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Research Article

Oscillation of Third-Order Neutral Delay Differential Equations

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The purpose of this paper is to examine oscillatory properties of the third-order neutral delay differential equation $[a(t)(b(t)(x(t)+p(t)x(\sigma(t)))')']'+q(t)x(\tau(t))=0$. Some oscillatory and asymptotic criteria are presented. These criteria improve and complement those results in the literature. Moreover, some examples are given to illustrate the main results.

1. Introduction

This paper is concerned with the oscillation and asymptotic behavior of the third-order neutral differential equation

$$\left[a(t)\left(b(t)\left(x(t)+p(t)x(\sigma(t))\right)'\right)'\right]'+q(t)x(\tau(t))=0. \tag{E}$$

We always assume that

(H1) $a(t), b(t), p(t), q(t) \in C([t_0,\infty)), a(t)>0, b(t)>0, q(t)>0$,

(H2) $\tau(t), \sigma(t) \in C([t_0,\infty)), \tau(t)\le t, \sigma(t)\le t, \lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\sigma(t)=\infty$.

We set $z(t) := x(t)+p(t)x(\sigma(t))$. By a solution of (E), we mean a nontrivial function $x(t)\in C([T_x,\infty)), T_x\ge t_0$, which has the properties $z(t)\in C^1([T_x,\infty)), b(t)z'(t)\in C^1([T_x,\infty))$, $a(t)(b(t)z'(t))'\in C^1([T_x,\infty))$ and satisfies (E) on $[T_x,\infty)$. We consider only those solutions $x(t)$ of (E) which satisfy $\sup\{|x(t)| : t\ge T\}>0$ for all $T\ge T_x$. We assume that (E) possesses such a solution. A solution of (E) is called oscillatory if it has arbitrarily large zeros on $[T_x,\infty)$;

otherwise, it is called nonoscillatory. Equation (E) is said to be almost oscillatory if all its solutions are oscillatory or convergent to zero asymptotically.

Recently, great attention has been devoted to the oscillation of differential equations; see, for example, the papers [1–30]. Hartman and Wintner [9], Hanan [10], and Erbe [8] studied a particular case of (E), namely, the third-order differential equation

$$x'''(t) + q(t)x(t) = 0. \tag{1.1}$$

Equation (E) with $p(t) = 0$ plays an important role in the study of the oscillation of third-order trinomial delay differential equation

$$x'''(t) + p(t)x'(t) + g(t)x(\tau(t)) = 0, \tag{1.2}$$

see [6, 12, 27]. Baculíková and Džurina [21, 22], Candan and Dahiya [25], Grace et al. [28], and Saker and Džurina [30] examined the oscillation behavior of (E) with $p(t) = 0$. It seems that there are few results on the oscillation of (E) with a neutral term. Baculíková and Džurina [23, 24] and Thandapani and Li [17] investigated the oscillation of (E) under the assumption

$$b(t) = 1, \qquad \int_{t_0}^{\infty} \frac{1}{a(t)} \mathrm{d}t = \infty, \quad a'(t) \geq 0. \tag{1.3}$$

Graef et al. [13] and Candan and Dahiya [26] considered the oscillation of

$$\left[a(t)\Big(b(t)\big(x(t) + p_1 x(t-\sigma)\big)'\Big)'\right]' + q(t)x(t-\tau) = 0, \quad 0 \leq p_1 < 1. \tag{E_1}$$

In this paper, we shall further the investigation of the oscillations of (E) and (E_1). Three cases:

$$\int_{t_0}^{\infty} \frac{1}{a(t)} \mathrm{d}t = \infty, \qquad \int_{t_0}^{\infty} \frac{1}{b(t)} \mathrm{d}t = \infty, \tag{1.4}$$

$$\int_{t_0}^{\infty} \frac{1}{a(t)} \mathrm{d}t < \infty, \qquad \int_{t_0}^{\infty} \frac{1}{b(t)} \mathrm{d}t = \infty, \tag{1.5}$$

$$\int_{t_0}^{\infty} \frac{1}{a(t)} \mathrm{d}t < \infty, \qquad \int_{t_0}^{\infty} \frac{1}{b(t)} \mathrm{d}t < \infty, \tag{1.6}$$

are studied.

In the following, all functional inequalities considered in this paper are assumed to hold eventually, that is, they are satisfied for all t large enough. Without loss of generality, we can deal only with the positive solutions of (E).

