Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2013, Article ID 173407, 10 pages
http://dx.doi.org/10.1155/2013/173407

Research Article

The M/M/N Repairable Queueing System with Variable Breakdown Rates

Shengli Lv[1] and Jingbo Li[2]

[1] *School of Science, Yanshan University, Qinhuangdao 066004, China*
[2] *College of Mathematics and Information, Hebei Normal University of Science and Technology, Qinhuangdao 066004, China*

Correspondence should be addressed to Shengli Lv; qhdddlsl@163.com

Received 6 October 2012; Accepted 31 December 2012

Academic Editor: Xiaochen Sun

This paper considers the M/M/N repairable queuing system. The customers' arrival is a Poisson process. The servers are subject to breakdown according to Poisson processes with different rates in idle time and busy time, respectively. The breakdown servers are repaired by repairmen, and the repair time is an exponential distribution. Using probability generating function and transform method, we obtain the steady-state probabilities of the system states, the steady-state availability of the servers, and the mean queueing length of the model.

1. Introduction

In queuing researches, many researchers have studied the queuing system with repairable servers. Most of the works of the repairable queuing system deal with the single-server models [1–9]. The works about multiserver repairable systems is not sufficient. Mitrany and Avi-Itzhak [10] analyzed the model with N units of servers and the same amount of repairmen, they obtained the steady-state mean queuing length of customers. Neuts and Lucantoni [11] studied the model with N units of repairable servers and c ($0 \le c \le N$) repairmen by matrix analysis method and obtained the steady-state properties of the model.

In recent years, many flexible policies have been introduced to the repairable systems. Gray et al. [5] studied the model with a single server which may take a vacation in idle times and may breakdown in busy times; they obtained the mean queue length. Altman and Yechiali [12] presented a comprehensive analysis of the M/M/1 and M/G/1 queues, as well as of the M/M/c queue with server vacations; they obtained various closed-form results for the probability generating function (PGF) of the number of the customers. Zhang and Hou [6] analyzed an M/G/1 queue with working vacations and vacation interruptions; they obtained the queue length distribution and steady-state service status probability. Yang et al. [7] analyzed an M/G/1 queuing system with second optional service, server breakdowns, and general startup times under (N, p)-policy, they obtained the explicit closed-form expression of the joint optimum threshold values of (N, p) at the minimum cost. Chang et al. [8] studied the optimal management problem of a finite capacity M/H2/1 queuing system, where the unreliable server operates F-policy, a cost model is developed to determine the optimal capacity K, the optimal threshold F, the optimal setup rate, and the optimal repair rate at a minimum cost. Wang [9] used a quasi-birth-and-death (QBD) modeling approach to model queuing-inventory systems with a single removable server, performance measures are obtained by using both hybrid and standard procedures; an optimal control policy is proposed and verified through numerical studies.

The most works of repairable queuing system assumed that the server breakdown rate is constant, but the breakdown rate of a server may be variable in a real system. It is well known that many kinds of machine are easy to breakdown at their busy times, and some equipments may be easy to fail after a long idle period. For example, the tires of the truck prefer to breakdown when the truck is running on the road. On the other hand, the storage battery in an automobile may not work if the automobile is idle for long period. For the actual demands of the above cases, we study a multiserver repairable queuing system in this paper, and assume that the unreliable servers have different breakdown rates in their busy times and idle times, respectively.

The rest of this paper is organized as follows. Section 2 describes the model and gives the balance equations. Section 3 presents the equations of PGF. The steady-state availability of the system is derived in Section 4. The steady-state probabilities of the system states and mean queuing length are obtained in Section 5. Case analysis is given in Section 6. Section 7 presents the conclusions.

2. Model Description

The model characteristics are as follows.

(1) There are N units of identical servers in the system. The servers are subject to breakdown according to Poisson processes with different rates which are ξ_1 in idle times and ξ_2 in busy times, respectively.

(2) Customers arrive according to a Poisson process with rate λ. The service discipline is first come first served (FCFS). The service which is interrupted by a server breakdown will become the first one of the queue of customers. The service time distribution is an exponential distribution with parameter μ.

(3) There are c $(1 \leq c \leq N)$ reliable repairmen to maintain the unreliable servers. The repair discipline is first come first repaired (FCFR). The repair time distribution is an exponential distributions with parameter η. A server is as good as a new one after repair.

We define

$X(t) \equiv$ the number of available servers in the system at the moment t $(0 \leqslant X(t) \leqslant N)$,

$Y(t) \equiv$ the number of customers in the system at the moment t $(0 \leqslant Y(t))$.

The stochastic process $\{X(t), Y(t), t \geq 0\}$ is a two-dimensional Markov process which is called quasi-birth-and-death (QBD) process [11] with state space $\{(i,j),\ 0 \leq i \leq N,\ j \geq 0\}$.

Let $P_{i,j}(t)$ denote the probability that the system is in a state of (i,j) at the moment t, and $P_{i,j}$ denote the steady-state probability of $P_{i,j}(t)$, then we have

$$P_{i,j} = \begin{cases} \lim\limits_{t\to\infty} P_{i,j}(t), & i = 0,1,2,\ldots,N,\ j = 0,1,2,\ldots, \\ 0, & \text{other}. \end{cases} \tag{1}$$

Assuming that the system is positive recurrent, the balance equations are as follows:

