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## Research Article

# General Common Fixed Point Theorems and Applications

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The main result is a common fixed point theorem for a pair of multivalued maps on a complete metric space extending a recent result of Đorić and Lazović (2011) for a multivalued map on a metric space satisfying Ćirić-Suzuki-type-generalized contraction. Further, as a special case, we obtain a generalization of an important common fixed point theorem of Ćirić (1974). Existence of a common solution for a class of functional equations arising in dynamic programming is also discussed.

## 1. Introduction

Consistent with Nadler [1, page 620],  $(X, d)$  will denote a metric space and  $CL(X)$ , the collection of all nonempty closed subsets of  $X$ . For  $A, B \in CL(X)$  and  $\varepsilon > 0$ ,

$$\begin{aligned} N(\varepsilon, A) &= \{x \in X : d(x, a) < \varepsilon \text{ for some } a \in A\}, \\ E_{A,B} &= \{\varepsilon > 0 : A \subseteq N(\varepsilon, B), B \subseteq N(\varepsilon, A)\}, \\ H(A, B) &= \begin{cases} \inf E_{A,B}, & \text{if } E_{A,B} \neq \emptyset \\ +\infty, & \text{if } E_{A,B} = \emptyset. \end{cases} \end{aligned} \tag{1.1}$$

The hyperspace  $(CL(X), H)$  is called the generalized Hausdorff metric space induced by the metric  $d$  on  $X$ .

For nonempty subsets  $A, B$  of  $X$ ,  $d(A, B)$  denotes the gap between the subsets  $A$  and  $B$ , while

$$\begin{aligned} \rho(A, B) &= \sup\{d(a, b) : a \in A, b \in B\}, \\ BN(X) &= \{A : \emptyset \neq A \subseteq X \text{ and the diameter of } A \text{ is finite}\}. \end{aligned} \quad (1.2)$$

As usual, we write  $d(x, B)$  (resp.  $\rho(x, B)$ ) for  $d(A, B)$  (resp.  $\rho(A, B)$ ) when  $A = \{x\}$ .

Let  $S, T : X \rightarrow CL(X)$ . Then  $u \in X$  is a fixed point of  $S$  if and only if  $u \in Su$  and a common fixed point of  $S$  and  $T$  if and only if  $u \in Su \cap Tu$ .

Let  $S$  and  $T$  be maps to be defined specifically in a particular context, while  $x$  and  $y$  are the elements of a metric space  $(X, d)$ :

$$M(Sx, Ty) = \max \left\{ d(x, y), d(x, Sx), d(y, Ty), \frac{d(x, Ty) + d(y, Sx)}{2} \right\}. \quad (1.3)$$

Recently Suzuki [2] and Kikkawa and Suzuki [3] obtained interesting generalizations of the Banach's classical fixed point theorem and other fixed point results by Nadler [4], Jungck [5], and Meir and Keeler [6]. These results have important outcomes (see, e.g., [7–14]). The following result, due to Ćirić and Lazović [9], extends and generalizes fixed point theorems from Ćirić [15], Kikkawa and Suzuki [3], Nadler [4], Reich [16], Rus [17], and others.

**Theorem 1.1.** *Define a nonincreasing function  $\varphi$  from  $[0, 1)$  onto  $(0, 1]$  by*

$$\varphi(r) = \begin{cases} 1 & \text{if } 0 \leq r < \frac{1}{2} \\ 1 - r & \text{if } \frac{1}{2} \leq r < 1. \end{cases} \quad (1.4)$$

*Let  $X$  be a complete metric space and  $T : X \rightarrow CL(X)$ . Assume there exists  $r \in [0, 1)$  such that for every  $x, y \in X$ ,*

$$\varphi(r)d(x, Tx) \leq d(x, y) \text{ implies } H(Tx, Ty) \leq rM(Tx, Ty). \quad (1.5)$$

*Then there exists  $z \in X$  such that  $z \in Tz$ .*

We remark that, for every  $x, y \in X$ , the generalized contraction  $H(Tx, Ty) \leq rM(Tx, Ty)$ ,  $0 \leq r < 1$ , was first studied by Ćirić [15]. The following important common fixed point theorem is due to Ćirić [18].