2. Main Results

In this section, we will give the main results.

Theorem 2.1. *Assume that* (1.4) *holds,* $0 \le p(t) \le p_1 < 1$*. If for some function* $\rho \in C^1([t_0,\infty),(0,\infty))$*, for all sufficiently large* $t_1 \ge t_0$ *and for* $t_3 > t_2 > t_1$*, one has*

$$\limsup_{t\to\infty}\int_{t_3}^{t}\left(\rho(s)q(s)(1-p(\tau(s)))\frac{\int_{t_2}^{\tau(s)}\left(\int_{t_1}^{v}(1/a(u))\mathrm{d}u/b(v)\right)\mathrm{d}v}{\int_{t_1}^{s}(1/a(u))\mathrm{d}u}-\frac{a(s)\left(\rho'(s)\right)^2}{4\rho(s)}\right)\mathrm{d}s=\infty, \tag{2.1}$$

$$\int_{t_0}^{\infty}\frac{1}{b(v)}\int_{v}^{\infty}\frac{1}{a(u)}\int_{u}^{\infty}q(s)\mathrm{d}s\,\mathrm{d}u\,\mathrm{d}v=\infty, \tag{2.2}$$

then (*E*) *is almost oscillatory.*

Proof. Assume that x is a positive solution of (E). Based on the condition (1.4), there exist two possible cases:

(1) $z(t)>0, z'(t)>0,\ (b(t)z'(t))'>0, [a(t)(b(t)z'(t))']'<0$,

(2) $z(t)>0, z'(t)<0, (b(t)z'(t))'>0, [a(t)(b(t)z'(t))']'<0$ for $t\ge t_1$, t_1 is large enough.

Assume that case (1) holds. We define the function ω by

$$\omega(t)=\rho(t)\frac{a(t)(b(t)z'(t))'}{b(t)z'(t)},\quad t\ge t_1. \tag{2.3}$$

Then, $\omega(t)>0$ for $t\ge t_1$. Using $z'(t)>0$, we have

$$x(t)\ge(1-p(t))z(t). \tag{2.4}$$

Since

$$b(t)z'(t)\ge\int_{t_1}^{t}\frac{a(s)(b(s)z'(s))'}{a(s)}\ge a(t)(b(t)z'(t))'\int_{t_1}^{t}\frac{1}{a(s)}\mathrm{d}s, \tag{2.5}$$

we have that

$$\left(\frac{b(t)z'(t)}{\int_{t_1}^{t}(1/a(s))\mathrm{d}s}\right)'\le 0. \tag{2.6}$$

Thus, we get

$$\begin{aligned}z(t)&=z(t_2)+\int_{t_2}^{t}\frac{b(s)z'(s)}{\int_{t_1}^{s}(1/a(u))\mathrm{d}u}\frac{\int_{t_1}^{s}(1/a(u))\mathrm{d}u}{b(s)}\mathrm{d}s\\&\ge\frac{b(t)z'(t)}{\int_{t_1}^{t}(1/a(u))\mathrm{d}u}\int_{t_2}^{t}\frac{\int_{t_1}^{s}(1/a(u))\mathrm{d}u}{b(s)}\mathrm{d}s,\end{aligned} \tag{2.7}$$

for $t \geq t_2 > t_1$. Differentiating (2.3), we obtain

$$\omega(t) = \rho'(t)\frac{a(t)(b(t)z'(t))'}{b(t)z'(t)} + \rho(t)\frac{\left(a(t)(b(t)z'(t))'\right)'}{b(t)z'(t)} - \rho(t)\frac{a(t)(b(t)z'(t))'(b(t)z'(t))'}{(b(t)z'(t))^2}. \quad (2.8)$$

It follows from (E), (2.3), and (2.4) that

$$\omega'(t) \leq \frac{\rho'(t)}{\rho(t)}\omega(t) - \rho(t)q(t)(1-p(\tau(t)))\frac{z(\tau(t))}{b(t)z'(t)} - \frac{\omega^2(t)}{\rho(t)a(t)}, \quad (2.9)$$

that is,

$$\omega'(t) \leq \frac{\rho'(t)}{\rho(t)}\omega(t) - \rho(t)q(t)(1-p(\tau(t)))\frac{z(\tau(t))}{b(\tau(t))z'(\tau(t))}\frac{b(\tau(t))z'(\tau(t))}{b(t)z'(t)} - \frac{\omega^2(t)}{\rho(t)a(t)}, \quad (2.10)$$