$$\begin{aligned}
&(\lambda + c\eta) P_{0,0} = \xi_1 P_{1,0}, \\
&(\lambda + c\eta) P_{0,j} = \lambda P_{0,j-1} + \xi_2 P_{1,j}, \quad j > 0, \\
&(\lambda + c\eta + i\xi_1) P_{i,0} = c\eta P_{i-1,0} + \mu P_{i,1} + (i+1)\xi_1 P_{i+1,0}, \\
&\qquad 0 < i \leq N-c,\ j = 0, \\
&P_{i,j}\left[\lambda + c\eta + j\mu + (i-j)\xi_1 + j\xi_2\right] = \lambda P_{i,j-1} + c\eta P_{i-1,j} \\
&\qquad + (j+1)\mu P_{i,j+1} + \left[(i+1-j)\xi_1 + j\xi_2\right] P_{i+1,j}, \\
&\qquad 0 < i \leq N-c,\ 0 < j < i, \\
&P_{i,j}\left[\lambda + c\eta + i\mu + i\xi_2\right] \\
&\qquad = \lambda P_{i,j-1} + c\eta P_{i-1,j} + i\mu P_{i,j+1} + (\xi_1 + j\xi_2) P_{i+1,j}, \\
&\qquad 0 < i \leq N-c,\ j = i, \\
&P_{i,j}\left[\lambda + c\eta + i\mu + i\xi_2\right] \\
&\qquad = \lambda P_{i,j-1} + c\eta P_{i-1,j} + i\mu P_{i,j+1} + (i+1)\xi_2 P_{i+1,j}, \\
&\qquad 0 < i \leq N-c,\ j > i, \\
&P_{i,0}\left[\lambda + (N-i)\eta + i\xi_1\right] \\
&\qquad = (N-i+1)\eta P_{i-1,0} + \mu P_{i,1} + (i+1)\xi_1 P_{i+1,0}, \\
&\qquad N-c < i < N,\ j = 0, \\
&P_{i,j}\left[\lambda + (N-i)\eta + j\mu + (i-j)\xi_1 + j\xi_2\right] \\
&\qquad = (N-i+1)\eta P_{i-1,j} + \lambda P_{i,j-1} + (j+1)\mu P_{i,j+1} \\
&\qquad + \left[(i+1-j)\xi_1 + j\xi_2\right] P_{i+1,j}, \\
&\qquad N-c < i < N,\ 0 < j < i, \\
&P_{i,j}\left[\lambda + (N-i)\eta + i\mu + i\xi_2\right] \\
&\qquad = \lambda P_{i,j-1} + (N-i+1)\eta P_{i-1,j} + i\mu P_{i,j+1} \\
&\qquad + (\xi_1 + j\xi_2) P_{i+1,j}, \quad N-c < i < N,\ j = i, \\
&P_{i,j}\left[\lambda + (N-i)\eta + i\mu + i\xi_2\right] \\
&\qquad = \lambda P_{i,j-1} + (N-i+1)\eta P_{i-1,j} + i\mu P_{i,j+1} \\
&\qquad + (i+1)\xi_2 P_{i+1,j}, \quad N-c < i < N,\ j > i, \\
&P_{N,0}(\lambda + N\xi_1) = \eta P_{N-1,0} + \mu P_{N,1}, \quad i = N,\ j = 0, \\
&P_{N,j}\left[\lambda + j\mu + (N-j)\xi_1 + j\xi_2\right] \\
&\qquad = \lambda P_{N,j-1} + \eta P_{N-1,j} + (j+1)\mu P_{N,j+1}, \\
&\qquad i = N,\ 0 < j < N, \\
&P_{N,j}(\lambda + N\mu + N\xi_2) = \lambda P_{N,j-1} + \eta P_{N-1,j} + N\mu P_{N,j+1}, \\
&\qquad i = N,\ j \geq N.
\end{aligned} \tag{2}$$

Here, we give the derivation of the second equation of (2). Since the process $\{X(t), Y(t), t \geq 0\}$ is a vector Markov process of continuous time, we write the equations of the state of $(0, j)$ by considering the transitions occurring between the moments t and $t + \Delta t$ $(\Delta t > 0)$ as follows:

$$P_{0,j}(t+\Delta t) = P_{0,j-1}(t)\lambda\Delta t + P_{1,j}(t)\xi_2\Delta t + P_{0,j}(t)\left[1-(\lambda+c\eta)\Delta t\right] + o(\Delta t), \tag{3}$$

then we have

$$\begin{aligned} P_{0,j}(t+\Delta t) - P_{0,j}(t) &= P_{0,j-1}(t)\lambda\Delta t + P_{1,j}(t)\xi_2\Delta t \\ &\quad - P_{0,j}(t)(\lambda+c\eta)\Delta t + o(\Delta t), \end{aligned} \tag{4}$$

$$\begin{aligned} \frac{P_{0,j}(t+\Delta t) - P_{0,j}(t)}{\Delta t} &= P_{0,j-1}(t)\lambda + P_{1,j}(t)\xi_2 \\ &\quad - P_{0,j}(t)(\lambda+c\eta) + \frac{o(\Delta t)}{\Delta t}. \end{aligned} \tag{5}$$

Letting $\Delta t \to 0$ in (5), we have

$$P_{0,j}(t)' = P_{0,j-1}(t)\lambda + P_{1,j}(t)\xi_2 - P_{0,j}(t)(\lambda+c\eta). \tag{6}$$

If the system is positive recurrent, we have the formulas $\lim_{t\to\infty} P_{0,j}(t)' = 0$ [13]. Letting $t \to 0$ in (6), we obtain the second equation of (2). The derivations of other formulas in (2) are similar.

3. Equations of Probability Generating Functions

The PGFs of the number of customers are defined as follows:

$$G_i(z) \equiv \sum_{j=0}^{\infty} z^j P_{i,j}, \qquad G(z) \equiv \sum_{i=0}^{N} G_i(z), \qquad 0 \leq i \leq N,\ |z| \leq 1. \tag{7}$$

Then

$$G_i(1) = \sum_{j=0}^{\infty} P_{i,j} \quad (i = 0, 1, 2, \dots, N), \tag{8}$$

where $G_i(1)$ is the steady-state probability that the number of the available servers of the system is i. Hence,

$$\sum_{i=0}^{N} G_i(1) = 1. \tag{9}$$

Multiplying the two sides of every equation of (2) by z^{j+1}, and summing over j $(j = 0, 1, 2, \dots)$ for every i, we obtain

$$\begin{aligned}
& z(\lambda - \lambda z + c\eta)G_0(z) - z\xi_2 G_1(z) = z(\xi_1 - \xi_2)P_{1,0}, \\
& -c\eta z G_{i-1}(z) + \left[z(i\mu + i\xi_2 + \lambda + c\eta) - \lambda z^2 - i\mu\right]G_i(z) \\
&\quad - (i+1)z\xi_2 G_{i+1}(z) \\
&\quad = \sum_{m=0}^{i-1}(\xi_2 - \xi_1)(i-m)P_{i,m}z^{m+1} \\
&\qquad + \sum_{m=0}^{i}(\xi_1 - \xi_2)(i+1-m)P_{i+1,m}z^{m+1} \\
&\qquad + (z-1)\sum_{m=0}^{i-1}\mu(i-m)P_{i,m}z^m, \quad 0 < i \leq N-c, \\
& -(N-i+1)\eta z G_{i-1}(z) \\
& + \left\{z\left[i\mu + i\xi_2 + \lambda + (N-i)\eta\right] - \lambda z^2 - i\mu\right\}G_i(z) \\
& - (i+1)z\xi_2 G_{i+1}(z) \\
&\quad = \sum_{m=0}^{i-1}(\xi_2 - \xi_1)(i-m)P_{i,m}z^{m+1} \\
&\qquad + \sum_{m=0}^{i}(\xi_1 - \xi_2)(i+1-m)P_{i+1,m}z^{m+1} \\
&\qquad + (z-1)\sum_{m=0}^{i-1}\mu(i-m)P_{i,m}z^m, \quad N-c < i < N, \\
& -\eta z G_{N-1}(z) + \left[z(N\mu + N\xi_2 + \lambda) - \lambda z^2 - N\mu\right]G_N(z) \\
&\quad = \sum_{m=0}^{N-1}(\xi_2 - \xi_1)(N-m)P_{N,m}z^{m+1} \\
&\qquad + (z-1)\sum_{m=0}^{N-1}\mu(N-m)P_{N,m}z^m.
\end{aligned} \tag{10}$$