**Theorem 1.2.** *Let  $X$  be a complete metric space and  $S, T : X \rightarrow X$ . Assume there exists  $r \in [0, 1)$  such that for every  $x, y \in X$ ,*

$$d(Sx, Ty) \leq rM(Sx, Ty). \quad (1.6)$$

*Then  $S$  and  $T$  have a unique common fixed point.*

For an excellent discussion on several special cases and variants of Theorem 1.2, one may refer to Rus [17]. However, the generality of Theorem 1.2 may be appreciated from the fact that (1.6) in Theorem 1.2 cannot be replaced by

$$d(Sx, Ty) \leq r \max\{d(x, y), d(x, Sx), d(y, Ty), d(x, Ty), d(y, Sx)\}. \quad (1.7)$$

Indeed, Sastry and Naidu [19, Example 5] have shown that maps  $S$  and  $T$  satisfying (1.7) need not have a common fixed point on a complete metric space. Notice that the condition (1.7) with  $S = T$  is the quasicontraction due to Ćirić [20].

The main result of this paper (cf. Theorem 2.2) generalizes Theorems 1.1 and 1.2. Further, a corollary of Theorem 2.2 is used to obtain a unique common fixed point theorem for multivalued maps on a metric space with values in  $BN(X)$ . As another application, we deduce the existence of a common solution for a general class of functional equations under much weaker conditions than those in [12, 14, 21–24].

## 2. Main Results

We shall need the following result essentially due to Nadler [4] (see also [15, 25], [26, page 4], [27], [17, page 76]).

**Lemma 2.1.** *If  $A, B \in \text{CL}(X)$  and  $a \in A$ , then for each  $\varepsilon > 0$ , there exists  $b \in B$  such that  $d(a, b) \leq H(A, B) + \varepsilon$ .*

**Theorem 2.2.** *Let  $X$  be a complete metric space and  $S, T : X \rightarrow \text{CL}(X)$ . Assume there exists  $r \in [0, 1)$  such that for every  $x, y \in X$ ,*

$$\varphi(r) \min\{d(x, Sx), d(y, Ty)\} \leq d(x, y) \text{ implies } H(Sx, Ty) \leq rM(Sx, Ty). \quad (2.1)$$

*Then there exists an element  $u \in X$  such that  $u \in Su \cap Tu$ .*

*Proof.* Obviously  $M(Sx, Ty) = 0$  iff  $x = y$  is a common fixed point of  $S$  and  $T$ . So, we may take without any loss of generality that  $M(Sx, Ty) > 0$  for distinct  $x, y \in X$ . Let  $\varepsilon > 0$  be such that  $\beta = r + \varepsilon < 1$ . Let  $u_0 \in X$  and  $u_1 \in Tu_0$ . Then by Lemma 2.1, there exists  $u_2 \in Su_1$  such that

$$d(u_2, u_1) \leq H(Su_1, Tu_0) + \varepsilon M(Su_1, Tu_0). \quad (2.2)$$

Similarly, there exists  $u_3 \in Tu_2$  such that

$$d(u_3, u_2) \leq H(Tu_2, Su_1) + \varepsilon M(Tu_2, Su_1). \quad (2.3)$$

Continuing in this manner, we find a sequence  $\{u_n\}$  in  $X$  such that

$$\begin{aligned} u_{2n+1} &\in Tu_{2n}, \quad u_{2n+2} \in Su_{2n+1} \text{ such that} \\ d(u_{2n+1}, u_{2n}) &\leq H(Tu_{2n}, Su_{2n-1}) + \varepsilon M(Tu_{2n}, Su_{2n-1}), \\ d(u_{2n+2}, u_{2n+1}) &\leq H(Su_{2n+1}, Tu_{2n}) + \varepsilon M(Su_{2n+1}, Tu_{2n}). \end{aligned} \quad (2.4)$$