which follows from (2.6) and (2.7) that

$$\begin{aligned}
\omega'(t) &\leq \frac{\rho'(t)}{\rho(t)}\omega(t) - \rho(t)q(t)(1-p(\tau(t)))\frac{\int_{t_2}^{\tau(t)}\left(\int_{t_1}^{s}(1/a(u))\mathrm{d}u/b(s)\right)\mathrm{d}s}{\int_{t_1}^{\tau(t)}(1/a(u))\mathrm{d}u}\frac{\int_{t_1}^{\tau(t)}(1/a(u))\mathrm{d}u}{\int_{t_1}^{t}(1/a(u))\mathrm{d}u} - \frac{\omega^2(t)}{\rho(t)a(t)} \\
&= \frac{\rho'(t)}{\rho(t)}\omega(t) - \rho(t)q(t)(1-p(\tau(t)))\frac{\int_{t_2}^{\tau(t)}\left(\int_{t_1}^{s}(1/a(u))\mathrm{d}u/b(s)\right)\mathrm{d}s}{\int_{t_1}^{t}(1/a(u))\mathrm{d}u} - \frac{\omega^2(t)}{\rho(t)a(t)}.
\end{aligned} \quad (2.11)$$

Hence, we have

$$\omega'(t) \leq -\rho(t)q(t)(1-p(\tau(t)))\frac{\int_{t_2}^{\tau(t)}\left(\int_{t_1}^{s}(1/a(u))\mathrm{d}u/b(s)\right)\mathrm{d}s}{\int_{t_1}^{t}(1/a(u))\mathrm{d}u} + \frac{a(t)\left(\rho'(t)\right)^2}{4\rho(t)}. \quad (2.12)$$

Integrating the last inequality from $t_3 (> t_2)$ to t, we get

$$\int_{t_3}^{t}\left(\rho(s)q(s)(1-p(\tau(s)))\frac{\int_{t_2}^{\tau(s)}\left(\int_{t_1}^{v}(1/a(u))\mathrm{d}u/b(v)\right)\mathrm{d}v}{\int_{t_1}^{s}(1/a(u))\mathrm{d}u} - \frac{a(s)\left(\rho'(s)\right)^2}{4\rho(s)}\right)\mathrm{d}s \leq \omega(t_3), \quad (2.13)$$

which contradicts (2.1).

Assume that case (2) holds. Using the similar proof of [23, Lemma 2], we can get $\lim_{t\to\infty} x(t) = 0$ due to condition (2.2). This completes the proof. □

Theorem 2.2. *Assume that* (1.5) *holds,* $0 \le p(t) \le p_1 < 1$. *Further, assume that for some function* $\rho \in C^1([t_0,\infty),(0,\infty))$, *for all sufficiently large* $t_1 \ge t_0$ *and for* $t_3 > t_2 > t_1$, *one has* (2.1) *and* (2.2). *If*

$$\limsup_{t\to\infty}\int_{t_2}^{t}\left(\delta(s)q(s)(1-p(\tau(s)))\int_{t_1}^{\tau(s)}\frac{dv}{b(v)}-\frac{1}{4\delta(s)a(s)}\right)ds=\infty, \tag{2.14}$$

where

$$\delta(t) := \int_t^{\infty}\frac{1}{a(s)}ds, \tag{2.15}$$

then (E) is almost oscillatory.

Proof. Assume that x is a positive solution of (E). Based on the condition (1.5), there exist three possible cases (1), (2) (as those of Theorem 2.1), and

(3) $z(t) > 0$, $z'(t) > 0$, $(b(t)z'(t))' < 0$, $[a(t)(b(t)z'(t))']' < 0$, for $t \ge t_1$, t_1 is large enough.

Assume that case (1) and case (2) hold, respectively. We can obtain the conclusion of Theorem 2.2 by applying the proof of Theorem 2.1.

