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Research Article

Positive Solutions to Nonlinear Higher-Order Nonlocal Boundary Value Problems for Fractional Differential Equations

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We study existence of positive solutions to nonlinear higher-order nonlocal boundary value problems corresponding to fractional differential equation of the type ${}^c\mathfrak{D}^{\delta}_{0+}u(t) + f(t, u(t)) = 0$, $t \in (0, 1)$, $0 < t < 1$. $u(1) = \beta u(\eta) + \lambda_2$, $u'(0) = \alpha u'(\eta) - \lambda_1$, $u''(0) = 0$, $u'''(0) = 0 \cdots u^{(n-1)}(0) = 0$, where, $n - 1 < \delta < n$, $n(\geq 3) \in \mathbb{N}$, $0 < \eta, \alpha, \beta < 1$, the boundary parameters $\lambda_1, \lambda_2 \in \mathbb{R}^+$ and ${}^cD^{\delta}_{0+}$ is the Caputo fractional derivative. We use the classical tools from functional analysis to obtain sufficient conditions for the existence and uniqueness of positive solutions to the boundary value problems. We also obtain conditions for the nonexistence of positive solutions to the problem. We include examples to show the applicability of our results.

1. Introduction

Fractional calculus goes back to the beginning of the theory of differential calculus and is developing since the 17th century through the pioneering work of Leibniz, Euler, Abel, Liouville, Riemann, Letnikov, Weyl, and many others. Fractional calculus is the generalization of ordinary integration and differentiation to an arbitrary order. For almost 300 years, it was seen as interesting but abstract mathematical concept. Nevertheless the applications of fractional calculus just emerged in the last few decades in various areas of physics, chemistry, engineering, biosciences, electrochemistry, and diffusion processes. For details, we refer the readers to [1–5].

The existence and uniqueness of solutions for fractional differential equations is well studied in [6–10] and references therein. It should be noted that most of the papers and books on fractional calculus are devoted to the solvability of initial value problems for fractional differential equations. In contrast, the theory of boundary value problems for nonlinear fractional differential equations has received attention quiet recently, and many aspects of the theory need to be further investigated.

There are some recent development dealing with the existence and multiplicity of positive solutions to nonlinear boundary value problems for fractional differential equations, see, for example, [11–18] and the reference therein. However, few results can be found in the literature concerning the existence of positive solutions to nonlinear three-point boundary value problems for fractional differential equations. For example, Li and coauthors [19] obtained sufficient conditions for the existence and multiplicity results to the following three point fractional boundary value problem

$$\begin{gathered}\mathfrak{D}_{0+}^{\alpha}u(t)+f(t,u(t))=0,\quad 0<t<1,\ 1<\alpha\leq 2,\\ u(0)=0,\quad \mathfrak{D}_{0+}^{\beta}u(1)=a\mathfrak{D}_{0+}^{\beta}u(\xi),\end{gathered}\tag{1.1}$$

where $\mathfrak{D}_{0+}^{\alpha}$ is standard Riemann-Liouville fractional order derivative.

Bai [20] studied the existence and uniqueness of positive solutions to the following three-point boundary value problem for fractional differential equations

$$\begin{gathered}\mathfrak{D}_{0+}^{\alpha}u(t)+f(t,u(t))=0,\quad 0<t<1,\ 1<\alpha\leq 2,\\ u(0)=0,\quad \beta u(\eta)=u(1),\end{gathered}\tag{1.2}$$

where $0<\beta\eta^{\alpha-1}<1$, $0<\eta<1$, $\mathfrak{D}_{0+}^{\alpha}$ is standard Riemann-Liouville fractional order derivative. The function f is assumed to be continuous on $[0,1]\times[0,\infty)$.

The purpose of the present work is to investigate sufficient conditions for the existence, uniqueness, and nonexistence of positive solutions to more general boundary value problems for higher-order nonlinear fractional differential equations

$$\begin{gathered}{}^{c}\mathfrak{D}_{0+}^{\delta}\,u(t)+f(t,u(t))=0,\quad t\in(0,1),\ n-1<\delta<n,\\ u(1)=\beta u(\eta)+\lambda_2,\quad u'(0)=\alpha u'(\eta)-\lambda_1,\quad u''(0)=0,\quad u'''(0)=0\cdots u^{(n-1)}(0)=0,\end{gathered}\tag{1.3}$$

where, $n-1<\delta<n$, $n=3,4,5,\ldots$; $0<\eta,\alpha,\beta<1$, the boundary parameters $\lambda_1,\lambda_2\in\mathbb{R}^{+}$ and ${}^{c}\mathfrak{D}_{0+}^{\delta}$, is the Caputo fractional derivative. The function f is assumed to be continuous and nonnegative on $[0,1]\times[0,\infty)$. To the best of our knowledge, existence and uniqueness of positive solution to the boundary value problem (1.3) have never been studied previously.

This paper is organized as follows: in Section 2, we recall some basic definitions and preliminary results which are needed for our main results. In Section 3, we study existence and uniqueness and nonexistence of positive solutions to the boundary value problem (1.3) under certain assumptions on the function f. Moreover, examples are provided to illustrate the applicability of main results.

2. Background Materials and Lemmas

For the convenience of the readers, in this section, we provide definitions of Riemann-Liouville fractional integral and fractional derivative and some of their basic properties which will be helpful in the forth coming investigations.

Definition 2.1 (see [2, 5]). For a function $\phi : (a, \infty) \to \mathbb{R}$, the Riemann-Liouville fractional integral of order $\alpha > 0$ is defined by

$$\mathcal{I}_{a+}^{\alpha}\phi(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (t-s)^{\alpha-1}\phi(s)ds, \tag{2.1}$$

The provided that the integral on the right hand side exists. For $\alpha, \beta \geq 0$, the fractional integral satisfies the semigroup property

$$\mathcal{I}_{0+}^{\alpha}\mathcal{I}_{0+}^{\beta}\phi(t) = \mathcal{I}_{0+}^{\alpha+\beta}\phi(t) = \mathcal{I}_{0+}^{\beta}\mathcal{I}_{0+}^{\alpha}\phi(t) \quad \text{almost everywhere on } [0,1]. \tag{2.2}$$

In addition, if $\phi \in C[0,1]$ or if $\alpha + \beta \geq 1$, then the identity is true for every $t \in [0,1]$.

