) \\ &\quad + i\sigma_2 \left(|u_1|^4 u_1 - |u_1|^4 u_1 + |u_1|^4 u_2 - |u_2|^4 u_2\right) \\ &= \sigma_1 h \left(\phi_1(u_1) + u_2 \phi_1'(\xi)\right) \\ &\quad + i\sigma_2 h \left(\phi_2(u_1) + u_2 \phi_2'(\eta)\right). \end{aligned} \tag{39}$$

Substituting (39) into (32), we obtain

$$\begin{aligned} &i h_t + \gamma_2 \Delta h + i\gamma h + \sigma_1 h \left(\phi_1(u_1) + u_2 \phi_1'(\xi)\right) \\ &\quad + i\sigma_2 h \left(\phi_2(u_1) + u_2 \phi_2'(\eta)\right) + g = 0, \end{aligned} \tag{40}$$

$$i g_t + \gamma_2 \Delta g + (i\Gamma - \chi) g + h = 0. \tag{41}$$

To prove the Theorem 4, we take the following four steps.

Step 1. Taking the inner product of (40) with $\overline{h}$ and (41) with $\overline{g}$, respectively, we have

$$\begin{aligned} &\left(i h_t, \overline{h}\right) + \left(\gamma_2 \Delta h, \overline{h}\right) + \left(i\gamma h, \overline{h}\right) \\ &\quad + \left(\sigma_1 h \left(\phi_1(u_1) + u_2 \phi_1'(\xi)\right), \overline{h}\right) \\ &\quad + \left(i\sigma_2 h \left(\phi_2(u_1) + u_2 \phi_2'(\eta)\right), \overline{h}\right) + \left(g, \overline{h}\right) = 0, \end{aligned} \tag{42}$$

$$(i g_t, \overline{g}) + (\gamma_2 \Delta g, \overline{g}) + ((i\Gamma - \chi) g, \overline{g}) + (h, \overline{g}) = 0, \tag{43}$$

using

$$\begin{aligned} \frac{d}{dt} \int_\Omega |u|^2 dx &= \frac{d}{dt} \int_\Omega u \overline{u}\, dx = \int_\Omega \left(u_t \overline{u} + u \overline{u_t}\right) dx \\ &= 2 \operatorname{Re} \int_\Omega u_t \overline{u}\, dx. \end{aligned} \tag{44}$$

Thus,

$$\begin{aligned} &\operatorname{Im} \left(i \int_\Omega u_t \overline{u}\, dx\right) = \frac{1}{2} \frac{d}{dt} \int_\Omega |u|^2 dx, \\ &\left(\gamma_2 \Delta h, \overline{h}\right) = -\gamma_2 \|h_x\|^2, \qquad \left(i\gamma h, \overline{h}\right) = i\gamma \|h\|^2, \end{aligned} \tag{45}$$

then taking the imaginary part of (42) and (43), respectively,

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|h\|^2 + \gamma \|h\|^2 + \sigma_1 \operatorname{Im} \int_\Omega u_2 \phi_1'(\xi) |h|^2 dx \\ &\quad + \sigma_2 \operatorname{Im} \int_\Omega \phi_2(u_1) |h|^2 dx \\ &\quad + \sigma_2 \operatorname{Re} \int_\Omega \phi_2'(\eta) |h|^2 dx + \operatorname{Im} \int_\Omega g \overline{h}\, dx = 0, \end{aligned} \tag{46}$$

$$\frac{1}{2} \frac{d}{dt} \|g\|^2 + \Gamma \|g\|^2 + \operatorname{Im} \int_\Omega h \overline{g}\, dx = 0, \tag{47}$$

by using the extend Hölder inequality, we can obtain

$$\begin{aligned} &\left|\operatorname{Im} \int_\Omega g \overline{h}\, dx\right| \le \frac{1}{2} \left(\|g\|^2 + \|h\|^2\right), \\ &\left|\sigma_1 \operatorname{Im} \int_\Omega u_2 \phi_1'(\xi) |h|^2 dx\right| \le |\sigma_1| \int_\Omega |u_2|\, \phi_1'(\xi) |h|^2 dx \\ &\qquad \le |\sigma_1|\, \|h\|^2 \|u_2\|_\infty \left\|\phi_1'(\xi)\right\|_\infty \\ &\qquad \le C \|h\|^2, \\ &\left|\sigma_2 \operatorname{Re} \int_\Omega u_2 \phi_2'(\eta) |h|^2 dx\right| \le |\sigma_2| \int_\Omega |u_2|\, \phi_2'(\xi) |h|^2 dx \\ &\qquad \le |\sigma_2|\, \|h\|^2 \|u_2\|_\infty \left\|\phi_2'(\eta)\right\|_\infty \\ &\qquad \le C \|h\|^2. \end{aligned} \tag{48}$$