Assume that case (3) holds. From $[a(t)(b(t)z'(t))']' < 0$, $a(t)(b(t)z'(t))'$ is decreasing. Thus, we get

$$a(s)\big(b(s)z'(s)\big)' \le a(t)\big(b(t)z'(t)\big)', \quad s \ge t \ge t_1. \tag{2.16}$$

Dividing the above inequality by $a(s)$ and integrating it from t to l, we obtain

$$b(l)z'(l) \le b(t)z'(t) + a(t)\big(b(t)z'(t)\big)'\int_t^{l}\frac{ds}{a(s)}. \tag{2.17}$$

Letting $l \to \infty$, we have

$$0 \le b(t)z'(t) + a(t)\big(b(t)z'(t)\big)'\int_t^{\infty}\frac{ds}{a(s)}, \tag{2.18}$$

that is,

$$-\int_t^{\infty}\frac{ds}{a(s)}\frac{a(t)(b(t)z'(t))'}{b(t)z'(t)} \le 1. \tag{2.19}$$

Define function ϕ by

$$\phi(t) := \frac{a(t)(b(t)z'(t))'}{b(t)z'(t)}, \quad t \ge t_1. \tag{2.20}$$

Then, $\phi(t) < 0$ for $t \geq t_1$. Hence, by (2.19) and (2.20), we get

$$-\delta(t)\phi(t) \leq 1. \tag{2.21}$$

Differentiating (2.20), we obtain

$$\phi'(t) = \frac{\left(a(t)\left(b(t)z'(t)\right)'\right)'}{b(t)z'(t)} - \frac{a(t)\left(b(t)z'(t)\right)'\left(b(t)z'(t)\right)'}{\left(b(t)z'(t)\right)^2}. \tag{2.22}$$

Using $z'(t) > 0$, we have (2.4). From (E) and (2.4), we have

$$\phi'(t) \leq -q(t)\left(1 - p(\tau(t))\right)\frac{z(\tau(t))}{b(t)z'(t)} - \frac{a(t)\left(b(t)z'(t)\right)'\left(b(t)z'(t)\right)'}{\left(b(t)z'(t)\right)^2}. \tag{2.23}$$

In view of (3), we see that

$$z(t) \geq b(t)\int_{t_1}^{t} \frac{\mathrm{d}s}{b(s)} z'(t). \tag{2.24}$$

Hence,

$$\left(\frac{z(t)}{\int_{t_1}^{t}(\mathrm{d}s/b(s))}\right)' \leq 0, \tag{2.25}$$

which implies that

$$\frac{z(\tau(t))}{z(t)} \geq \frac{\int_{t_1}^{\tau(t)}(\mathrm{d}s/b(s))}{\int_{t_1}^{t}(\mathrm{d}s/b(s))}. \tag{2.26}$$

By (2.20) and (2.23), (2.24), and (2.26), we obtain

$$\phi'(t) \leq -q(t)\left(1 - p(\tau(t))\right)\int_{t_1}^{\tau(t)} \frac{\mathrm{d}s}{b(s)} - \frac{\phi^2(t)}{a(t)}. \tag{2.27}$$

Multiplying the last inequality by $\delta(t)$ and integrating it from $t_2(> t_1)$ to t, we have

$$\phi(t)\delta(t)-\phi(t_2)\delta(t_2)+\int_{t_2}^{t}\delta(s)q(s)\left(1-p(\tau(s))\right)\int_{t_1}^{\tau(s)}\frac{\mathrm{d}v}{b(v)}\mathrm{d}s+\int_{t_2}^{t}\frac{\phi^2(s)\delta(s)}{a(s)}\mathrm{d}s+\int_{t_2}^{t}\frac{\phi(s)}{a(s)}\mathrm{d}s\leq 0, \tag{2.28}$$

which follows that

$$\int_{t_2}^{t}\left(\delta(s)q(s)(1-p(\tau(s)))\int_{t_1}^{\tau(s)}\frac{dv}{b(v)}-\frac{1}{4\delta(s)a(s)}\right)ds \le 1+\phi(t_2)\delta(t_2) \tag{2.29}$$

due to (2.21), which contradicts (2.14). This completes the proof. □

Theorem 2.3. *Assume that* (1.6) *holds,* $0 \le p(t) \le p_1 < 1$. *Further, assume that for some function* $\rho \in C^1([t_0,\infty),(0,\infty))$, *for all sufficiently large* $t_1 \ge t_0$ *and for* $t_3 > t_2 > t_1$, *one has* (2.1), (2.2), *and* (2.14). *If*

$$\int_{t_1}^{\infty}\frac{1}{b(v)}\int_{t_1}^{v}\frac{1}{a(u)}\int_{t_1}^{u}\eta(s)q(s)\xi(\tau(s))ds\,du\,dv=\infty, \tag{2.30}$$

where

$$\eta(t) := 1-p(\tau(t))\frac{\xi(\sigma(\tau(t)))}{\xi(\tau(t))} > 0, \qquad \xi(t) := \int_t^{\infty}\frac{1}{b(s)}ds, \tag{2.31}$$

then (E) is almost oscillatory.