Now, we consider two cases and show that for any  $n \in N$ ,

$$d(u_{2n+1}, u_{2n}) \leq \beta d(u_{2n-1}, u_{2n}). \quad (2.5)$$

Case 1. If  $d(u_{2n-1}, Su_{2n-1}) \geq d(u_{2n}, Tu_{2n})$ , then

$$\varphi(r) \min\{d(u_{2n-1}, Su_{2n-1}), d(u_{2n}, Tu_{2n})\} \leq d(u_{2n-1}, u_{2n}). \quad (2.6)$$

Therefore by the assumption,

$$H(Su_{2n-1}, Tu_{2n}) \leq rM(Su_{2n-1}, Tu_{2n}). \quad (2.7)$$

Case 2. If  $d(u_{2n}, Tu_{2n}) \geq d(u_{2n-1}, Su_{2n-1})$ , then

$$\varphi(r) \min\{d(u_{2n-1}, Su_{2n-1}), d(u_{2n}, Tu_{2n})\} \leq d(u_{2n-1}, u_{2n}). \quad (2.8)$$

So by the assumption,

$$H(Su_{2n-1}, Tu_{2n}) \leq rM(Su_{2n-1}, Tu_{2n}). \quad (2.9)$$

Hence in either case we obtain by (2.7) and (2.9),

$$\begin{aligned} & d(u_{2n}, u_{2n+1}) \\ & \leq H(Su_{2n-1}, Tu_{2n}) + \varepsilon M(Su_{2n-1}, Tu_{2n}) \\ & \leq rM(Su_{2n-1}, Tu_{2n}) + \varepsilon M(Su_{2n-1}, Tu_{2n}) = \beta M(Su_{2n-1}, Tu_{2n}) \\ & = \beta \max\left\{d(u_{2n-1}, u_{2n}), d(u_{2n-1}, Su_{2n-1}), d(u_{2n}, Tu_{2n}), \frac{d(u_{2n-1}, Tu_{2n}) + d(u_{2n}, Su_{2n-1})}{2}\right\} \\ & \leq \beta \max\{d(u_{2n-1}, u_{2n}), d(u_{2n}, u_{2n+1})\}. \end{aligned} \quad (2.10)$$

This yields (2.5). Analogously, we obtain  $d(u_{2n+2}, u_{2n+1}) \leq \beta d(u_{2n+1}, u_{2n})$ , and conclude that for any  $n \in N$ ,

$$d(u_{n+1}, u_n) \leq \beta d(u_n, u_{n-1}). \quad (2.11)$$

Therefore  $\{u_n\}$  is a Cauchy sequence and has a limit in  $X$ . Call it  $u$ .

Now we show that for any  $y \in X - \{u\}$ ,

$$d(u, Ty) \leq r \max\{d(u, y), d(y, Ty)\}, \quad (2.12)$$

$$d(u, Sy) \leq r \max\{d(u, y), d(y, Sy)\}. \quad (2.13)$$

Since  $u_n \rightarrow u$ , there exists  $n_0 \in \mathbb{N}$  (natural numbers) such that

$$d(u, u_n) \leq \frac{1}{3}d(u, y) \quad \text{for } y \neq u \text{ and all } n \geq n_0. \quad (2.14)$$

Then as in [2, page 1862],

$$\begin{aligned} \varphi(r)d(u_{2n-1}, Su_{2n-1}) &\leq d(u_{2n-1}, Su_{2n-1}) \leq d(u_{2n-1}, u_{2n}) \leq d(u_{2n-1}, u) + d(u, u_{2n}) \\ &\leq \frac{2}{3}d(y, u) = d(y, u) - \frac{1}{3}d(y, u) \leq d(y, u) - d(u_{2n-1}, u) \\ &\leq d(u_{2n-1}, y). \end{aligned} \quad (2.15)$$

Therefore

$$\varphi(r)d(u_{2n-1}, Su_{2n-1}) \leq d(u_{2n-1}, y). \quad (2.16)$$

Now either  $d(u_{2n-1}, Su_{2n-1}) \leq d(y, Ty)$  or  $d(y, Ty) \leq d(u_{2n-1}, Su_{2n-1})$ .