Combining (46) and (47), then we infer that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left(\|h\|^2 + \|g\|^2\right) + \gamma \|h\|^2 + \gamma \|g\|^2 \\ &\quad + \sigma_2 \int_\Omega \phi_2(u_1) |h|^2 dx \le C \|h\|^2 + \|g\|^2. \end{aligned} \tag{49}$$

Step 2. Taking the inner product of (40) with $-\overline{h_{xx}}$ and (41) with $-\overline{g_{xx}}$, respectively, we have

$$\begin{aligned} &\left(i h_t, \overline{h_{xx}}\right) + \left(\gamma_2 \Delta h, -\overline{h_{xx}}\right) + \left(i\gamma h, -\overline{h_{xx}}\right) \\ &\quad + \left(\sigma_1 h \left(\phi_1(u_1) + u_2 \phi_1'(\xi)\right), -\overline{h_{xx}}\right) \\ &\quad + \left(i\sigma_2 h \left(\phi_2(u_1) + u_2 \phi_2'(\eta)\right), -\overline{h_{xx}}\right) \\ &\quad + \left(g, -\overline{h_{xx}}\right) = 0, \end{aligned} \tag{50}$$

$$\begin{aligned} &\left(i g_t, -\overline{g_{xx}}\right) + \left(\gamma_2 \Delta g, -\overline{g_{xx}}\right) + \left((i\Gamma - \chi) g, -\overline{g_{xx}}\right) \\ &\quad + \left(h, -\overline{g_{xx}}\right) = 0. \end{aligned} \tag{51}$$

Note that

$$\begin{aligned} &\left(i h_t, -\overline{h_{xx}}\right) = i \int_\Omega h_{xt} \overline{h_x} dx, \\ &\left(g, -\overline{h_{xx}}\right) = \int_\Omega g_x \overline{h_x} dx, \\ &\left(\gamma_2 \Delta h, -\overline{h_{xx}}\right) = \|\Delta h\|^2, \\ &\left(i\gamma h, -\overline{h_{xx}}\right) = i\gamma \|h_x\|^2, \end{aligned}$$

$$\begin{aligned}&\left(\sigma_1 h\left(\phi_1(u_1)+u_2\phi_1'(\xi)\right),-\overline{h_{xx}}\right)\\&=\sigma_1\int_\Omega\left[|h_x|^2\left(\phi_1(u_1)+u_2\phi_1'(\xi)\right)\right.\\&\qquad\left.+h\overline{h_x}\left(\phi_1(u_1)+u_2\phi_1'(\xi)\right)_x\right]dx,\\&\left(i\sigma_2 h\left(\phi_2(u_1)+u_2\phi_2'(\eta)\right),-\overline{h_{xx}}\right)\\&=i\sigma_2\int_\Omega\left[|h_x|^2\left(\phi_2(u_1)+u_2\phi_2'(\eta)\right)\right.\\&\qquad\left.+h\overline{h_x}\left(\phi_2(u_1)+u_2\phi_2'(\eta)\right)_x\right]dx,\end{aligned}\tag{52}$$

then taking the imaginary part of (50) and (51), respectively,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|h_x\|^2+\gamma\|h_x\|^2\\&+\sigma_1\operatorname{Im}\int_\Omega\left(u_2\phi_1'(\xi)|h_x|^2\right.\\&\qquad\left.+h\overline{h_x}\left(\phi_1(u_1)+u_2\phi_1'(\xi)\right)_x\right)dx\\&+\sigma_2\operatorname{Im}\int_\Omega\phi_2(u_1)|h_x|^2dx\\&+\sigma_2\operatorname{Re}\int_\Omega\left(u_2\phi_2'(\eta)|h_x|^2\right.\\&\qquad\left.+h\overline{h_x}\left(\phi_2(u_1)+u_2\phi_2'(\eta)\right)_x\right)dx\\&+\operatorname{Im}\int_\Omega g_x\overline{h_x}dx=0,\end{aligned}\tag{53}$$

$$\frac{1}{2}\frac{d}{dt}\|g\|^2+\Gamma\|g\|^2+\operatorname{Im}\int_\Omega h\overline{g}\,dx=0.\tag{54}$$