Definition 2.2 (see [2, 5]). The standard Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $\phi : (a, \infty) \to \mathbb{R}$ is given by $\mathfrak{D}_{a+}^{\alpha}\phi(t) = (d/dt)^{n}\mathcal{I}_{a+}^{n-\alpha}\phi(t)$, where $a \in \mathbb{R}$, $n = [s] + 1$, provided that the right hand side is pointwise defined on (a, ∞).

Definition 2.3. For a given function $\phi : (a, \infty) \to \mathbb{R}$, the Caputo fractional derivative of order $\alpha > 0$ is defined by ${}^{c}\mathfrak{D}_{a+}^{\alpha}\,\phi(t) = \mathcal{I}_{a+}^{n-\alpha}\phi^{(n)}(t)$, where $a \in \mathbb{R}$, $n = [s] + 1$.

Lemma 2.4 (see [2]). *If $\alpha > \beta > 0$, then ${}^{c}\mathfrak{D}_{0+}^{\beta}\,\mathcal{I}_{0+}^{\alpha}\phi(t) = \mathcal{I}_{0+}^{\alpha-\beta}\phi(t)$. In particular, if m is positive integer and $\delta > m$, then $(d^{m}/dt^{m})(\mathcal{I}_{0+}^{\delta}\phi(t)) = \mathcal{I}_{0+}^{\delta-m}\phi(t)$.*

The following two lemmas play a fundamental role to obtain an equivalent integral representation to the boundary value problem (1.3).

Lemma 2.5 (see [2]). *Let $\alpha > 0$, then*

$$\mathcal{I}_{0+}^{\alpha}\,{}^{c}\mathfrak{D}_{0+}^{\alpha}\,\phi(t) = \phi(t) - \sum_{k=0}^{n-1}\frac{\phi^{k}(0)}{k!}t^{k} = \phi(t) + c_{1} + c_{2}t + c_{3}t^{2} + \cdots + c_{n}t^{n-1}, \tag{2.3}$$

where $c_i = \phi^{k-1}(0)/(k-1)!$, $i = 1, 2, \ldots, n$ and $n - 1 < \alpha \leq n$.

Lemma 2.6 (see [2]). *For $\alpha > 0$, the fractional differential equation ${}^{c}\mathfrak{D}_{0+}^{\alpha}\,\phi(t) = 0$ has a general solution $\phi(t) = c_{1} + c_{2}t + c_{3}t^{2} + \cdots + c_{n}t^{n-1}$, where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$ and $n - 1 < \alpha \leq n$.*

For the existence of positive solutions to the boundary value problem (1.3), we use the following fixed point theorem due to Krasnosel'skii.

Theorem 2.7 (see [21]). *Let $\mathcal{E}$ be a Banach space, and let $\mathcal{P} \subset \mathcal{E}$ be a cone. Assume that Ω_1, Ω_2 are open subsets of $\mathcal{E}$ such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Let $\mathcal{A} : \mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{E}$ be completely continuous* *such that either*

(i) $\|\mathcal{A}u\| \leq \|u\|$, *for* $u \in \mathcal{P} \cap \partial\Omega_1$ *and* $\|\mathcal{A}u\| \geq \|u\|$, *for* $u \in \mathcal{P} \cap \partial\Omega_2$, *or*

(ii) $\|\mathcal{A}u\| \geq \|u\|$, *for* $u \in \mathcal{P} \cap \partial\Omega_1$ *and* $\|\mathcal{A}u\| \leq \|u\|$, *for* $u \in \mathcal{P} \cap \partial\Omega_2$.

Then, $\mathcal{A}$ *has a fixed point in* $\mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Main Results

Lemma 3.1. *Let* $h \in C[0,1]$, *then the unique solution of the linear problem*

$$ {}^{c}\mathfrak{D}_{0+}^{\delta} u(t) + h(t) = 0, \quad t \in (0,1), \; n-1 < \delta < n, \; n(\geq 3) \in \mathbb{N}, \tag{3.1}$$

$$ u(1) = \beta u(\eta) + \lambda_2, \quad u'(0) = \alpha u'(\eta) - \lambda_1, \quad u''(0) = 0, \quad u'''(0) = 0 \cdots u^{(n-1)}(0) = 0 \tag{3.2}$$

is given by

$$ u(t) = \int_0^1 G(t,s)h(s)ds + \int_0^1 H(t;\eta,s)h(s)ds + \psi(t), \tag{3.3}$$

where,

$$ G(t,s) = \begin{cases} \dfrac{(1-s)^{\delta-1} - (t-s)^{\delta-1}}{\Gamma(\delta)}, & s \leq t, \\ \dfrac{(1-s)^{\delta-1}}{\Gamma(\delta)}, & t \leq s, \end{cases}$$
$$ H(t;\eta,s) = \begin{cases} \dfrac{\beta\left[(1-s)^{\delta-1} - (\eta-s)^{\delta-1}\right]}{(1-\beta)\Gamma(\delta)} + \dfrac{\alpha[1-\beta\eta-(1-\beta)t](\eta-s)^{\delta-2}}{(1-\alpha)(1-\beta)\Gamma(\delta-1)}, & s \leq \eta, \\ \dfrac{\beta(1-s)^{\delta}}{(1-\beta)\Gamma(\delta-1)}, & \eta \leq s, \end{cases} \tag{3.4}$$

and $\psi(t) = ((1-\beta\eta-(1-\beta)t)/(1-\alpha)(1-\beta))\lambda_1 + \lambda_2/(1-\beta)$.