So in either case by (2.16),

$$\varphi(r) \min\{d(u_{2n-1}, Su_{2n-1}), d(y, Ty)\} \leq d(u_{2n-1}, y). \quad (2.17)$$

Hence by the assumption (2.1),

$$\begin{aligned} d(u_{2n}, Ty) &\leq H(Su_{2n-1}, Ty) \leq rM(Su_{2n-1}, Ty) \\ &\leq r \max\left\{d(u_{2n-1}, y), d(u_{2n-1}, Su_{2n-1}), d(y, Ty), \frac{d(u_{2n-1}, Ty) + d(y, Su_{2n-1})}{2}\right\}. \end{aligned} \quad (2.18)$$

Making  $n \rightarrow \infty$ ,

$$\begin{aligned} d(u, Ty) &\leq r \max\left\{d(u, y), d(u, u), d(y, Ty), \frac{d(u, Ty) + d(y, u)}{2}\right\} \\ &\leq r \max\{d(u, y), d(y, Ty), d(u, Ty)\}. \end{aligned} \quad (2.19)$$

This yields (2.12). Similarly, we can show (2.13).

Now, we show that  $u \in Su \cap Tu$ .

For  $0 \leq r < 1/2$ , the following cases arise.

*Case 1.* Suppose  $u \notin Su$  and  $u \notin Tu$ . Then as in [8, page 6], let  $a \in Tu$  be such that

$$2rd(a, u) < d(u, Tu), \quad (2.20)$$

and  $a \in Su$  be such that  $2rd(a, u) < d(u, Su)$ .

Since  $a \in Tu$  implies  $a \neq u$ , we have from (2.12) and (2.13),

$$d(u, Ta) \leq r \max\{d(u, a), d(a, Ta)\}, \quad (2.21)$$

$$d(u, Sa) \leq r \max\{d(u, a), d(a, Sa)\}. \quad (2.22)$$

On the other hand, since  $\varphi(r)d(u, Tu) \leq d(u, Tu) \leq d(a, u)$ ,

$$\varphi(r) \min\{d(a, Sa), d(u, Tu)\} \leq d(a, u). \quad (2.23)$$

Therefore by the assumption (2.1),

$$\begin{aligned} d(Sa, a) \leq H(Sa, Tu) &\leq r \max\left\{d(a, u), d(u, Tu), d(a, Sa), \frac{d(u, Sa) + d(a, Tu)}{2}\right\} \\ &= r \max\left\{d(a, u), d(a, Sa), \frac{1}{2}d(u, Sa)\right\}. \end{aligned} \quad (2.24)$$

This gives  $d(a, Sa) \leq H(Sa, Tu) \leq rd(a, u) < d(a, u)$ .

So by (2.22),  $d(Sa, u) \leq rd(a, u)$ . Thus

$$\begin{aligned} d(u, Tu) &\leq d(u, Sa) + H(Sa, Tu) \\ &\leq rd(a, u) + rd(a, u) = 2rd(a, u) < d(u, Tu) \quad (\text{by the assumption of Case 1}). \end{aligned} \quad (2.25)$$

This contradicts  $u \notin Tu$ . Consequently  $u \in Tu$ . Similarly  $u \in Su$ .

*Case 2.* Let  $u \in Su$  and  $u \notin Tu$ . Then as in the previous case, let  $a \in Tu$  be such that

$$2rd(a, u) < d(u, Tu). \quad (2.26)$$

Since  $a \neq u$ , we have from (2.13),

$$d(u, Sa) \leq r \max\{d(u, a), d(a, Sa)\}. \quad (2.27)$$

On the other hand, Since  $\varphi(r)d(u, Tu) \leq d(u, Tu) \leq d(a, u)$ ,

$$\varphi(r) \min\{d(a, Sa), d(u, Tu)\} \leq d(a, u). \quad (2.28)$$

Therefore by the assumption (2.1),

$$\begin{aligned} d(Sa, a) &\leq H(Sa, Tu) \leq r \max \left\{ d(a, u), d(u, Tu), d(a, Sa), \frac{d(u, Sa) + d(a, Tu)}{2} \right\} \\ &= r \max \left\{ d(a, u), d(a, Sa), \frac{1}{2}d(u, Sa) \right\}. \end{aligned} \quad (2.29)$$

This gives  $d(a, Sa) \leq H(Sa, Tu) \leq rd(a, u) < d(a, u)$ .