Note the following inequalities:

$$\begin{aligned}&\left|\sigma_1\operatorname{Im}\int_\Omega\left(u_2\phi_1'(\xi)|h_x|^2+h\overline{h_x}\left(\phi_1(u_1)+u_2\phi_1'(\xi)\right)_x\right)dx\right|\\&\le|\sigma_1|\operatorname{Im}\int_\Omega\left(|u_2|\left|\phi_1'(\xi)\right||h_x|^2\right.\\&\qquad+|h|\left|\overline{h_x}\right|\left(\left|\phi_1(u_1)_x\right|+|u_{2x}|\left|\phi_1'(\xi)\right|\right.\\&\qquad\left.\left.+|u_2|\left|\phi_1'(\xi)_x\right|\right)\right)dx\\&\le C\|h_x\|^2+|\sigma_1|\,\|h\|\,\left\|\overline{h_x}\right\|\left(\left\|\phi_1(u_1)_x\right\|_\infty+\|u_{2x}\|_\infty\left\|\phi_1'(\xi)\right\|_\infty\right.\\&\qquad\left.+\|u_2\|_\infty\left\|\phi_1'(\xi)_x\right\|_\infty\right),\\&\le C\|h_x\|^2+c\|h\|^2,\\&\sigma_2\operatorname{Re}\int_\Omega\left(u_2\phi_2'(\eta)|h_x|^2+h\overline{h_x}\left(\phi_2(u_1)+u_2\phi_2'(\eta)\right)_x\right)dx\\&\le C\|h_x\|^2+c\|h\|^2,\end{aligned}\tag{55}$$

Combining (53) and (54), one can obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|h_x\|^2+\|g_x\|^2\right)+\gamma\|h_x\|^2+\gamma\|g_x\|^2\\&+\sigma_2\int_\Omega\phi_2(u_1)|h_x|^2dx\le C\|h_x\|^2+\|g_x\|^2+c\|h\|^2.\end{aligned}\tag{56}$$

Step 3. Taking the inner product of (40) with $\overline{h_{xxxx}}$ and (41) with $\overline{g_{xxxx}}$, respectively, we have

$$\begin{aligned}&\left(ih_t,\overline{h_{xxxx}}\right)+\left(\gamma_2\Delta h,\overline{h_{xxxx}}\right)+\left(i\gamma h,\overline{h_{xxxx}}\right)\\&\quad+\left(\sigma_1 h\left(\phi_1(u_1)+u_2\phi_1'(\xi)\right),\overline{h_{xxxx}}\right)\\&\quad+\left(i\sigma_2 h\left(\phi_2(u_1)+u_2\phi_2'(\eta)\right),\overline{h_{xxxx}}\right)\\&\quad+\left(g,\overline{h_{xxxx}}\right)=0,\\&\left(ig_t,\overline{g_{xxxx}}\right)+\left(\gamma_2\Delta g,\overline{g_{xxxx}}\right)+\left((i\Gamma-\chi)g,\overline{g_{xxxx}}\right)\\&\quad+\left(h,\overline{g_{xxxx}}\right)=0,\end{aligned}\tag{57}$$

using

$$\begin{aligned}&\left(ih_t,\overline{h_{xxxx}}\right)=i\int_\Omega h_{xxt}\overline{h_{xx}}dx,\\&\left(g,\overline{h_{xxxx}}\right)=\int_\Omega g_{xx}\overline{h_{xx}}dx,\\&\left(\gamma_2\Delta h,\overline{h_{xxxx}}\right)=\|h_{xxx}\|^2,\\&\left(i\gamma h,\overline{h_{xxxx}}\right)=i\gamma\|h_{xx}\|^2,\\&\left(\sigma_1 h\left(\phi_1(u_1)+u_2\phi_1'(\xi)\right),\overline{h_{xxxx}}\right)\\&\quad=\sigma_1\left(\left(h\left(\phi_1(u_1)+u_2\phi_1'(\xi)\right)\right)_{xx},\overline{h_{xx}}\right)\\&\quad=\sigma_1\left(h_{xx}\phi_1(u_1)+\psi_1,\overline{h_{xx}}\right),\\&\left(i\sigma_2 h\left(\phi_2(u_1)+u_2\phi_2'(\eta)\right),\overline{h_{xxxx}}\right)\\&\quad=i\sigma_2\left(\left(h\left(\phi_2(u_1)+u_2\phi_2'(\eta)\right)\right)_{xx},\overline{h_{xx}}\right)\\&\quad=i\sigma_2\left(h_{xx}\phi_2(u_1)+\psi_2,\overline{h_{xx}}\right),\end{aligned}\tag{58}$$