Proof. Assume that x is a positive solution of (E). Based on the condition (1.6), there exist four possible cases (1), (2), and (3) (as those of Theorem 2.2), and

(4) $z(t) > 0$, $z'(t) < 0$, $(b(t)z'(t))' < 0$, $[a(t)(b(t)z'(t))']' < 0$, for $t \ge t_1$, t_1 is large enough.

Assume that case (1), case (2), and case (3) hold, respectively. We can obtain the conclusion of Theorem 2.3 by using the proof of Theorem 2.2.

Assume that case (4) holds. Since $(b(t)z'(t))' < 0$, we get

$$z'(s) \le \frac{b(t)z'(t)}{b(s)}, \quad s \ge t, \tag{2.32}$$

which implies that

$$z(t) \ge -\xi(t)b(t)z'(t) \ge L\xi(t) \tag{2.33}$$

for some constant $L > 0$. By (2.33), we obtain

$$\left(\frac{z(t)}{\xi(t)}\right)' \ge 0. \tag{2.34}$$

Using (2.34), we see that

$$x(t) = z(t)-p(t)x(\sigma(t)) \ge z(t)-p(t)z(\sigma(t)) \ge \left(1-p(t)\frac{\xi(\sigma(t))}{\xi(t)}\right)z(t). \tag{2.35}$$

From (E), (2.33), and (2.35), we have

$$\left[a(t)\left(b(t)z'(t)\right)'\right]' + Lq(t)\left(1 - p(\tau(t))\frac{\xi(\sigma(\tau(t)))}{\xi(\tau(t))}\right)\xi(\tau(t)) \le 0. \tag{2.36}$$

Integrating the last inequality from t_1 to t, we get

$$a(t)\left(b(t)z'(t)\right)' + L\int_{t_1}^{t} q(s)\left(1 - p(\tau(s))\frac{\xi(\sigma(\tau(s)))}{\xi(\tau(s))}\right)\xi(\tau(s))\mathrm{d}s \le 0. \tag{2.37}$$

Integrating again, we have

$$b(t)z'(t) + L\int_{t_1}^{t}\frac{1}{a(u)}\int_{t_1}^{u} q(s)\left(1 - p(\tau(s))\frac{\xi(\sigma(\tau(s)))}{\xi(\tau(s))}\right)\xi(\tau(s))\mathrm{d}s\,\mathrm{d}u \le 0. \tag{2.38}$$

Integrating again, we obtain

$$z(t_1) \ge L\int_{t_1}^{t}\frac{1}{b(v)}\int_{t_1}^{v}\frac{1}{a(u)}\int_{t_1}^{u} q(s)\left(1 - p(\tau(s))\frac{\xi(\sigma(\tau(s)))}{\xi(\tau(s))}\right)\xi(\tau(s))\mathrm{d}s\,\mathrm{d}u\,\mathrm{d}v + z(t), \tag{2.39}$$

which contradicts (2.30). This completes the proof. □

Theorem 2.4. *Assume that* (1.6) *holds,* $0 \le p(t) \le p_1 < 1$. *Further, assume that for some function* $\rho \in C^1([t_0,\infty),(0,\infty))$, *for all sufficiently large* $t_1 \ge t_0$ *and for* $t_3 > t_2 > t_1$, *one has* (2.1), (2.2) *and* (2.14). *If*

$$\int_{t_1}^{\infty}\frac{1}{b(v)}\int_{t_1}^{v}\frac{1}{a(u)}\int_{t_1}^{u} q(s)\mathrm{d}s\,\mathrm{d}u\,\mathrm{d}v = \infty, \tag{2.40}$$

then (E) is almost oscillatory.

Proof. Assume that x is a positive solution of (E). Based on the condition (1.6), there exist four possible cases (1), (2), (3), and (4) (as those of Theorem 2.3).

Assume that case (1), case (2), and case (3) hold, respectively. We can obtain the conclusion of Theorem 2.4 by using the proof of Theorem 2.2.