So by (2.22),  $d(Sa, u) \leq rd(a, u)$ . Thus

$$\begin{aligned} d(u, Tu) &\leq d(u, Sa) + H(Sa, Tu) \\ &\leq rd(a, u) + rd(a, u) = 2rd(a, u) < d(u, Tu) \quad (\text{by the assumption of Case 2}). \end{aligned} \quad (2.30)$$

This contradicts  $u \notin Tu$ . Consequently  $u \in Tu$ .

*Case 3.*  $u \in Tu$  and  $u \notin Su$ . As in the previous case, it follows that  $u \in Su$ .

Now we consider the case  $1/2 \leq r < 1$ .

First we show that

$$H(Sx, Tu) \leq r \max \left\{ d(x, u), d(x, Sx), d(u, Tu), \frac{d(x, Tu) + d(u, Sx)}{2} \right\}. \quad (2.31)$$

Assume that  $x \neq u$ . Then for every  $n \in \mathbb{N}$ , there exists  $z_n \in Sx$  such that

$$d(u, z_n) \leq d(u, Sx) + \frac{1}{n}d(x, u). \quad (2.32)$$

Therefore

$$\begin{aligned} d(x, Sx) &\leq d(x, z_n) \leq d(x, u) + d(u, z_n) \\ &\leq d(x, u) + d(u, Sx) + \frac{1}{n}d(x, u). \end{aligned} \quad (2.33)$$

Using (2.13) with  $y = x$ , (2.33) implies

$$d(x, Sx) \leq d(x, u) + r \max\{d(x, u), d(x, Sx)\} + \frac{1}{n}d(u, x). \quad (2.34)$$

If  $d(x, u) \geq d(x, Sx)$ , then (2.34) gives

$$\begin{aligned} d(x, Sx) &\leq d(x, u) + rd(x, u) + \frac{1}{n}d(u, x) \\ &= \left(1 + r + \frac{1}{n}\right)d(x, u). \end{aligned} \quad (2.35)$$

Making  $n \rightarrow \infty$ ,

$$d(x, Sx) \leq (1 + r)d(x, u). \quad (2.36)$$

Thus  $\varphi(r)d(x, Sx) = (1 - r)d(x, Sx) \leq (1/(1 + r))d(x, Sx) \leq d(x, u)$ .

Then  $\varphi(r) \min\{d(x, Sx), d(u, Tu)\} \leq d(x, u)$ , and by the assumption (2.1),

$$H(Sx, Tu) \leq r \max\left\{d(x, u), d(x, Sx), d(u, Tu), \frac{d(x, Tu) + d(u, Sx)}{2}\right\}. \quad (2.37)$$

If  $d(x, u) < d(x, Sx)$ , then (2.34) gives

$$d(x, Sx) \leq d(x, u) + rd(x, Sx) + \frac{1}{n}d(u, x), \quad (2.38)$$

that is,  $(1 - r)d(x, Sx) \leq (1 + (1/n))d(x, u)$ . □

Making  $n \rightarrow \infty$ ,

$$\varphi(r)d(x, Sx) \leq d(x, u). \quad (2.39)$$

Then  $\varphi(r) \min\{d(x, Sx), d(u, Tu)\} \leq d(x, u)$ , and by the assumption, we get (2.37).

Taking  $x = u_{2n+1}$  in (2.37) and passing to the limit, we obtain

$$d(u, Tu) \leq rd(u, Tu). \quad (2.40)$$

This gives  $u \in Tu$ . Analogously,  $u \in Su$ .

The following result generalizes Theorem 1.2.