Proof. In view of Lemma 2.5, (3.1) is equivalent to the integral equation

$$ u(t) = -I_{0+}^{\delta} h(t) + c_1 + c_2 t + c_3 t^2 + \cdots + c_n t^{n-1}. \tag{3.5}$$

Using Lemma 2.4, we obtain

$$ \begin{aligned} u'(t) &= -I_{0+}^{\delta-1} h(t) + c_2 + 2c_3 t + \cdots + (n-1)c_n t^{n-2}, \\ u''(t) &= -I_{0+}^{\delta-2} h(t) + 2c_3 + \cdots + (n-1)(n-2)c_n t^{n-3}, \\ &\vdots \\ u^{(n-1)}(t) &= -I_{0+}^{\delta-(n-1)} h(t) + (n-1)!c_n, \end{aligned} \tag{3.6}$$

where $c_1, c_2, \ldots, c_n \in \mathbb{R}$. The boundary conditions, $u''(0) = 0$, $u'''(0) = 0, \ldots, u^{(n-1)}(0) = 0$, lead to $c_3 = 0, c_4 = 0, \ldots, c_n = 0$. Using the boundary conditions $u'(0) = \alpha u'(\eta) - \lambda_1$, $u(1) = \beta u(\eta) + \lambda_2$ and the fact that $I_{0+}^{\delta-1} h(0) = 0$, we obtain

$$(1-\alpha)c_2 = -\alpha I_{0+}^{\delta-1} h(\eta) - \lambda_1, \qquad (1-\beta)c_1 + (1-\beta\eta)c_2 = I_{0+}^{\delta} h(1) - \beta I_{0+}^{\delta} h(\eta) + \lambda_2, \tag{3.7}$$

which implies that $c_2 = -(\alpha/(1-\alpha)) I_{0+}^{\delta-1} h(\eta) - (\lambda_1/(1-\alpha))$ and

$$c_1 = \frac{1}{1-\beta} I_{0+}^{\delta} h(1) - \frac{\beta}{1-\beta} I_{0+}^{\delta} h(\eta) + \frac{\alpha(1-\beta\eta)}{(1-\alpha)(1-\beta)} I_{0+}^{\delta-1} h(\eta) + \frac{(1-\beta\eta)\lambda_1}{(1-\alpha)(1-\beta)} + \frac{\lambda_2}{(1-\beta)}. \tag{3.8}$$

Hence, the unique solution of the linear fractional boundary value problem (3.1), (3.2) is given by

$$\begin{aligned}
u(t) &= -I_{0+}^{\delta} h(t) + \frac{1}{1-\beta} I_{0+}^{\delta} h(1) - \frac{\beta}{1-\beta} I_{0+}^{\delta} h(\eta) \\
&\quad + \frac{\alpha(1-\beta\eta) - \alpha(1-\beta)t}{(1-\alpha)(1-\beta)} I_{0+}^{\delta-1} h(\eta) + \left(\frac{1-\beta\eta}{(1-\alpha)(1-\beta)} - \frac{t}{1-\alpha} \right)\lambda_1 + \frac{\lambda_2}{1-\beta} \\
&= -I_{0+}^{\delta} h(t) + I_{0+}^{\delta} h(1) + \frac{\beta}{1-\beta} I_{0+}^{\delta} h(1) - \frac{\beta}{1-\beta} I_{0+}^{\delta} h(\eta) \\
&\quad + \frac{\alpha(1-\beta\eta) - \alpha(1-\beta)t}{(1-\alpha)(1-\beta)} I_{0+}^{\delta-1} h(\eta) + \left(\frac{1-\beta\eta}{(1-\alpha)(1-\beta)} - \frac{t}{1-\alpha} \right)\lambda_1 + \frac{\lambda_2}{1-\beta} \\
&= \int_0^t \frac{(1-s)^{\delta-1} - (t-s)^{\delta-1}}{\Gamma(\delta)} h(s)\,ds + \int_t^1 \frac{(1-s)^{\delta-1}}{\Gamma(\delta)} h(s)\,ds \\
&\quad + \int_0^\eta \left(\frac{\beta(1-s)^{\delta-1} - \beta(\eta-s)^{\delta-1}}{(1-\beta)\Gamma(\delta)} + \frac{(\alpha(1-\beta\eta) - \alpha(1-\beta)t)(\eta-s)^{\delta-2}}{(1-\alpha)(1-\beta)\Gamma(\delta-1)} \right) h(s)\,ds \\
&\quad + \int_\eta^1 \frac{\beta(1-s)^{\delta-1}}{(1-\beta)\Gamma(\delta)} h(s)\,ds + \left(\frac{1-\beta\eta}{(1-\alpha)(1-\beta)} - \frac{t}{1-\alpha} \right)\lambda_1 + \frac{\lambda_2}{1-\beta} \\
&= \int_0^1 G(t,s)h(s)\,ds + \int_0^1 H(t;\eta,s)h(s)\,ds + \psi(t).
\end{aligned} \tag{3.9}$$

□

Lemma 3.2. *The functions $G(t,s)$ and $H(t;\eta,s)$ satisfy the following properties:*

(i) $G(t,s) \geq 0$, $H(t;\eta,s) \geq 0$ *and* $G(t,s) \leq G(s,s)$ *for all* $0 \leq s, t \leq 1$,

(ii) *For* $s \in (0,1)$, $0 < \xi < \tau < 1$, $\min_{\xi \leq t \leq \tau} G(t,s) \geq (1-\tau^{\delta-1}) \max_{0 \leq t \leq 1} G(t,s) = (1-\tau^{\delta-1}) G(s,s)$,

(iii) $\beta(1-\eta^{\delta-1})(1-s)^{\delta-1} \leq (1-\beta)\Gamma(\delta) H(t;\eta,s) < 2(\delta-1)(1-s)^{\delta-2}$,

(iv) *for* $s \in (0,1)$, $0 < \xi < \tau < 1$, $\min_{\xi \leq t \leq \tau} H(t;\eta,s) \geq (1-\tau) \max_{0 \leq t \leq 1} H(t;\eta,s)$.