We give some explanations of (10), the first equation of (2) multiplied by z, we get

$$(\lambda + c\eta)P_{0,0}z = \xi_1 P_{1,0}z. \tag{11}$$

The second equation of (2) multiplied by z^{j+1}, we get

$$(\lambda + c\eta)P_{0,j}z^{j+1} = \lambda P_{0,j-1}z^{j+1} + \xi_2 P_{1,j}z^{j+1}, \quad j > 0. \tag{12}$$

Summing (11) and (12) over j and using (7), we obtain the first equation of (10). The other equations of (10) are obtained in the same way.

In order to simplify (10), we defined the following notations:

$$f_i(z) \equiv (i\mu + i\xi_2 + \lambda + c\eta) z - \lambda z^2 - i\mu, \quad i = 0, 1, 2, \ldots, N - c,$$

$$f_i(z) \equiv [i\mu + i\xi_2 + \lambda + (N - i)\eta] z - \lambda z^2 - i\mu, \quad i = N - c + 1, N - c + 2, \ldots, N, \tag{13}$$

$$A(z) \equiv \begin{bmatrix} f_0(z) & -\xi_2 z & & & & & & & \\ -c\eta z & f_1(z) & -2\xi_2 z & & & & & & \\ & \ddots & \ddots & \ddots & & & & & \\ & & -c\eta z & f_{N-c+1}(z) & -(N-c+2)\xi_2 z & & & & \\ & & & (c-1)\eta z & f_{N-c+2}(z) & -(N-c+3)\xi_2 z & & & \\ & & & & \ddots & \ddots & \ddots & & \\ & & & & & \ddots & \ddots & \ddots & \\ & & & & & & -2\eta z & f_{N-1}(z) & -N\xi_2 z \\ & & & & & & & -\eta z & f_N(z) \end{bmatrix}, \tag{14}$$

$$\begin{aligned} b_0(z) &\equiv z(\xi_1 - \xi_2) P_{1,0}, \\ b_i(z) &\equiv \sum_{m=0}^{i-1} (\xi_2 - \xi_1)(i - m) P_{i,m} z^{m+1} \\ &\quad + \sum_{m=0}^{i} (\xi_1 - \xi_2)(i + 1 - m) P_{i+1,m} z^{m+1} \\ &\quad + (z - 1) \sum_{m=0}^{i-1} \mu (i - m) P_{i,m} z^m, \quad 0 < i < N, \\ b_N(z) &\equiv \sum_{m=0}^{N-1} (\xi_2 - \xi_1)(N - m) P_{N,m} z^{m+1} \\ &\quad + (z - 1) \sum_{m=0}^{N-1} \mu (N - m) P_{N,m} z^m, \end{aligned}$$

$$\overline{b}(z) \equiv \begin{bmatrix} b_0(z) \\ b_1(z) \\ \vdots \\ b_N(z) \end{bmatrix}, \qquad \overline{g}(z) \equiv \begin{bmatrix} G_0(z) \\ G_1(z) \\ \vdots \\ G_N(z) \end{bmatrix}. \tag{15}$$

Using the above notations, (10) is rewritten as follows:

$$A(z)\overline{g}(z) = \overline{b}(z). \tag{16}$$

Using Cramer's rule we obtain

$$|A(z)| G_i(z) = |A_i(z)|, \quad i = 0, 1, 2, \ldots, N, \tag{17}$$

where $|A(z)|$ denotes the determinant of $A(z)$, and $A_i(z)$ is a matrix obtained by replacing the $(i+1)$th column of $A(z)$ with $\overline{b}(z)$. In (17), the functions of z are continuous and bounded in the interval $[0, 1]$, so the equations in (17) are valid in the interval $[0, 1]$ no matter $|A(z)| = 0$ or not.

4. Steady-State Availability

In this section, we discuss the steady-state availabilities $G_i(1)$ $(i = 0, 1, 2, \ldots, N)$.

Letting $z = 1$ in (10) we obtain

$$\begin{aligned} &c\eta G_0(1) - \xi_2 G_1(1) = (\xi_1 - \xi_2) P_{1,0}, \\ &c\eta G_{i-1}(1) - i\xi_2 G_i(1) - [c\eta G_i(1) - (i + 1)\xi_2 G_{i+1}(1)] \\ &\quad = (\xi_1 - \xi_2) \sum_{m=1}^{i} m P_{i,i-m} - (\xi_1 - \xi_2) \sum_{m=1}^{i+1} m P_{i+1,i+1-m}, \\ &\qquad 0 < i \le N - c, \\ &[N - (i - 1)] \eta G_{i-1}(1) - i\xi_2 G_i(1) \\ &\quad - [(N - i)\eta G_i(1) - (i + 1)\xi_2 G_{i+1}(1)] \\ &\quad = (\xi_1 - \xi_2) \sum_{m=1}^{i} m P_{i,i-m} - (\xi_1 - \xi_2) \sum_{m=1}^{i+1} m P_{i+1,i+1-m}, \\ &\qquad N - c < i < N, \\ &\eta G_{N-1}(1) - N\xi_2 G_N(1) = (\xi_1 - \xi_2) \sum_{m=1}^{N} m P_{N,N-m}. \end{aligned} \tag{18}$$

The $(N+1)$ equations of (18) are simplified to N independent equations, joined with (9), we have

$$\begin{aligned} &c\eta G_i(1) - (i + 1)\xi_2 G_{i+1}(1) \\ &\quad = (\xi_1 - \xi_2) \sum_{m=1}^{i+1} m P_{i+1,i+1-m}, \quad 0 \le i \le N - c, \end{aligned}$$

$$
\begin{aligned}
&(N-i)\,\eta G_i(1) - (i+1)\,\xi_2 G_{i+1}(1) \\
&\quad = (\xi_1 - \xi_2)\sum_{m=1}^{i+1} m P_{i+1,i+1-m}, \quad N-c \le i \le N-1, \\
&\qquad\qquad \sum_{i=0}^{N} G_i(1) = 1.
\end{aligned}
\tag{19}
$$