**Corollary 2.3.** *Let  $X$  be a complete metric space and  $S, T$  maps from  $X$  into  $X$ . Suppose there exists  $r \in [0, 1)$  such that for every  $x, y \in X$ ,*

$$\varphi(r) \min\{d(x, Sx), d(y, Ty)\} \leq d(x, y) \text{ implies } d(Sx, Ty) \leq rM(Sx, Ty). \quad (2.41)$$

*Then  $S$  and  $T$  have a unique common fixed point.*

*Proof.* For single-valued maps  $S$  and  $T$ , it comes from Theorem 2.2 that they have a common fixed point. The uniqueness of the common fixed point follows easily. □

*Remark 2.4.* Theorem 1.1 is obtained as a particular case of Theorem 2.2 when  $S = T$ .

Now we derive the following result due to Đorić and Lazović [9, Corollary 2.3].



**Corollary 2.5.** Let  $X$  be a complete metric space and  $T$  a map from  $X$  into  $X$ . Suppose there exists  $r \in [0, 1)$  such that for every  $x, y \in X$ ,

$$\varphi(r)d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq rM(Tx, Ty). \quad (2.42)$$

Then  $T$  has a unique fixed point.

*Proof.* It comes from Corollary 2.3 when  $S = T$ .  $\square$

The following example shows the generality of our results.

*Example 2.6.* Let  $X = \{(0, 0), (0, 4), (4, 0), (0, 5), (5, 0), (4, 5), (5, 4)\}$  be endowed with the metric  $d$  defined by

$$d[(x_1, x_2), (y_1, y_2)] = |x_1 - y_1| + |x_2 - y_2|. \quad (2.43)$$

Let  $S$  and  $T$  be such that

$$S(x_1, x_2) = \begin{cases} (x_1, 0) & \text{if } x_1 \leq x_2 \\ (0, 0) & \text{if } x_1 > x_2, \end{cases} \quad T(x_1, x_2) = \begin{cases} (x_2, 0) & \text{if } x_1 \leq x_2 \\ (0, x_2) & \text{if } x_1 > x_2. \end{cases} \quad (2.44)$$

Then  $S$  and  $T$  do not satisfy the condition (1.6) of Theorem 1.2 at  $x = (4, 5)$ ,  $y = (5, 4)$ . However, this is readily verified that all the hypotheses of Corollary 2.3 are satisfied for the maps  $S$  and  $T$ .

**Theorem 2.7.** Let  $X$  be a complete metric space and  $P, Q : X \rightarrow BN(X)$ . Assume there exists  $r \in [0, 1)$  such that for every  $x, y \in X$ ,

$$\varphi(r) \min\{\rho(x, Px), \rho(y, Qy)\} \leq d(x, y) \quad (2.45)$$

implies

$$\rho(Px, Qy) \leq r \max\left\{d(x, y), \rho(x, Px), \rho(y, Qy), \frac{d(x, Qy) + d(y, Px)}{2}\right\}. \quad (2.46)$$

Then there exists a unique point  $z \in X$  such that  $z \in Pz \cap Qz$ .

*Proof.* Choose  $\lambda \in (0, 1)$ . Define single-valued maps  $S, T : X \rightarrow X$  as follows. For each  $x \in X$ , let  $Sx$  be a point of  $Px$  which satisfies

$$d(x, Sx) \geq r^\lambda \rho(x, Px). \quad (2.47)$$

Similarly, for each  $y \in X$ , let  $Ty$  be a point of  $Qy$  such that

$$d(y, Ty) \geq r^\lambda \rho(y, Qy). \quad (2.48)$$

Since  $Sx \in Px$  and  $Ty \in Qy$ ,

$$d(x, Sx) \leq \rho(x, Px), \quad d(y, Ty) \leq \rho(y, Qy). \quad (2.49)$$

So, (2.45) gives

$$\varphi(r) \min\{d(x, Sx), d(y, Ty)\} \leq \varphi(r) \min\{\rho(x, Px), \rho(y, Qy)\} \leq d(x, y), \quad (2.50)$$

and this implies (2.46). Therefore

$$\begin{aligned} d(Sx, Ty) &\leq \rho(Px, Qy) \\ &\leq r \cdot r^{-\lambda} \max\left\{r^\lambda d(x, y), r^\lambda \rho(x, Px), r^\lambda \rho(y, Qy), \frac{r^\lambda d(x, Qy) + r^\lambda d(y, Px)}{2}\right\} \\ &\leq r^{1-\lambda} \max\left\{d(x, y), d(x, Sx), d(y, Ty), \frac{d(x, Ty) + d(y, Sx)}{2}\right\}. \end{aligned} \quad (2.51)$$