Assume that case (4) holds. Then, $\lim_{t\to\infty} z(t) = l \ge 0$($l$ is finite). Assume that $l > 0$. Then, from the proof of [23, Lemma 2], we see that there exists a constant $k > 0$ such that

$$x(t) \ge kl. \tag{2.41}$$

The rest of the proof is similar to that of Theorem 2.3 and hence is omitted. □

3. Examples

In this section, we will present some examples to illustrate the main results.

Example 3.1. Consider the third-order neutral delay differential equation

$$\left(t\left(x(t)+p_1x\left(\frac{t}{2}\right)\right)''\right)'+\frac{\lambda}{t^2}x(t)=0,\quad \lambda>0,\ t\geq 1, \tag{3.1}$$

where $p_1\in[0,1)$.

Let $\rho(t)=t$. It follows from Theorem 2.1 that every solution x of (3.1) is either oscillatory or $\lim_{t\to\infty}x(t)=0$, if $\lambda>1/(4k(1-p_1))$ for some $k\in(1/4,1)$.

Note that (3.1) is almost oscillatory, if $\lambda>2/(1-p_1)$ due to [23, Corollary 3].

Example 3.2. Consider the third-order neutral delay differential equation

$$\left(\frac{1}{t}\left(t^{1/2}\left(x(t)+\frac{1}{2}x(t-\pi)\right)'\right)'\right)'+\left(\frac{t^{-1/2}}{2}+\frac{3}{8}t^{-5/2}\right)x\left(t-\frac{7\pi}{2}\right)=0, \tag{3.2}$$

$t\geq 1$.

Let $\rho(t)=1$. It follows from Theorem 2.1 that every solution x of (3.2) is almost oscillatory. One such solution is $x(t)=\sin t$.

Example 3.3. Consider the third-order neutral delay differential equation

$$\left(t^{4/3}\left(x(t)+p_1x\left(\frac{t}{2}\right)\right)''\right)'+\frac{\lambda}{t^{5/3}}x(t)=0,\quad \lambda>0,\ t\geq 1, \tag{3.3}$$

where $p_1\in[0,1)$.

Let $\rho(t)=1$. It follows from Theorem 2.2 that every solution x of (3.3) is either oscillatory or $\lim_{t\to\infty}x(t)=0$, if $\lambda>1/(36k(1-p_1))$ for some $k\in(1/4,1)$.

Note that [22, Theorem 1] cannot be applied to (3.3) when $p_1=0$.

Example 3.4. Consider the third-order neutral delay differential equation

$$\left(t^2\left(t^2\left(x(t)+\frac{1}{3}x\left(\frac{t}{2}\right)\right)'\right)'\right)'+\lambda t^2x(t)=0,\quad \lambda>0, t\geq 1. \tag{3.4}$$

Let $\rho(t)=1$. It follows from Theorem 2.3 that every solution x of (3.4) is either oscillatory or $\lim_{t\to\infty}x(t)=0$, if $\lambda>0$.

4. Remarks

Remark 4.1. In [3], Agarwal et al. established a well-known result; see [4, Lemma 6.1]. Using [4, Lemma 6.1] and defining the function ω as in Theorem 2.1 with $\rho(t) = 1$, we can replace condition (2.1) with

$$\left(a(t)y'(t)\right)' + q(t)\left(1 - p(\tau(t))\right)\frac{\int_{t_2}^{\tau(t)}\left(\int_{t_1}^{s}(1/a(u))\mathrm{d}u/b(s)\right)\mathrm{d}s}{\int_{t_1}^{t}(1/a(u))\mathrm{d}u}y(t) = 0 \tag{4.1}$$

that is oscillatory. Similarly, we can replace condition (2.14) by

$$\left(a(t)y'(t)\right)' + q(t)\left(1 - p(\tau(t))\right)\int_{t_1}^{\tau(t)}\frac{\mathrm{d}s}{b(s)}y(t) = 0 \tag{4.2}$$

that is oscillatory.

Remark 4.2. The results for (E) can be extended to the nonlinear differential equations.

Remark 4.3. It is interesting to find a method to study (E) for the case when

$$\int_{t_0}^{\infty}\frac{1}{a(t)}\mathrm{d}t = \infty, \qquad \int_{t_0}^{\infty}\frac{1}{b(t)}\mathrm{d}t < \infty. \tag{4.3}$$

Remark 4.4. It is interesting to find other methods to present some sufficient conditions which guarantee that every solution of (E) is oscillatory.

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