Using (2), $P_{i,j}$ $(0 \le j \le i-1, 1 \le i \le N)$ are reduced to $P_{i,0}$ $(1 \le i \le N)$ which will be solved in Section 5. Given $P_{i,0}$ $(1 \le i \le N)$, we obtain $G_i(1)$ $(i = 0, 1, 2, \ldots, N)$ by solving (19), then the steady-state availability of the system is as follows:

$$
A = 1 - G_0(1). \tag{20}
$$

5. Steady-State Probabilities of the System States and Mean Queuing Length

5.1. The Roots of $|A(z)|$ *in the Interval* $(0,1)$. In order to get the steady-state probabilities $P_{i,0}$ $(1 \le i \le N)$, we need all N independent linear equations. Further, for getting the linear equations of $P_{i,0}$ $(1 \le i \le N)$ we need the roots of $|A(z)|$ in the interval $(0,1)$, so we discuss the roots of $|A(z)|$ as follows.

We define the following notations:

$$
Q_0(z) \equiv 1, \qquad Q_1(z) \equiv f_N(z), \tag{21}
$$

$$
(z) \equiv \begin{vmatrix} f_{N-1}(z) & -N\xi_2 z \\ -\eta z & f_N(z) \end{vmatrix}, \\
\vdots \qquad \vdots
$$

$$
Q_N(z) \equiv \begin{vmatrix}
f_1(z) & -2\xi_2 z & & & & & & \\
-c\eta z & f_2(z) & -3\xi_2 z & & & & & \\
 & \ddots & \ddots & \ddots & & & & \\
 & & -c\eta z & f_{N-c+1}(z) & -(N-c+2)\,\xi_2 z & & & \\
 & & & (c-1)\,\eta z & f_{N-c+2}(z) & -(N-c+3)\,\xi_2 z & & \\
 & & & & \ddots & \ddots & \ddots & \\
 & & & & & -2\eta z & f_{N-1}(z) & -N\xi_2 z \\
 & & & & & & -\eta z & f_N(z)
\end{vmatrix}, \tag{22}
$$

$$
Q_{N+1}(z) \equiv |A(z)|, \quad (0 \le z \le \infty), \tag{23}
$$

where $Q_i(z)$ $(i = 1, 2, \ldots, N)$ is formed by the last i rows and last i columns of $|A(z)|$. According to (14) we have

$$
\begin{aligned}
Q_{k+1}(z) &= f_{N-k}(z)\,Q_k(z) - (N-k+1)\,\xi_2 k\eta z^2 Q_{k-1}(z), \\
&\qquad 1 \le k \le c-1, \\
Q_{k+1}(z) &= f_{N-k}(z)\,Q_k(z) - (N-k+1)\,\xi_2 c\eta z^2 Q_{k-1}(z), \\
&\qquad c \le k \le N.
\end{aligned}
\tag{24}
$$

The properties of $Q_i(z)$ $(i = 1, 2, \ldots, N)$ and the necessary proofs are as follows.

(a) $Q_0(z)$ has no roots.

(b) $Q_k(z)$ and $Q_{k+1}(z)$ have no common roots in the interval $(0, \infty)$ $(k = 1, 2, \ldots, N)$.

Proof. Suppose that (b) is not true and z_0 (> 0) is a common roots of $Q_k(z)$ and $Q_{k+1}(z)$, then $Q_{k-1}(z_0) = 0$ due to (24). Similarly, $Q_{k-2}(z_0) = 0$, so we get $Q_0(z_0) = 0$ which contradicts the statement (a). □

(c) If z_0 is a positive root of $Q_k(z)$ $(k = 1, 2, \ldots, N)$, then $Q_{k-1}(z_0)$ and $Q_{k+1}(z_0)$ are opposite in sign. This property readily follows from (24).

(d) $Q_k(1) > 0$, $k = 0, 1, 2, \ldots, N$.

Proof. Substituting $z = 1$ into $Q_k(z)$, we get $Q_k(1) = (N-k+1)(N-k+2)\cdots(N-1)N\xi_2^k > 0$, $k = 1, 2, \ldots, N$, and $Q_0(1) = 1$. □

(e) $Q_{N+1}(1) = |A(1)| = 0$, and $Q_{N+1}(0) = |A(0)| = 0$.

Proof. The first row of $|A(z)|$ has a common factor z, so $Q_{N+1}(0) = |A(0)| = 0$. The sum of every column of $|A(z)|$ has a common factor $(z-1)$. Replacing every element of the last row of $|A(z)|$ with the sum of the corresponding column and extracting the common factor $(z-1)$, $|A(z)|$ is written as follows:

$$|A(z)| = z(z-1) \times \begin{bmatrix} \lambda + c\eta - \lambda z & -\xi_2 & 0 & \cdots & 0 & 0 \\ -c\eta z & f_1(z) & -2\xi_2 z & \cdots & 0 & 0 \\ 0 & -c\eta & f_2(z) & \cdots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \cdots & f_{N-1}(z) & -N\xi_2 z \\ -\lambda z & -\lambda z + \mu & -\lambda z + 2\mu & \cdots & -\lambda z + (N-1)\mu & -\lambda z + N\mu \end{bmatrix}. \quad (25)$$

If we define

$$D(z) \equiv \begin{vmatrix} \lambda + c\eta - \lambda z & -\xi_2 & 0 & \cdots & 0 & 0 \\ -c\eta z & f_1(z) & -2\xi_2 z & \cdots & 0 & 0 \\ 0 & -c\eta & f_2(z) & \cdots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \cdots & f_{N-1}(z) & -N\xi_2 z \\ -\lambda z & -\lambda z + \mu & -\lambda z + 2\mu & \cdots & -\lambda z + (N-1)\mu & -\lambda z + N\mu \end{vmatrix}, \quad (26)$$

then

$$|A(z)| = z(z-1)D(z). \quad (27)$$

□

(f) $\text{Sign}[Q_k(0)] = (-1)^k$ $(k = 0, 1, 2, \ldots, N)$.

Proof. From the definitions of $f_i(z)$ $(i = 0, 1, 2, \ldots, N)$, we got $f_0(0) = 0$ and $f_k(0) < 0$ $(k = 1, 2, \ldots, N)$, so we get this property from (24). □

(g) $\text{Sign}[Q_k(+\infty)] = (-1)^k$ $(k = 0, 1, 2, \ldots, N+1)$.