So (2.50), namely,  $\varphi(r') \min\{d(x, Sx), d(y, Ty)\} \leq d(x, y)$  implies

$$d(Sx, Ty) \leq r' \max\left\{d(x, y), d(x, Sx), d(y, Ty), \frac{d(x, Ty) + d(y, Sx)}{2}\right\}, \quad (2.52)$$

where  $r' = r^{1-\lambda} < 1$ .

Hence by Theorem 2.2,  $S$  and  $T$  have a unique point  $z \in X$  such that  $Sz = Tz = z$ . This implies  $z \in Pz \cap Qz$ .  $\square$

**Corollary 2.8.** *Let  $X$  be a complete metric space and  $P : X \rightarrow BN(X)$ . Assume there exists  $r \in [0, 1)$  such that for every  $x, y \in X$ ,*

$$\begin{aligned} \rho(x, Px) &\leq (1+r)d(x, y) \text{ implies} \\ \rho(Px, Py) &\leq r \max\left\{d(x, y), \rho(x, Px), \rho(y, Py), \frac{d(x, Py) + d(y, Px)}{2}\right\}. \end{aligned} \quad (2.53)$$

Then there exists a unique point  $z \in X$  such that  $z \in Pz$ .

*Proof.* It comes from Theorem 2.7 when  $Q = P$ .  $\square$

### 3. Applications

Throughout this section, we assume that  $Y$  and  $Z$  are Banach spaces,  $W \subseteq Y$  and  $D \subseteq Z$ . Let  $R$  denotes the field of reals,  $g_1, g_2 : W \times D \rightarrow R$  and  $G_1, G_2 : W \times D \times R \rightarrow R$ . Taking  $W$  and  $D$

as the state and decision spaces, respectively, the problem of dynamic programming reduces to the problem of solving functional equations:

$$p_i = \sup_{y \in D} \{g_i(x, y) + H_i(x, y, p_i(x, y))\}, \quad x \in W, \quad i = 1, 2. \quad (3.1)$$

In the multistage process, some functional equations arise in a natural way (cf. [22, 23]; see also [21, 24, 28, 29]). In this section, we study the existence of common solution of the functional equations (3.1) arising in dynamic programming.

Let  $B(W)$  denotes the set of all bounded real-valued functions on  $W$ . For an arbitrary  $h \in B(W)$ , define  $\|h\| = \sup_{x \in W} |h(x)|$ . Then  $(B(W), \|\cdot\|)$  is a Banach space. Suppose that the following conditions hold:

(DP-1)  $H_1, H_2, g_1,$  and  $g_2$  are bounded.

(DP-2) There exists  $r \in [0, 1)$  such that for every  $(x, y) \in W \times D, h, k \in B(W)$  and  $t \in W$ ,

$$\varphi(r) \min\{|h(t) - A_1 h(t)|, |k(t) - A_2 k(t)|\} \leq |h(t) - k(t)| \quad (3.2)$$

implies

$$\begin{aligned} & |H_1(x, y, h(t)) - H_2(x, y, k(t))| \\ & \leq r \max \left\{ |h(t) - k(t)|, |h(t) - A_1 h(t)|, |k(t) - A_2 k(t)|, \frac{|h(t) - A_2 k(t)| + |k(t) - A_1 h(t)|}{2} \right\}, \end{aligned} \quad (3.3)$$

where  $A_1, A_2$  are defined as follows:

$$A_i h(x) = \sup_{y \in D} H_i(x, y, h(x, y)), \quad x \in W, \quad h \in B(W), \quad i = 1, 2. \quad (3.4)$$

**Theorem 3.1.** *Assume the conditions (DP-1) and (DP-2). Then the functional equations (3.1),  $i = 1, 2$ , have a unique common solution in  $B(W)$ .*

*Proof.* For any  $h, k \in B(W)$ , let  $d(h, k) = \sup\{|h(x) - k(x)| : x \in W\}$ . Then  $(B(W), d)$  is a complete metric space.