Proof. (i) For $\delta > 1$, in view of the expression for $G(t,s)$, it follows that $G(t,s) \geq 0$ and $G(t,s) \leq G(s,s)$ for all $0 \leq s, t \leq 1$.
For $s \leq \eta$ and $0 < \beta < 1$, we have $1 - \beta\eta - (1-\beta)t > (1-\beta\eta)(1-t)$. Hence, $H(t,\eta,s) > 0$. Also, we note that $H(t,\eta,s) > 0$ for $s \geq \eta$.
(ii) Now,

$$\max_{t\in[0,1]} G(t,s) = \frac{(1-s)^{\delta-1}}{\Gamma(\delta)}. \tag{3.10}$$

Since

$$\begin{aligned} G(t,s) &= \frac{1}{\Gamma(\delta)}\left[(1-s)^{\delta-1} - (t-s)^{\delta-1}\right] = \max_{t\in[0,1]} G(t,s) - \frac{(t-s)^{\delta-1}}{\Gamma(\delta)} \\ &\geq \max_{t\in[0,1]} G(t,s) - \tau^{\delta-1}\frac{(1-s/\tau)^{\delta-1}}{\Gamma(\delta)}, \qquad t\in[\xi,\tau], \end{aligned} \tag{3.11}$$

which implies that

$$\min_{t\in[\xi,\tau]} G(t,s) \geq \left(1-\tau^{\delta-1}\right)\max_{t\in[0,1]} G(t,s), \quad \text{for } s\in(0,1). \tag{3.12}$$

Now, for $s \leq \eta$, we have

$$\begin{aligned} H(t;\eta,s) &= \frac{\beta\left[(1-s)^{\delta-1} - (\eta-s)^{\delta-1}\right]}{(1-\beta)\Gamma(\delta)} + \frac{\alpha[1-\beta\eta-(1-\beta)t](\eta-s)^{\delta-2}}{(1-\alpha)(1-\beta)\Gamma(\delta-1)} \\ &\leq \frac{\beta(1-s)^{\delta-1}}{(1-\beta)\Gamma(\delta)} + \frac{\alpha(\eta-s)^{\delta-2}}{(1-\alpha)(1-\beta)\Gamma(\delta)} < \frac{2(\delta-1)(1-s)^{\delta-2}}{(1-\alpha)(1-\beta)\Gamma(\delta)}. \end{aligned} \tag{3.13}$$

For $s \geq \eta$, obviously, $H(\eta,s) < 2(\delta-1)(1-s)^{\delta-2}/(1-\alpha)(1-\beta)\Gamma(\delta)$. From the expression of $H(\eta,s)$, it clearly follows that

$$H(t;\eta,s) \geq \frac{\beta(1-\eta^{\delta-1})(1-s)^{\delta-1}}{(1-\beta)\Gamma(\delta)}. \tag{3.14}$$

(iii) From the definition of $H(t;\eta,s)$, we have

$$\frac{\partial}{\partial t}(H(t;\eta,s)) = \frac{-(\eta-s)^{\delta-2}}{(1-\alpha)\Gamma(\delta-1)} \leq 0. \tag{3.15}$$

Therefore, $H(t;\eta,s)$ is nonincreasing in t, so its minimum value occurs at $t=\tau$ for $t\in[\xi,\tau]$, and its maximum value occurs at $t=0$ for $t\in[0,1]$. That is,

$$\min_{\xi\le t\le\tau} H(t;\eta,s)=\frac{\beta\left[(1-s)^{\delta-1}-(\eta-s)^{\delta-1}\right]}{(1-\beta)\Gamma(\delta)}+\frac{\alpha[1-\beta\eta-(1-\beta)\tau](\eta-s)^{\delta-2}}{(1-\beta)\Gamma(\delta)}, \tag{3.16}$$

$$\max_{0\le t\le 1} H(t;\eta,s)=\frac{\beta\left[(1-s)^{\delta-1}-(\eta-s)^{\delta-1}\right]}{(1-\beta)\Gamma(\delta)}+\frac{\alpha(1-\beta\eta)(\eta-s)^{\delta-2}}{(1-\beta)\Gamma(\delta)}. \tag{3.17}$$

Since $(1-s)^{\delta-1}-(\eta-s)^{\delta-1}\ge 0$ and $1-\tau<1$, therefore

$$(1-s)^{\delta-1}-(\eta-s)^{\delta-1}\ge(1-\tau)\left((1-s)^{\delta-1}-(\eta-s)^{\delta-1}\right). \tag{3.18}$$

Also, as $1-\beta\le 1-\beta\eta$, therefore

$$1-\beta\eta-(1-\beta)\tau\ge 1-\beta\eta-(1-\beta\eta)\tau=(1-\tau)(1-\beta\eta). \tag{3.19}$$

Subletting (3.18) and (3.19) in (3.16), we have

$$\begin{aligned}\min_{\xi\le t\le\tau} H(t;\eta,s)&\ge(1-\tau)\left\{\frac{\beta\left[(1-s)^{\delta-1}-(\eta-s)^{\delta-1}\right]}{(1-\beta)\Gamma(\delta)}+\frac{\alpha(1-\beta\eta)(\eta-s)^{\delta-2}}{(1-\beta)\Gamma(\delta)}\right\}\\&=(1-\tau)\max_{0\le t\le 1} H(t;\eta,s).\end{aligned} \tag{3.20}$$

□

Remark 3.3. For $\lambda_1,\lambda_2>0$, $\min_{\xi\le t\le\tau}\psi(t)\ge(1-\tau)\max_{0\le t\le 1}\psi(t)$.

Proof. As $(d/dt)\psi(t)=-\lambda_1/(1-\alpha)<0$, therefore $\psi(t)$ is a decreasing function. Hence,

$$\begin{aligned}\max_{0\le t\le 1}\psi(t)&=\psi(0)=\frac{(1-\beta\eta)\lambda_1}{(1-\alpha)(1-\beta)}+\frac{\lambda_2}{(1-\beta)},\\\min_{\xi\le t\le\tau}\psi(t)&=\psi(\tau)=\frac{[1-\beta\eta-(1-\beta)\tau]\lambda_1}{(1-\alpha)(1-\beta)}+\frac{\lambda_2}{(1-\beta)}\\&\ge\frac{1-\tau}{1-\beta}\left(\frac{1-\beta\eta}{1-\alpha}\lambda_1+\lambda_2\right)\end{aligned}$$

$$= (1-\tau)\max_{0\le t\le 1}\varphi(t). \tag{3.21}$$

Thus, we have $\min_{\xi\le t\le\tau}\varphi(t) \ge (1-\tau)\max_{0\le t\le 1}\varphi(t)$.