where

$$\begin{aligned}\psi_1&=h_{xx}u_2\phi_1'(\xi)\\&\quad+2h_x\left(\phi_1(u_1)_x+u_{2x}\phi_1'(\xi)+u_2\phi_1'(\xi)_x\right)\\&\quad+h\left(\phi_1(u_1)_{xx}+u_{2xx}\phi_1'(\xi)+2u_{2x}\phi_1'(\xi)_x\right.\\&\quad\left.+u_2\phi_1'(\xi)_{xx}\right),\end{aligned}$$

$$\psi_2 = h_{xx}u_2\phi_2'(\eta) + 2h_x\left(\phi_2(u_1)_x + u_{2x}\phi_2'(\eta) + u_2\phi_2'(\eta)_x\right) + h\left(\phi_2(u_1)_{xx} + u_{2xx}\phi_2'(\eta) + 2u_{2x}\phi_2'(\eta)_x + u_2\phi_2'(\eta)_{xx}\right), \tag{59}$$

then taking the imaginary part of (50) and (51), respectively,

$$\frac{1}{2}\frac{d}{dt}\|h_{xx}\|^2 + \gamma\|h_{xx}\|^2 + \sigma_1 \operatorname{Im}\left(\psi_1, \overline{h_{xx}}\right) + \sigma_2\left(h_{xx}\phi_2(u_1), \overline{h_{xx}}\right) + \sigma_2 \operatorname{Re}\left(\psi_2, \overline{h_{xx}}\right) + \operatorname{Im}\int_\Omega g_{xx}\overline{h_{xx}}dx = 0, \tag{60}$$

$$\frac{1}{2}\frac{d}{dt}\|g_{xx}\|^2 + \Gamma\|g_{xx}\|^2 + \operatorname{Im}\int_\Omega h_{xx}\overline{g_{xx}}dx = 0. \tag{61}$$

Note the following inequalities:

$$\left|\operatorname{Im}\left(\psi_1, \overline{h_{xx}}\right)\right| \le C\|h\|_{H^2}^2, \qquad \left|\sigma_2 \operatorname{Re}\left(\psi_2, \overline{h_{xx}}\right)\right| \le C\|h\|_{H^2}^2. \tag{62}$$

Combining (60) and (61), one can obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|h_{xx}\|^2 + \|g_{xx}\|^2\right) + \gamma\|h_{xx}\|^2 + \Gamma\|g_{xx}\|^2 + \sigma_2\int_\Omega \phi_2(u_1)|h_{xx}|^2dx \le C\|h\|_{H^2}^2 + \|g_{xx}\|^2. \tag{63}$$

Step 4. Combining (49), (56) and (63), we get

$$\frac{1}{2}\frac{d}{dt}\left(\|h\|_{H^2}^2 + \|g\|_{H^2}^2\right) + \gamma\|h\|_{H^2}^2 + \gamma\|g\|_{H^2}^2 + \sigma_2\int_\Omega \phi_2(u_1)\left(|h|^2 + |h_x|^2 + |h_{xx}|^2\right)dx \le C\left(\|h\|^2 + \|h_x\|^2 + \|h\|_{H^2}^2\right) + \|g\|^2 + \|g_x\|^2 + \|g_{xx}\|^2. \tag{64}$$

Taking $\mu = \min(\Gamma, \gamma), C_0 = \max(C, 1)$ and noting that

$$\sigma_2\int_\Omega\left(|h|^2 + |h_x|^2 + |h_{xx}|^2\right)dx \ge 0, \tag{65}$$

so (64) can be reduced to

$$\frac{1}{2}\frac{d}{dt}\left(\|h\|_{H^2}^2 + \|g\|_{H^2}^2\right) + \mu\left(\|h\|_{H^2}^2 + \|g\|_{H^2}^2\right) \le C_0\left(\|h\|_{H^2}^2 + \|g\|_{H^2}^2\right). \tag{66}$$

By Gronwall's inequality

$$\|h\|_{H^2}^2 + \|g\|_{H^2}^2 \le \exp(2C_0t)\left(\|h(0)\|_{H^2}^2 + \|g(0)\|_{H^2}^2\right), \tag{67}$$

that is,

$$\|w_1(t) - w_2(t)\|_{V_2} \le \exp(2C_0t)\|w_{10} - w_{20}\|_{V_2}. \tag{68}$$

Meanwhile, it indicates that the Lipschitz constant $l(t) \le \exp(2C_0t)$. This completes the proof.