Let  $\lambda$  be any arbitrary positive number and  $h_1, h_2 \in B(W)$ . Pick  $x \in W$  and choose  $y_1, y_2 \in D$  such that

$$A_i h_i < H_i(x, y_i, h_i(x_i)) + \lambda, \quad (3.5)$$

where  $x_i = (x, y_i)$ ,  $i = 1, 2$ .

Further,

$$A_1 h_1 \geq H_1(x, y_2, h_1(x_2)), \quad (3.6)$$

$$A_2 h_2 \geq H_2(x, y_1, h_2(x_1)). \quad (3.7)$$

Therefore, the first inequality in (DP-2) becomes

$$\varphi(r) \min\{|h_1(x) - A_1 h_1(x)|, |h_2(x) - A_2 h_2(x)|\} \leq |h_1(x) - h_2(x)|, \quad (3.8)$$

and this together with (3.5) and (3.7) implies

$$\begin{aligned} A_1 h_1 - A_2 h_2 &< H_1(x, y_1, h_1(x_1)) - H_2(x, y, h_2(x_1)) + \lambda \\ &\leq |H_1(x, y_1, h_1(x_1)) - H_2(x, y_1, h_2(x_1))| + \lambda \\ &\leq rM(H_1 h_1, H_2 h_2) + \lambda. \end{aligned} \quad (3.9)$$

Similarly, (3.5), (3.6), and (3.8) imply

$$A_2 h_2(x) - A_1 h_1(x) \leq rM(A_1 h_1, A_2 h_2) + \lambda. \quad (3.10)$$

So, from (3.10) and (3.11), we obtain

$$|A_1 h_1(x) - A_2 h_2(x)| \leq r M(A_1 h_1, A_2 h_2) + \lambda. \quad (3.11)$$

Since this inequality is true for any  $x \in W$ , and  $\lambda > 0$  is arbitrary, on taking supremum, we find from (3.8) and (3.11) that

$$\varphi(r) \min\{d(h_1, A_1 h_1), d(h_2, A_2 h_2)\} \leq d(h_1, h_2) \quad (3.12)$$

implies

$$d(A_1 h_1, A_2 h_2) \leq rM(A_1 h_1, A_2 h_2). \quad (3.13)$$

Therefore, Corollary 2.3 applies, wherein  $A_1$  and  $A_2$  correspond, respectively, to the maps  $S$  and  $T$ . So  $A_1$  and  $A_2$  have a unique common fixed point  $h^*$ , that is,  $h^*(x)$  is the unique bounded common solution of the functional equations (3.1),  $i = 1, 2$ .  $\square$

The following result generalizes a recent result of Singh and Mishra [12, Corollary 4.2] which in turn extends certain results from [21, 23, 24].

**Corollary 3.2.** *Suppose that the following conditions hold.*

- (i)  $G$  and  $g$  are bounded.
- (ii) There exists  $r \in [0, 1)$  such that for every  $x, y \in W \times D$ ,  $h, k \in B(W)$  and  $t \in W$ ,

$$\begin{aligned} \varphi(r)|h(t) - Kh(t)| &\leq |h(t) - k(t)| \text{ implies} \\ |G(x, y, h(t)) - G(x, y, k(t))| &\leq r \max M(K, h(t), k(t)), \end{aligned} \quad (3.14)$$

where  $K$  is defined as

$$Kh(t) = \sup_{y \in D} \{g(t, y) + G(t, y, h(t, y))\}, \quad t \in W, h \in B(W). \quad (3.15)$$

Then the functional equation (3.1) with  $H_1 = H_2 = G$  and  $g_1 = g_2 = g$  possesses a unique bounded solution in  $W$ .

*Proof.* It comes from Theorem 3.1 when  $g_1 = g_2 = g$  and  $H_1 = H_2 = G$ .  $\square$

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