Let $\mathcal{B} = C[0,1]$ be endowed with the norm $\|u\| = \max_{0\le t\le 1}|u(t)|$. Define a cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \left\{u \in \mathcal{B} : \min_{\xi\le t\le\tau} u(t) \ge (1-\tau)\|u\|\right\}, \tag{3.22}$$

and an operator $\mathcal{A} : \mathcal{B} \to \mathcal{B}$ by

$$\mathcal{A}u(t) = \int_0^1 G(t,s)f(s,u(s))ds + \int_0^1 H(t;\eta,s)f(s,u(s))ds + \varphi(t). \tag{3.23}$$

By Lemma 3.1, the boundary value problem (1.3) has a solution if and only if $\mathcal{A}$ has a fixed point. □

Lemma 3.4. *The operator $\mathcal{A} : \mathcal{P} \to \mathcal{P}$ is completely continuous.*

Proof. Firstly, we show that $\mathcal{A}(P) \subset P$. From (3.23), Lemma 3.2, and Remark 3.3, we have

$$\begin{aligned}
\min_{\xi\le t\le\tau}(\mathcal{A}u(t)) &\ge \left(1-\tau^{\delta-1}\right)\int_0^1 G(s,s)f(s,u(s))ds + (1-\tau)\int_0^1 \max_{t\in[0,1]} H(t;\eta,s)f(s,u(s))ds \\
&\quad + (1-\tau)\max_{t\in[0,1]}\varphi(t) \\
&\ge (1-\tau)\int_0^1 G(s,s)f(s,u(s))ds + \int_0^1 \max_{t\in[0,1]} H(t;\eta,s)f(s,u(s))ds + \max_{t\in[0,1]}\varphi(t) \\
&\ge (1-\tau)\|\mathcal{A}u\|.
\end{aligned} \tag{3.24}$$

Hence, we have $\mathcal{A}(P) \subset P$.

Next, we show that $\mathcal{A}$ is uniformly bounded. For fixed $\ell > 0$, consider a bounded subset $\mathcal{M}$ of $\mathcal{P}$ defined by

$$\mathcal{M} = \{u \in \mathcal{P} : \|u\| \le \ell,\ \ell \in \mathbb{R}^+\}, \tag{3.25}$$

and define $\mathcal{K} = \max_{0\le u\le \ell} f(t,u(t)) + 1$, then for $u \in \mathcal{M}$, we have

$$\begin{aligned}
|\mathcal{A}u(t)| &\le \int_0^1 G(t,s)\left|f(s,u(s))\right|ds + \int_0^1 H(t;\eta,s)\left|f(s,u(s))\right|ds + \max_{t\in[0,1]}\psi(t) \\
&\le \frac{\mathcal{K}}{\Gamma(\delta)}\int_0^1 (1-s)^{\delta-1}ds + \frac{2\mathcal{K}(\delta-1)}{(1-\alpha)(1-\beta)\Gamma(\delta)}\int_0^1 (1-s)^{\delta-2}ds + \frac{(1-\beta\eta)\lambda_1 + (1-\alpha)\lambda_2}{(1-\alpha)(1-\beta)} \\
&\le \frac{\mathcal{K}}{(1-\alpha)(1-\beta)}\left(\mathcal{I}^{\delta}_{0+}(1) + 2\mathcal{I}^{\delta}_{0+}(1)\right) + \frac{(1-\beta\eta)\lambda_1 + (1-\alpha)\lambda_2}{(1-\alpha)(1-\beta)} \\
&= \frac{(3\delta-1)\mathcal{K}}{\delta(1-\alpha)(1-\beta)(\delta-1)} + \frac{(1-\beta\eta)\lambda_1 + (1-\alpha)\lambda_2}{(1-\alpha)(1-\beta)}.
\end{aligned} \tag{3.26}$$

Hence, $\mathcal{A}(\mathcal{M})$ is bounded.

Finally, we show that $\mathcal{A}$ is equicontinuous. Define $\sigma = \delta(1-\alpha)\Gamma(\delta)\varepsilon/\mathcal{K}[\delta(1-\alpha) + \alpha\eta^{\alpha-1} + \lambda_1]$ and choose $t > \tau$ such that $t - \tau < \sigma$. Then, for all $\varepsilon > 0$ and $u \in \mathcal{M}$, we have

$$\begin{aligned}
|\mathcal{A}u(t) - \mathcal{A}u(\tau)| &= \int_0^1 (G(t,s) - G(\tau,s))f(s,u(s))ds + \int_0^1 \left(H(t;\eta,s) - H(t;\eta,s)\right)f(s,u(s))ds \\
&\quad - \frac{\lambda_1}{1-\alpha}(t-\tau) \\
&\le \mathcal{K}\int_0^1 G(t,s) - G(\tau,s)ds + \int_0^1 H(t;\eta,s) - H(t;\eta,s)ds + \frac{\lambda_1}{1-\alpha}(t-\tau) \\
&= \mathcal{K}\frac{1}{\Gamma(\delta)}\int_0^t \left((t-s)^{\delta-1} - (\tau-s)^{\delta-1}\right)ds + \frac{\alpha(t-\tau)}{(1-\alpha)\Gamma(\delta-1)} \\
&\quad \times \int_0^{\eta} (\eta-s)^{\delta-2}ds + \frac{\lambda_1}{1-\alpha}(t-\tau) \\
&= \frac{\mathcal{K}}{\delta\Gamma(\delta)}\left[t^{\delta} - \tau^{\delta} + \frac{\alpha\eta^{\delta-1} + \lambda_1}{1-\alpha}(t-\tau)\right].
\end{aligned} \tag{3.27}$$