Now, we intend to show the squeezing property for semigroup $S(t)$. To this end, we introduce the operator $A = -(\partial/\partial x^2)$ from $D(A)$ to H with domain

$$D(A) = \left\{u \in H^2(\Omega)\right\}. \tag{69}$$

Obviously, A is an unbounded self-adjoint positive operator and the inverse A^{-1} is compact. Thus, there exists an orthonormal basis $\{w_i\}_{i=1}^{\infty}$ $i = 1$ of H consisting of eigenvectors of A such that

$$Aw_i = \lambda_i w_i, \qquad 0 \le \lambda_1 < \lambda_1 \cdots \lambda_i < \cdots \longrightarrow +\infty, \quad \text{when } i \longrightarrow \infty. \tag{70}$$

For all N denote by $P = P_n : H \to \operatorname{span}\{w_1, w_2, \ldots, w_n\}$ the projector $Q = Q_N = I - P_N$. In the following, we will use

$$\left\|A^{(1/2)}u\right\| = \left\|\frac{\partial u}{\partial x}\right\|, \quad \left\|A^{(1/2)}u\right\| \ge \lambda_{N+1}^{(1/2)}, \quad u \in Q_NH, \quad \|Q_Nu\| \le \|u\|, \quad u \in H, \quad \|AQ_Nu\| = \|Q_NAu\| \le \|Au\|, \quad u \in D(A). \tag{71}$$

Decompose h, g as

$$h = Ph + Qh, \qquad g = Pg + Qg. \tag{72}$$

Applying Q to (32) and (33) we find that

$$iQh_t + \gamma_2\Delta Qh + i\gamma h + Qf(u_1, u_2) + Qg = 0, \tag{73}$$

$$iQg_t + \gamma_2\Delta Qg + (i\Gamma - \chi)Qg + Qh = 0. \tag{74}$$

Take the inner product of (73) with $\overline{Qh}$ and (74) with $\overline{Qg}$, respectively. Then like Step 1, we can get

$$\frac{1}{2}\frac{d}{dt}\left(\|Qh\|^2 + \|Qg\|^2\right) + \gamma\|Qh\|^2 + \Gamma\|Qg\|^2 + \sigma_2\int_\Omega Q\phi_2(u_1)|Qh|^2dx \le C\|Qh\|^2 + \|Qg\|^2. \tag{75}$$

Take the inner product of (73) with $-\overline{Qh_{xx}}$ and (74) with $-\overline{Qg_{xx}}$, respectively. Then like Step 2, we can get

$$\frac{1}{2}\frac{d}{dt}\left(\|Qh_x\|^2 + \|Qg_x\|^2\right) + \gamma\|Qh_x\|^2 + \Gamma\|Qg_x\|^2 + \sigma_2\int_\Omega Q\phi_2(u_1)|Qh_x|^2dx \le C\|Qh_x\|^2 + \|Qg_x\|^2 + c\|Qh\|^2. \tag{76}$$

Take the inner product of (73) with $\overline{Qh_{xxxx}}$ and (74) with $\overline{Qg_{xxxx}}$, respectively. Then like Step 3, we can get

$$\frac{1}{2}\frac{d}{dt}\left(\|Qh_{xx}\|^2+\|Qg_{xx}\|^2\right)+\gamma\|Qh_{xx}\|^2+\Gamma\|Qg_{xx}\|^2 + \sigma_2\int_\Omega Q\phi_2(u_1)|Qh_{xx}|^2dx \le C\|Qh\|_{H^2}^2+\|Qg_{xx}\|^2. \quad (77)$$

Combining (75), (76), and (77), we get

$$\frac{1}{2}\frac{d}{dt}\left(\|Qh\|_{H^2}^2+\|Qg\|_{H^2}^2\right)+\mu\left(\|Qh\|_{H^2}^2+\|Qg\|_{H^2}^2\right) \le C_0\left(\|Qh\|_{H^2}^2+\|Qg\|_{H^2}^2\right). \quad (78)$$