Proof. It is since the highest power term of $Q_k(z)$ is $(-\lambda z^2)^k$ $(k = 0, 1, 2, \ldots, N+1)$ and the sign of $Q_k(+\infty)$ is determined by its highest power term. □

Theorem 1. *If $D(1) > 0$, the polynomial $|A(z)|$ has exactly $(N-1)$ distinct roots in the interval $(0, 1)$.*

Proof. Since $Q_1(z) = f_N(z) = [N(\mu + \xi_2) + \lambda]z - \lambda z^2 - N\mu$, $Q_1(z)$ is a 2-power polynomial of z. Further, we find that $Q_1(1) = \xi_2 > 0$ and $Q_1(0) = -\mu < 0$, so $Q_1(z)$ has two distinct roots which are denoted by $z_{1,1}$ $(0 < z_{1,1} < 1)$ and $z_{1,2}$ (> 1).

With the fact that $z_{1,1}$ and $z_{1,2}$ are roots of $Q_1(z)$, and $Q_0(z) = 1 > 0$, according to the property (c) or (24), we get $Q_2(z_{1,1}) < 0$ and $Q_2(z_{1,2}) < 0$.

$Q_2(z)$ is a 4-power polynomial of z. From the properties (c), (d), (f), and (g), we find that $Q_2(z)$ has one and only one root in each interval of $(0, z_{1,1})$, $(z_{1,1}, 1)$, $(1, z_{1,2})$, and $(z_{1,2}, +\infty)$.

So on, we find that $Q_N(z)$ is a $2N$-power polynomial of z, it has N distinct roots in the interval $(0, 1)$ and N distinct roots in the interval $(1, \infty)$. We denote the $2N$ roots of $Q_N(z)$ by $z_{N,i}$ $(i = 1, 2, \ldots, 2N)$ orderly.

From the properties (c), (d), (e), and (f), we find that $|A(z)|$ has one and only one root in each interval $(z_{N,i}, z_{N,i+1})$ $(i = 1, 2, \ldots, N-1, N+1, \ldots, 2N-1)$, all of them are $2(N-1)$ distinct roots of $|A(z)|$.

Since $|A(1)| = 0$ and $D(1) = (|A(z)|)'_{z=1} > 0$, it has a real number ε (>0) satisfies $1 + \varepsilon < z_{N,N+1}$ and $|A(1+\varepsilon)| > 0$. On the other hand, from (c) and (d) we get

$$\text{Sign}[Q_{N+1}(z_{N,i})] = (-1)^{N+i}, \quad i = N+1, N+2, \ldots, 2N, \quad (28)$$

then $\text{Sign}[Q_{N+1}(z_{N,N+1})] = (-1)^{N+N+1}$ or $|A(z_{N,N+1})| = Q_{N+1}(z_{N,N+1}) < 0$, so $|A(z)|$ has at least one root in the interval $(1, z_{N,N+1})$.

From (28), we get $\text{Sign}[|A(z_{N,2N})|] = \text{Sign}[Q_{N+1}(z_{N,2N})] = (-1)^{N+2N} = (-1)^{3N} = (-1)^N$. From the property (g), we get $\text{Sign}[|A(+\infty)|] = \text{Sign}[Q_{N+1}(+\infty)] = (-1)^{N+1}$. So we know that $|A(z)|$ has at least one root in the interval $(z_{N,2N}, \infty)$.

From the properties (e) and (f), we know that 0 and 1 are roots of $|A(z)|$.

In conclusion, $|A(z)|$ is a $2(N+1)$-power polynomial of z, it has $2(N+1)$ distinct roots at most. Now we make certain all roots of $|A(z)|$ and find that it has $N-1$ distinct roots in the interval $(0, 1)$. □

From the proof, we find that the $(N-1)$ distinct roots in the interval $(0, 1)$ of $|A(z)|$ are also the roots of $D(z)$.

5.2. Steady-State Probabilities. Assuming that the system parameters meet $D(1) > 0$. Letting z_k $(k = 1, 2, \ldots, N-1)$ denote the roots of $|A(z)|$ in the interval $(0, 1)$. Substituting z_1 in (17), we obtain a set of linear equations about the steady-state probabilities of $P_{i,0}$ $(i = 1, 2, \ldots, N)$, but these equations are similar to each other. However, the equations belong to different z_k $(k = 1, 2, \ldots, N-1)$ are independent mutually, so we obtain $(N-1)$ independent equations by the $(N-1)$ different roots of z_k, respectively.

In the following, we discuss about the Nth-independent linear equation of $P_{i,0}$ $(i = 1, 2, \ldots, N)$. Similar to (27), $|A_i(z)|$ is written as follows:

$$|A_i(z)| = z(z-1)D_i(z), \quad i = 0, 1, 2, \ldots, N, \quad (29)$$

where

$$D_i(z) = \begin{vmatrix} c\eta + \lambda(1-z) & -\xi_2 & \cdots & (\xi_1 - \xi_2) P_{1,0} & \cdots & 0 \\ -c\eta z & f_1(z) & \cdots & b_1(z) & \cdots & 0 \\ 0 & -c\eta z & \cdots & b_2(z) & \cdots & 0 \\ & & \vdots & & \vdots & \\ 0 & 0 & \cdots & b_{N-1}(z) & \cdots & -N\xi_2 z \\ -\lambda z & -\lambda z + \mu & \cdots & \underbrace{\sum_{i=1}^{N}\sum_{m=1}^{i} m\mu p_{i,i-m} z^{i-m}}_{(i+1)\text{th column}} & \cdots & -\lambda z + N\mu \end{vmatrix}. \quad (30)$$

Substituting (27) and (29) into (17), we obtain

$$D(z) G_i(z) = D_i(z), \quad i = 0, 1, 2, \ldots, N. \quad (31)$$

Substituting $z = 1$ in (31), we obtain

$$D(1) G_i(1) = D_i(1), \quad i = 0, 1, 2, \ldots, N. \quad (32)$$

From (19), we know that $G_i(1)$ $(i = 0, 1, 2, \ldots, N)$ can be expressed by $P_{i,0}$ $(i = 1, 2, \ldots, N)$, so (32) are linear equations of $P_{i,0}$ $(i = 1, 2, \ldots, N)$, but they are similar to each other. However, every equation of (32) is independent with the $(N-1)$ linear equations obtained by the roots of $|A_i(z)|$ in the interval $(0, 1)$. Then all independent linear equations of $P_{i,0}$ $(i = 1, 2, \ldots, N)$ are as follows:

$$\begin{aligned} |A_0(z_k)| &= 0, \quad k = 1, 2, \ldots, N-1, \\ D(1) G_0(1) &= D_0(1). \end{aligned} \quad (33)$$

Further, from (29), (33) is equivalent to the follows:

$$\begin{aligned} D_0(z_k) &= 0, \quad k = 1, 2, \ldots, N-1, \\ D(1) G_0(1) &= D_0(1). \end{aligned} \quad (34)$$

The steady-state probabilities of $P_{i,0}$ $(i = 0, 1, 2, \ldots, N)$ are obtained by solving (34). Using $P_{i,0}$ $(i = 0, 1, 2, \ldots, N)$ and (2), we obtain the other steady-state probabilities of $P_{i,j}$ $(i = 0, 1, 2, \ldots, N, j = 1, 2, \ldots)$.