Using the mean value theorem, we obtain $t^{\delta} - \tau^{\delta} \le \delta(t-\tau) < \delta\sigma$. Hence, it follows that

$$|\mathcal{A}u(t) - \mathcal{A}u(\tau)| < \frac{\mathcal{K}\sigma\left[\delta(1-\alpha) + \alpha\eta^{\alpha-1} + \lambda_1\right]}{\delta(1-\alpha)\Gamma(\delta)} < \varepsilon. \tag{3.28}$$

By means of Arzela-Ascoli theorem $\mathcal{A} : \mathcal{D} \to \mathcal{D}$ is completely continuous operator. □

For convenience, we introduce following notations:

$$\mathcal{K}^1_{\alpha,\beta,\delta} = \left(3I^{\delta}_{0+}\varphi_1(1) + \frac{6I^{\delta-1}_{0+}\varphi_1(1)}{(1-\alpha)(1-\beta)}\right)^{-1}, \qquad \mathcal{K}^2_{\xi,\beta,\tau,\delta} = \left(\frac{(1-\tau^{\delta-1})}{1-\beta}\mathcal{I}^{\delta}_{\xi+}\varphi_2(\tau)\right)^{-1},$$
$$c_\alpha = \frac{1}{3}(1-\alpha), \qquad c_\beta = \frac{1}{3}(1-\beta), \qquad \gamma = 1-\tau^{\delta-1}. \tag{3.29}$$

Theorem 3.5. *If there exist constants $\rho_1, \rho_2 \in \mathbb{R}^+$ such that $\rho_2 > \rho_1$ and functions $\varphi_1, \varphi_2 \in L[0,1]$ such that*

(i) $f(t,u) \geq \rho_2 \mathcal{K}^2_{\xi,\beta,\tau,\delta}\varphi_2(t)$, *for* $(t,u) \in [0,1] \times [\gamma\rho_2, \rho_2]$,

(ii) $f(t,u) \leq \rho_1 \mathcal{K}^1_{\alpha,\beta,\eta}\varphi_1(t)$, *for* $(t,u) \in [0,1] \times [0, \rho_1]$,

then the boundary value problem (1.3) *has at least one positive solution for λ_1, λ_2 small enough and has no positive solution for λ_1, λ_2 large enough.*

Proof. By Lemma 3.4, the operator $\mathcal{A}$ is completely continuous. The proof is divided into two steps.

Step 1. We prove that the boundary value problem (1.3) has at least one positive solution.

Define $\Omega_{\rho_1} = \{u \in \mathcal{B} : \|u\| < \rho_1\}$ an open subset of $\mathcal{B}$ and choose λ_1, λ_2 such that $\lambda_1 \leq c_\alpha \rho_1$ and $\lambda_2 \leq c_\beta \rho_1$. Then, for any $u \in \mathcal{P} \cap \partial\Omega_{\rho_1}$, we have $\|u\| = \rho_1$ and in view of Lemma 3.2 and (3.23), it follows that

$$\begin{aligned}
|\mathcal{A}u(t)| &= \left|\int_0^1 G(t,s)f(s,u(s))ds + \int_0^1 H(t;\eta,s)f(s,u(s))ds + \left(\frac{1-\beta\eta-(1-\beta)t}{(1-\alpha)(1-\beta)}\right)\lambda_1 + \frac{\lambda_2}{1-\beta}\right| \\
&\leq I^{\delta}_{0+}f(1,u(1)) + \frac{2}{(1-\alpha)(1-\beta)}I^{\delta-1}_{0+}f(1,u(1)) + \frac{(1-\beta\eta)\lambda_1 + (1-\alpha)\lambda_2}{(1-\alpha)(1-\beta)} \\
&\leq \mathcal{K}^1_{\alpha,\beta,\delta}\rho_1\left[I^{\delta}_{0+}\varphi_1(1) + \frac{2I^{\delta-1}_{0+}\varphi_1(1)}{(1-\alpha)(1-\beta)}\right] + \frac{\lambda_1}{1-\alpha} + \frac{\lambda_2}{1-\beta} \\
&\leq \frac{\rho_1}{3} + \frac{\rho_1}{3} + \frac{\rho_1}{3} = \|u\|,
\end{aligned} \tag{3.30}$$

which implies that $\|\mathcal{A}u\| \leq \|u\|$, for $u \in \mathcal{P} \cap \partial\Omega_{\rho_1}$.

Define $\Omega_{\rho_2} = \{u \in \mathcal{P} : \|u\| < \rho_2\}$. For any $t \in [\xi, \tau]$ and $u \in \mathcal{P} \cap \partial\Omega_{\rho_2}$, using Lemma 3.2, we have

$$\min_{\xi \leq t < \tau} u(t) \geq \left(1 - \tau^{\delta-1}\right)\|u\|. \tag{3.31}$$

Therefore, by (3.3) and Remark 3.3, we have the following estimate:

$$|Au(t)| = \left| \int_0^1 G(t,s) f(s,u(s)) ds + \int_0^1 H(t;\eta,s) f(s,u(s)) ds + \left(\frac{1-\beta\eta-(1-\beta)t}{(1-\alpha)(1-\beta)} \right) \lambda_1 + \frac{\lambda_2}{1-\beta} \right|$$

$$\geq \left[\left(1-\tau^{\delta-1}\right) + \frac{\beta\left(1-\eta^{\delta-1}\right)}{1-\beta} \right] I_{\xi+}^{\delta} f(\tau,u(\tau)) + \left(\frac{1-\beta\eta-(1-\beta)t}{(1-\alpha)(1-\beta)} \right) \lambda_1 + \frac{\lambda_2}{1-\beta}$$

$$\geq \rho_2 \mathcal{K}^2_{\xi,\beta,\tau,\delta} \left[\left(1-\tau^{\delta-1}\right) + \frac{\beta\left(1-\tau^{\delta-1}\right)}{1-\beta} \right] I_{\xi+}^{\delta} \varphi_2(\tau)$$

$$= \rho_2 \mathcal{K}^2_{\xi,\beta,\tau,\delta} \left(\frac{\left(1-\tau^{\delta-1}\right)}{1-\beta} \right) I_{\xi+}^{\delta} \varphi_2(\tau) = \rho_2 = \|u\|, \tag{3.32}$$

which implies that $\|\mathcal{A}u\| \geq \|u\|$ for $u \in \mathcal{P} \cap \partial\Omega_{\rho_2}$

Hence, by Theorem 2.7, it follows that $\mathcal{A}$ has a fixed point u in $\mathcal{P} \cap (\overline{\Omega}_{\rho_2} \setminus \Omega_{\rho_1})$.