Using the G-N inequality

$$\|u_x\|^2\le\|u\|\,\|u_{xx}\|\le\frac{1}{2}\left(\|u\|^2+\|u_{xx}\|^2\right), \quad (79)$$

from (78), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\left(\|Qh\|_{H^2}^2+\|Qg\|_{H^2}^2\right)+\mu\left(\|Qh\|_{H^2}^2+\|Qg\|_{H^2}^2\right)\\&\le\frac{3C_0}{2}\left(\|Qh\|+\|Qh_{xx}\|+\|Qg\|+\|Qg_{xx}\|\right)\\&\le\frac{3C_0}{2}\lambda_{N+1}^{-1}\left(\|Qh_{xx}\|+\|Qg_{xx}\|\right)\\&\le\frac{3C_0}{2}\lambda_{N+1}^{-1}\left(\|Qh\|_{H^2}^2+\|Qg\|_{H^2}^2\right)\\&\le\frac{3C_0}{2}\lambda_{N+1}^{-1}\exp(2C_0t)\left(\|h(0)\|_{H^2}^2+\|g(0)\|_{H^2}^2\right).\end{aligned} \quad (80)$$

By Gronwall lemma we get

$$\begin{aligned}\|Qh\|_{H^2}^2&+\|Qg\|_{H^2}^2\\&\le\left(\|h(0)\|_{H^2}^2+\|g(0)\|_{H^2}^2\right)\exp(-2\mu t)\\&\quad+\overline{C}\lambda_{N+1}^{-1}\exp(2C_0t)\left(\|h(0)\|_{H^2}^2+\|g(0)\|_{H^2}^2\right)\\&\le\left[\exp(-2\mu t)+\overline{C}\lambda_{N+1}^{-1}\exp(2C_0t)\right]\\&\quad\times\left(\|h(0)\|_{H^2}^2+\|g(0)\|_{H^2}^2\right).\end{aligned} \quad (81)$$

Letting $t_*>0$ be fixed we take $w(t)=w_1(t)-w_2(t)=(h(t),g(t))$ and assume that

$$\exp(-2\mu t_*)\le\frac{1}{256}. \quad (82)$$

Then we choose N large enough so that

$$\overline{C}\lambda_{N+1}^{-1}\exp(2C_0t)\le\frac{1}{256}, \quad (83)$$

that is,

$$\lambda_{N+1}\ge256\overline{C}\exp(2C_0t). \quad (84)$$

From (82) and (84), we obtain

$$\|Qh\|_{H^2}^2+\|Qg\|_{H^2}^2\le\frac{1}{128}\left(\|h(0)\|_{H^2}^2+\|g(0)\|_{H^2}^2\right). \quad (85)$$

This shows that when $t_*>0$ is fixed, Lipschitz constant for $S(t)$ in B is equal to $\exp(2C_0t_*)$ and N satisfies

$$\lambda_{N+1}\ge256\overline{C}\exp(2C_0t_*). \quad (86)$$

We have

$$\|Qw\|_{V_2}\le\|Qw(0)\|_{V_2}. \quad (87)$$

So when

$$\begin{aligned}&\|Qw(t_*)\|_{V_2}>\|Pw(t_*)\|_{V_2},\\ \|w(t_*)\|_{V_2}&=\|Qw(t_*)\|_{V_2}+\|Pw(t_*)\|_{V_2}\\&<2\|Qw(t_*)\|_{V_2}\le\frac{1}{64}\|Qw(0)\|_{V_2}\\&\le\frac{1}{64}\|w(0)\|_{V_2}.\end{aligned} \quad (88)$$

This completes the proof of Theorem 4. □

Theorem 11. *The semigroup $S(t)$ associated with problem* (1)-(2) *is squeezing in B. Now we conclude this section by giving our main result.*

Theorem 12. *Suppose that problem* (1)-(2) *satisfies Theorem 9; there exist $t_*\ge(1/2\mu)\ln(256)$ and N large enough such that*

$$\lambda_{N+1}\ge256\overline{C}\exp(2C_0t_*). \quad (89)$$

Then for the nonlinear semigroup $S(t)$ defined in (4) *and* (5), *$S(t)_{t\le0}$; B admits an exponential attractor M in V_2 and*

$$\begin{aligned}d_F(M)&\le1+CN_0,\\ \operatorname{dist}_{V_2}(S(t)B,M)&\le C_0\exp(-C_1t),\end{aligned} \quad (90)$$

where C_0, C_1, C are constants independent of the solution of the equation.

4. Conclusions

In this paper, we have studied the coupled Ginzburg-Landau equations which describe Bose-Einstein condensates and nonlinear optical waveguides and cavities with periodic initial boundary; the existence of the exponential attractors is obtained by showing Lipschitz continuity and the squeezing property. For exponential attractor, N is only large enough such that

$$\lambda_{N+1}\ge256\overline{C}\exp(2C_0t_*). \quad (91)$$

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