5.3. Mean Queuing Length. After getting the probabilities $P_{i,j}$ $(0 \le j \le i-1, 1 \le i \le N)$ in (19) we obtain $G_i(1)$ $(i = 0, 1, 2, \ldots, N)$ by solving (19) and obtain the steady-state availability (A) of the model by (20).

From (31), we obtain

$$G_i(z) = \frac{D_i(z)}{D(z)}, \quad D(z) \neq 0, \ |z| \le 1, \ i = 0, 1, 2, \ldots, N. \quad (35)$$

The PGF of $G(z)$ is obtained by (7). Using the property of PGF [13] we obtain the steady-state mean queuing length is as follows:

$$L = \left. \frac{dG(z)}{dz} \right|_{z=1}. \quad (36)$$

6. Case Analysis

We analyze the case of $N = 2$ and $c = 1$ in this section. According to the above discussion, the determinant $|A(z)|$ (or $D(z)$) of this case has only one root z_1 in the interval $(0, 1)$. The notations of this case are as follows:

$$\begin{aligned}
f_0(z) &= (\lambda + \eta) z - \lambda z^2, \\
f_1(z) &= (\mu + \xi_2 + \lambda + \eta) z - \lambda z^2 - \mu, \\
f_2(z) &= (2\mu + 2\xi_2 + \lambda) z - \lambda z^2 - 2\mu, \\
b_0(z) &= (\xi_1 - \xi_2) P_{1,0} z, \\
b_1(z) &= (\mu + \xi_2 - \xi_1) P_{1,0} z \\
&\quad + \sum_{m=1}^{2} m (\xi_1 - \xi_2) P_{2,2-m} z^{3-m} - \mu P_{1,0} \\
&= \mu P_{1,0}(z-1) + z(\xi_1 - \xi_2)(P_{2,1} z + 2P_{2,0} - P_{1,0}), \\
b_2(z) &= \sum_{m=1}^{2} m (\mu + \xi_2 - \xi_1) P_{2,2-m} z^{3-m} \\
&\quad - \sum_{m=1}^{2} m\mu P_{2,2-m} z^{2-m}, \\
A(z) &= \begin{bmatrix} f_0(z) & -\xi_2 z & 0 \\ -\eta z & f_1(z) & -2\xi_2 z \\ 0 & -\eta z & f_2(z) \end{bmatrix}, \\
|A(z)| &= z(z-1) \\
&\quad \times \begin{vmatrix} \lambda + \eta - \lambda z & -\xi_2 & 0 \\ -\eta z & (\mu + \xi_2 + \lambda + \eta) z - \lambda z^2 - \mu & -2\xi_2 z \\ -\lambda z & -\lambda z + \mu & -\lambda z + 2\mu \end{vmatrix}, \\
D(z) &= \begin{vmatrix} \lambda + \eta - \lambda z & -\xi_2 & 0 \\ -\eta z & (\mu + \xi_2 + \lambda + \eta) z - \lambda z^2 - \mu & -2\xi_2 z \\ -\lambda z & -\lambda z + \mu & -\lambda z + 2\mu \end{vmatrix} \\
&= -(\eta + \lambda - z\lambda)(z\lambda - 2\mu)[-\mu + z(\eta + \lambda - z\lambda + \mu)] \\
&\quad + z\xi_2 [z\lambda(-2\eta - 3\lambda + 3z\lambda) \\
&\qquad + 2(\eta - 2z\lambda + 2\lambda)\mu - 2z\lambda\xi_2],
\end{aligned} \quad (37)$$

then

$$D(1)=\begin{vmatrix}\eta & -\xi_2 & 0\\ -\eta & \eta+\xi_2 & -2\xi_2\\ -\lambda & -\lambda+\mu & -\lambda+2\mu\end{vmatrix} = 2\mu\left(\xi_2\eta+\eta^2\right)-\lambda\left(2\xi_2^2+2\xi_2\eta+\eta^2\right), \tag{38}$$

and $D(1)>0$ is equivalent to

$$2\mu\left(\xi_2\eta+\eta^2\right)-\lambda\left(2\xi_2^2+2\xi_2\eta+\eta^2\right)>0, \tag{39}$$

or

$$\frac{\lambda}{\mu}<\frac{2\xi_2\eta+2\eta^2}{2\xi_2^2+2\xi_2\eta+\eta^2}=\frac{2\eta/\left(\xi_2+\eta\right)}{1+\left(\xi_2/\left(\xi_2+\eta\right)\right)^2}. \tag{40}$$

The left of (40) is the mean service quantities that all customers need per unit time. The right of (40) is the mean service quantities that the two servers provide per unit time. So (40) is the necessary and sufficient condition of recurrence of the system.

Equations (19) in this case are as follows:

$$\begin{gathered}\eta G_0(1)-\xi_2G_1(1)=\left(\xi_1-\xi_2\right)P_{1,0},\\ \eta G_1(1)-2\xi_2G_2(1)=\sum_{m=1}^{2}m\left(\xi_1-\xi_2\right)P_{2,2-m},\\ \sum_{i=0}^{2}G_i(1)=1.\end{gathered} \tag{41}$$

Equations (2) in this case are as follows:

$$\begin{gathered}(\lambda+\eta)P_{0,0}=\xi_1P_{1,0},\quad i=0,\ j=0,\\ (\lambda+\eta)P_{0,j}=\lambda P_{0,j-1}+\xi_2P_{1,j},\quad i=0,\ j>0,\\ \left(\lambda+\eta+\xi_1\right)P_{1,0}=\eta P_{0,0}+\mu P_{1,1}+2\xi_1P_{2,0},\\ i=1,\ j=0,\\ \left(\lambda+\eta+\xi_2+\mu\right)P_{1,1}\\ =\lambda P_{1,0}+\eta P_{0,1}+\mu P_{1,2}+\left(\xi_1+\xi_2\right)P_{2,1},\quad i=1,\ j=1,\\ \left(\lambda+\eta+\xi_2+\mu\right)P_{1,j}\\ =\lambda P_{1,j-1}+\eta P_{0,j}+\mu P_{1,j+1}+2\xi_2P_{2,j},\quad i=1,\ j>1,\\ \left(\lambda+2\xi_1\right)P_{2,0}=\eta P_{1,0}+\mu P_{2,1},\quad i=2,\ j=0,\\ \left(\lambda+\xi_1+\xi_2+\mu\right)P_{2,1}=\lambda P_{2,0}+\eta P_{1,1}+2\mu P_{2,2},\\ i=2,\ j=1,\\ \left(\lambda+2\mu+2\xi_2\right)P_{2,j}=\lambda P_{2,j-1}+\eta P_{1,j}+2\mu P_{2,j+1},\\ i=2,\ j\geq 2.\end{gathered} \tag{42}$$