Step 2. Now, we prove that for large values of λ_1, λ_2, the boundary value problem (1.3) has no positive solution. Otherwise, for $i = 1,2$, there exists $0 < \lambda_{i1} < \lambda_{i2} \cdots < \lambda_{in} < \cdots$, with $\lim_{n\to\infty} \lambda_{in} = +\infty$, such that for any positive integer n, the boundary value problem

$$ {}^{c}\mathfrak{D}_{0+}^{\delta} u(t) + f(t,u(t)) = 0, \quad t \in (0,1),\ 0 < t < 1,$$

$$u(1) = \beta u(\eta) + \lambda_{2n}, \quad u'(0) = \alpha u'(\eta) - \lambda_{1n}, \quad u''(0) = 0, \quad u'''(0) = 0 \cdots u^{(n-1)}(0) = 0 \tag{3.33}$$

has a positive solution given by

$$u_n(t) = \int_0^1 G(t,s) f(s,u_n(s)) ds + \int_0^1 H(t,s) f(s,u_n(s)) ds + \left(\frac{1-\beta\eta-(1-\beta)t}{(1-\alpha)(1-\beta)} \right) \lambda_{1n} + \frac{\lambda_{2n}}{1-\beta}. \tag{3.34}$$

If $t > n$, then $u_n(t) \geq (1-t)\lambda_{1n}/(1-\alpha)(1-\beta) + \lambda_{2n}/(1-\beta)$ and if $t \leq n$, then $u_n(t) \geq (1-\eta)\lambda_{1n}/(1-\alpha)(1-\beta) + \lambda_{2n}/(1-\beta)$. Therefore, $u_n \to \infty$ as $n \to \infty$.

Let $\widehat{\rho}_1 = \min((1/2)\rho_2, \lambda_1/c_\alpha, \lambda_2/c_\beta)$ and define $\Omega_{\widehat{\rho}_1} = \{u \in \mathcal{P} : \rho_1 < \|u_n\| < \widehat{\rho}_1\}$. For $u \in \Omega_{\widehat{\rho}_1}$, using (3.3) and Remark 3.3, we have

$$\begin{aligned}
\|u_n(t)\| &\geq \int_0^1 G(t,s) f(s,u_n(s))ds + \int_0^1 H(t;\eta,s) f(s,u_n(s))ds \\
&\quad + \left(\frac{1-\beta\eta-(1-\beta)t}{(1-\alpha)(1-\beta)}\right)\lambda_{1n} + \frac{\lambda_{2n}}{1-\beta} \\
&\geq \frac{1}{2}\left[\left(1-\tau^{\delta-1}\right) + \frac{\beta(1-\eta^{\delta-1})}{1-\beta}\right]\mathcal{I}^{\delta}_{\xi+} f(1,u_n(1)) \\
&\geq \frac{\rho_2 \mathcal{K}^2_{\alpha,\beta,\eta}}{1-\beta}\left[(1-\beta)\left(1-\tau^{\delta-1}\right) + \beta\left(1-\eta^{\delta-1}\right)\right]\mathcal{I}^{\delta}_{\xi+}\psi_2(1) \\
&> 2\widehat{\rho}_1 > 2\|u_n\|,
\end{aligned} \tag{3.35}$$

which is a contradiction. Hence, the boundary value problem (1.3) have no positive solution for λ_1, λ_2 large enough. □

Theorem 3.6. *Assume that* (A) *is satisfied and there exists real-valued function* $\phi(t) \in L[0,1]$ *such that*

$$\begin{gathered}
\left|f(t,u) - f(t,v)\right| \leq \phi(t)|u-v|, \quad \text{for } t \in [0,1],\ u,v \in [0,\infty), \\
\mathcal{I}^{\delta}_{0+}\phi(1) + \frac{2}{(1-\alpha)(1-\beta)}\mathcal{I}^{\delta-1}_{0+}\phi(1) < 1,
\end{gathered} \tag{3.36}$$

then the boundary value problem (1.3) *has a unique positive solution.*

Proof. For $u, v \in \mathcal{P}$, using (3.23) and Lemma 3.2, we obtain

$$\begin{aligned}
|\mathcal{A}u(t) - \mathcal{A}v(t)| &\leq \int_0^1 G(t,s)\left|f(s,u(s)) - f(s,v(s))\right|ds \\
&\quad + \int_0^1 H(t;\eta,s)\left|f(s,u(s)) - f(s,v(s))\right|ds \\
&< \|u-v\|\left[\int_0^1 \frac{(1-s)^{\delta-1}}{\Gamma(\delta)}\phi(s)ds + \int_0^1 \frac{2(\delta-1)(1-s)^{\delta-2}}{(1-\alpha)(1-\beta)\Gamma(\delta)}\phi(s)ds\right] \\
&= \|u-v\|\left[\mathcal{I}^{\delta}_{0+}\phi(1) + \frac{2}{(1-\alpha)(1-\beta)}\mathcal{I}^{\delta-1}_{0+}\phi(1)\right] < \|u-v\|,
\end{aligned} \tag{3.37}$$

where we have taken into account $(1-s)^{\delta-1} \leq (1-s)^{\delta-2}$, we obtain the relation $\mathcal{I}_{0+}^{\delta}\phi(1) \leq (1/(\delta-1))\mathcal{I}_{0+}^{\delta-1}\phi(1)$. Hence, it follows by Banach contraction principle that the boundary value problem (1.3) has a unique positive solution. □

Example 3.7. Consider the boundary value problem

$$
\begin{gathered}
{}^{c}\mathfrak{D}_{0+}^{\delta} u(t) = \frac{t^2 e^{(3/2)t}+1}{8(1+t^2)} + \frac{t^2u^2(1+\sin t)}{16\pi}, \quad t \in (0,1), \\
u(1) = \beta u(\eta) + \lambda_2, \quad u'(0) = \alpha u'(\eta) - \lambda_1, \quad u''(0) = 0, \quad u'''(0) = 0,
\end{gathered} \tag{3.38}
$$

where, $\delta = 7/2$, $\eta = 1/2$, $\alpha = 2/3$, and $\beta = 1/3$. Let $f(t,u) = (t^2e^{(3/2)t}+1)/8(1+t^2) + t^2u^2(1+\sin t)/16\pi$, $(t,u) \in [0,1] \times [0,\infty]$. For $\rho_1 = 1$, $\rho_2 = 550$, we observe that

$$
\begin{gathered}
f(t,u) \leq \frac{1}{8}\left(e + \frac{1}{\pi}\right)t^2, \quad \text{for } (t,u) \in [0,1] \times [0,1], \\
f(t,u) \geq \left(\frac{1}{\sqrt{e}}\right)\frac{t^2}{1+t^2}, \quad \text{for } (t,u) \in [0,1] \times [283,550].
\end{gathered} \tag{3.39}
$$

Taking $\psi_1(t) = t^2$ and $\psi_2(t) = t^2/(1+t^2)$. For $\xi = 1/4$, $\tau = 3/4$, by computations, we find that $c_\alpha = 1/9$, $c_\beta = 2/9$, $I_{\xi+}^{\delta}\psi_2(\tau) = (1/60\sqrt{\pi})(49\sqrt{7}\tan^{-1}(\sqrt{2/7}) - \tan^{-1}(\sqrt{2}) - 1132\sqrt{2}/21) \approx 0.001291$, also $\mathcal{I}_{0+}^{\delta}\psi_1(1) = 128/10395\sqrt{\pi}$. Therefore, $\mathcal{K}_{\alpha,\beta,\delta}^{1} \approx 0.583566$ and $\mathcal{K}_{\xi,\beta,\tau,\delta}^{2} \approx 0.000993874$. Hence,

$$
\begin{gathered}
f(t,u) \leq \rho_1\mathcal{K}_{\alpha,\beta,\delta}^{1}\psi_1(t) \approx 0.583566t^2, \quad \text{for } (t,u) \in [0,1] \times [0,1], \\
f(t,u) \geq \rho_2\mathcal{K}_{\xi,\beta,\tau,\delta}^{2}\psi_2(t) \approx 0.546631\left(\frac{t^2}{1+t^2}\right), \quad \text{for } (t,u) \in [0,1] \times [283,550].
\end{gathered} \tag{3.40}
$$

Assumptions (i) and (ii) of the Theorem 3.5 are satisfied. Therefore, the boundary value problem (3.38) has a positive solution for $\lambda_1 \in [0,1/9]$, $\lambda_2 \in [0,2/9]$ and no positive solution for $\lambda_1 > 1/9$, $\lambda_2 > 2/9$.

Example 3.8. Consider the boundary value problem

$$
{}^{c}\mathfrak{D}_{0+}^{\delta} u(t) = \frac{e^{-33t}\left(11\sqrt{\pi}e^{33t}\cos(25t) + 500\sin^2 t\right)u^2}{(14\sqrt{\pi} + 425e^{(1/8)t})(1+u)}, \tag{3.41}
$$

$$
u(1) = \beta u(\eta) + \lambda_2, \quad u'(0) = \alpha u'(\eta) - \lambda_1, \quad u''(0) = 0, \quad u'''(0) = 0, \tag{3.42}
$$

where $\delta = 7/2$, $\alpha = 2/3$, $\beta = 3/5$, $\eta = 1/2$, and $\lambda_1, \lambda_2 \in \mathbb{R}$. Let $f(t,u) = e^{-33t}(11\sqrt{\pi}e^{33t}\cos(25t) + 500\sin^2 t)u^2/(14\sqrt{\pi} + 425e^{(1/8)t})(1+u)$, $(t,u) \in [0,1] \times [0,\infty]$. For $u, v \in \mathcal{P}$, $t \in [0,1]$, we have the following estimate:

$$\begin{aligned}
\left|f(t,u) - f(t,v)\right| &\leq \frac{e^{-33t}(11\sqrt{\pi}e^{33t} + 500)}{(14\sqrt{\pi} + 425e^{(1/8)t})}\left(\left|\frac{u^2}{1+u} + \frac{v^2}{1+v}\right|\right) \\
&\leq \frac{e^{-33t}(11\sqrt{\pi}e^{33t} + 500)}{(14\sqrt{\pi} + 425e^{(1/8)t})}\left(\frac{|u-v|(u+v+uv)}{(1+u)(1+v)}\right) \\
&\leq \frac{e^{-33t}(11\sqrt{\pi}e^{33t} + 500)}{(14\sqrt{\pi} + 425e^{(1/8)t})}(|u-v|).
\end{aligned} \tag{3.43}$$

Let $\phi(t) = e^{-33t}(11\sqrt{\pi}e^{33t} + 500)/(14\sqrt{\pi} + 425e^{(1/8)t})$, then by computations, we have $\mathcal{I}_{0+}^{\delta-1}\phi(1) \approx 0.013469$, $(2\delta-\alpha-1)\mathcal{I}_{0+}^{\delta-1}\phi(1)/(\delta-1)(1-\alpha)(1-\beta) \approx 0.215504 < 1$. All the conditions of Theorem 3.6 are satisfied. Therefore, by Theorem 3.6, the boundary value problem (3.41), (3.41) has a unique positive solution.

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