Using (42), we obtain

$$P_{2,1}=\frac{\left(2\xi_1+\lambda\right)P_{2,0}-\eta P_{1,0}}{\mu}. \tag{43}$$

Using (41) and (43), $G_0(1)$, $G_1(1)$, and $G_2(1)$ are expressed in an algebraic expressions of $P_{1,0}$ and $P_{2,0}$.

If (40) is satisfied, we obtain z_1 by solving

$$D(z)=0. \tag{44}$$

Equations (34) in this case are as follows:

$$\begin{gathered}D_0\left(z_1\right)=0,\\ D(1)G_0(1)=D_0(1).\end{gathered} \tag{45}$$

Solving (45), we obtain $P_{1,0}$ and $P_{2,0}$.

Using (42), we obtain $P_{i,j}$ $(i=0,1,2,\ j=1,2,\ldots)$.

Using (20) and (41), we obtain the steady-state availability A.

For the mean queuing lengths, we have

$$\begin{gathered}D_0(z)=\begin{vmatrix}\left(\xi_1-\xi_2\right)P_{1,0} & -\xi_2 & 0\\ b_1(z) & \left(\mu+\xi_2+\lambda+\eta\right)z-\lambda z^2-\mu & -2\xi_2\\ \mu\left(P_{2,1}z+2P_{2,0}+P_{1,0}\right) & -\lambda z+\mu & \lambda z+2\mu\end{vmatrix},\\ D_1(z)=\begin{vmatrix}\lambda+\eta-\lambda z & \left(\xi_1-\xi_2\right)P_{1,0} & 0\\ -\eta z & b_1(z) & -2\xi_2\\ -\lambda z & \mu\left(P_{2,1}z+2P_{2,0}+P_{1,0}\right) & \lambda z+2\mu\end{vmatrix},\\ D_2(z)=\begin{vmatrix}\lambda+\eta-\lambda z & -\xi_2 & \left(\xi_1-\xi_2\right)P_{1,0}\\ -\eta z & \left(\mu+\xi_2+\lambda+\eta\right)z-\lambda z^2-\mu & b_1(z)\\ -\lambda z & -\lambda z+\mu & \mu\left(P_{2,1}z+2P_{2,0}+P_{1,0}\right)\end{vmatrix},\end{gathered} \tag{46}$$

$$G(z)=\frac{D_0(z)+D_1(z)+D_2(z)}{D(z)}. \tag{47}$$

Using (36), we obtain the mean queuing lengths L is as follows:

Table 1: The availability A and mean queuing length L ($N = 2$, $c = 1$, and $\lambda = 1$).

μ	$\xi_1 = 0$, $\xi_2 = 0.5$, and $\eta = 1.2$		$\xi_1 = 0.3$, $\xi_2 = 0.5$, and $\eta = 1$		$\xi_1 = 0.5$, $\xi_2 = 0.5$, and $\eta = 1$	
	L	A	L	A	L	A
1.1	3.0797	0.8951	4.6956	0.8209	4.8505	0.8
1.2	2.3810	0.9058	3.5018	0.8266	3.6549	0.8
1.3	1.9425	0.9147	2.8136	0.8315	2.9650	0.8
1.4	1.6411	0.9224	2.3641	0.8354	2.5138	0.8
1.5	1.4207	0.9290	2.0464	0.8383	2.1947	0.8
1.6	1.2524	0.9348	1.8095	0.8429	1.9563	0.8
1.7	1.1195	0.9399	1.6256	0.8458	1.7710	0.8
1.8	1.0119	0.9443	1.4784	0.8484	1.6225	0.8
μ	$\xi_1 = 0.5$, $\xi_2 = 0.5$, and $\eta = 0.8$		$\xi_1 = 0.5$, $\xi_2 = 1$, and $\eta = 1.5$		$\xi_1 = 1$, $\xi_2 = 0.8$, and $\eta = 1.5$	
	L	A	L	A	L	A
1.1	9.4410	0.7423	9.9094	0.7412	5.0503	0.7734
1.2	6.1455	0.7423	5.7928	0.7509	3.7017	0.7700
1.3	4.6345	0.7423	4.1374	0.7608	2.9584	0.7671
1.4	3.7636	0.7423	3.2405	0.7687	2.4853	0.7646
1.5	3.1952	0.7423	2.6761	0.7756	2.1565	0.7623
1.6	2.7937	0.7423	2.2873	0.7817	1.9140	0.7603
1.7	2.4944	0.7423	2.0025	0.7872	1.7273	0.7585
1.8	2.2389	0.7423	1.7846	0.7920	1.5788	0.7569

$$\begin{aligned}
L &= \left.\frac{dG(z)}{dz}\right|_{z=1} \\
&= \frac{1}{\left[\lambda\eta^2 - 2\mu\eta^2 + 2\xi_2\left(\eta\lambda - \eta\mu + \lambda\xi_2\right)\right]^2} \\
&\quad \times \Big\{-\Big[\eta\left(-2\eta\lambda + 2\lambda^2 + 2\eta\mu - 5\lambda\mu + 2\mu^2\right) \\
&\qquad\quad +\xi_2\left(-4\eta\lambda + 3\lambda^2 + 2\eta\mu - 4\lambda\mu - 4\lambda\xi_2\right)\Big] \\
&\qquad \times \Big[\mu\left(\eta^2 + 2\xi_1\eta + 2\xi_1\xi_2\right)P_{1,0} \\
&\qquad\quad +\mu\left(\eta^2 + \eta\xi_2 + \eta\xi_1 + 2\xi_1\xi_2\right)\left(2P_{2,0} + P_{2,1}\right)\Big] \\
&\qquad + \mu\left[-\eta^2\lambda + 2\mu\eta^2 - 2\xi_2\left(\eta\lambda - \eta\mu + \lambda\xi_2\right)\right] \\
&\qquad \times \Big[\Big(\eta^2 - 2\eta\lambda + 2\eta\mu - 2\lambda\xi_2 + 2\xi_1\eta \\
&\qquad\qquad -\lambda\xi_1 + 2\mu\xi_1 + 2\xi_1\xi_2\Big)P_{1,0} \\
&\qquad\quad + \left(\eta^2 + \eta\xi_2 + \eta\xi_1 + 2\xi_1\xi_2\right)P_{2,1} \\
&\qquad\quad + \Big(\eta^2 - 2\eta\lambda + \eta\mu + \eta\xi_2 - 2\lambda\xi_2 + \eta\xi_1 \\
&\qquad\qquad -\lambda\xi_1 + 2\xi_1\xi_2\Big)\left(2P_{2,0} + P_{2,1}\right)\Big]\Big\}.
\end{aligned} \tag{48}$$

Numerical Example. Letting $N = 2$, $c = 1$, $\lambda = 1$, $\xi_1 = 0.5$, $\xi_2 = 1$, $\eta = 1$, and $\mu = 2$, we have

$$\frac{\lambda}{\mu} = \frac{1}{2} < \frac{4}{5} = \frac{2\eta/\left(\xi_2 + \eta\right)}{1 + \left(\xi_2/\left(\xi_2 + \eta\right)\right)^2}. \tag{49}$$

The roots of $D(z) = 0$ are as follows:

$$\begin{aligned} z_1 &= 0.349123, \qquad z_2 = 1.84513, \\ z_3 &= 4.46896, \qquad z_4 = 8.33679, \end{aligned} \tag{50}$$

and only z_1 in the interval $(0, 1)$.

Solving (45), we obtain

$$P_{1,0} = 0.151375, \qquad P_{2,0} = 0.297974. \tag{51}$$

Using (42) we obtain

$$P_{0,0} = 0.037844, \qquad P_{2,1} = 0.146599. \tag{52}$$

Solving (41), we obtain

$$\begin{aligned} &G_0(1) = 0.280333, \qquad G_1(1) = 0.35602, \\ &G_2(1) = 0.363647. \end{aligned} \tag{53}$$

Finally, the availability and mean queuing lengths of this example are as follows:

$$A = 0.719667, \qquad L = 6.04039. \tag{54}$$

The other numerical results are shown in Table 1. All the system parameters in Table 1 satisfy (40).

We find that the mean queuing length (L) decreases with the increasing of the parameter μ in Table 1, it is because of the greater service rate the less customers in the system. Furthermore, we find that the availability (A) increases with the increasing of the parameter μ, where $\xi_1 < \xi_2$ (the cases:

$\xi_1 = 0, \xi_2 = 0.5$; $\xi_1 = 0.3, \xi_2 = 0.5$; $\xi_1 = 0.5, \xi_2 = 1$); on the contrary, the availability decreases with the increasing of the parameter μ, where $\xi_1 > \xi_2$ (the case: $\xi_1 = 1, \xi_2 = 0.8$); otherwise, the availability is constant, where $\xi_1 = \xi_2$ (the case: $\xi_1 = 0.5, \xi_2 = 0.5$).

7. Conclusions

In Section 5.1, the inequality $D(1) > 0$ of Theorem 1 is the necessary and sufficient condition for the system to be positive recurrent, and a probability explanation of this condition is given by (40).

We find that the idle time breakdown rate ξ_1 does not appear in (40). This is because the busy time breakdown rate ξ_2 is at work when the number of the customers is greater than or equal to the number of the available servers, and the criteria of positive recurrence depends on the busy time breakdown rate.

A case analysis is given to illustrate the analysis of this paper, and the numerical results indicate that the variation of breakdown rates has a significant effect on the steady-state availability and steady-state queue length of the system.

Acknowledgments

This paper is supported by the National Natural Science Foundation of China no. 71071133 and the Natural Science Foundation of Hebei Province no. G2012203136.

References

[1] A. Federgruen and K. C. So, "Optimal maintenance policies for single-server queueing systems subject to breakdowns," *Operations Research*, vol. 38, no. 2, pp. 330–343, 1990.

[2] K. H. Wang, "Optimal operation of a Markovian queueing system with a removable and non-reliable server," *Microelectronics Reliability*, vol. 35, no. 8, pp. 1131–1136, 1995.

[3] Y. H. Tang, "Downtime of the service station in M/G/1 queueing system with repairable service station," *Microelectronics Reliability*, vol. 36, no. 2, pp. 199–202, 1996.

[4] Y. Lam, Y. L. Zhang, and Q. Liu, "A geometric process model for $M/M/1$ queueing system with a repairable service station," *European Journal of Operational Research*, vol. 168, no. 1, pp. 100–121, 2005.

[5] W. J. Gray, P. P. Wang, and M. Scott, "A vacation queueing model with service breakdowns," *Applied Mathematical Modelling*, vol. 24, no. 5-6, pp. 391–400, 2000.

[6] M. Zhang and Z. Hou, "Performance analysis of $M/G/1$ queue with working vacations and vacation interruption," *Journal of Computational and Applied Mathematics*, vol. 234, no. 10, pp. 2977–2985, 2010.

[7] D. Y. Yang, K. H. Wang, and W. L. Pearn, "Optimization on (N, p)-policy for an unreliable queue with second optional service and start-up," *Journal of the Chinese Institute of Industrial Engineers*, vol. 28, pp. 411–424, 2011.

[8] C. J. Chang, J. C. Ke, and H. I. Huang, "The optimal management of a queueing system with controlling arrivals," *Journal of the Chinese Institute of Industrial Engineers*, vol. 28, no. 3, pp. 226–236, 2011.

[9] F. F. Wang, "A generic modeling framework for queueing-inventory systems with a single removable server," *Journal of the Chinese Institute of Industrial Engineers*, vol. 29, pp. 1–15, 2012.

[10] I. L. Mitrany and B. Avi-Itzhak, "Many-server queue with service interruptions," *Operations Research*, vol. 16, pp. 628–638, 1968.

[11] M. F. Neuts and D. M. Lucantoni, "A Markovian queue with N servers subject to breakdowns and repairs," *Management Science. Journal of the Institute of Management Science. Application and Theory Series*, vol. 25, no. 9, pp. 849–861, 1979.

[12] E. Altman and U. Yechiali, "Analysis of customers' impatience in queues with server vacations," *Queueing Systems*, vol. 52, no. 4, pp. 261–279, 2006.

[13] S. M. Ross, *Stochastic Processes*, John Wiley & Sons, New York, NY, USA, 1983.