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# Existence of solutions to nonlinear fractional differential equations with boundary conditions on an infinite interval in Banach spaces

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**Abstract**
In this paper, we consider a system of nonlinear differential equations in a Banach space with boundary conditions on an infinite interval and provide sufficient conditions for the existence of solutions of the system. Our method relies upon the properties of the Kuratowski noncompactness measure and the Sadovskii fixed point theorem. An example is given to illustrate the main results.

**Keywords:** fractional differential equation; boundary value problem; fixed point theorem; Banach space

## 1 Introduction

Fractional differential equations are important mathematical models of some practical problems in many fields such as polymer rheology, chemistry physics, heat conduction, fluid flows, electrical networks, and many other branches of science (see [1–4]). Consequently, the fractional calculus and its applications in various fields of science and engineering have received much attention, and many papers and books on fractional calculus, fractional differential equations have appeared (see [5–9]). It should be noted that the theory of nonlinear fractional differential equation boundary value problems receives more and more attention (see [10–16]). Many authors discussed the existence of solutions in scalar spaces. However, according to the authors' knowledge, there are few papers to deal with the existence of solutions to the systems of fractional differential equations in a Banach space, especially with the boundary conditions on an infinite interval. Boundary value problems in infinite intervals arise naturally in the study of radially symmetric solutions of nonlinear elliptic equations and various physical phenomena (see [17, 18]).

In scalar case, Xu *et al.* [19] considered the nonlinear Dirichlet-type boundary value problems of the fractional differential equation, and the existence results were established by using the Leray-Shauder nonlinear alternative and a fixed point theorem on cone. For a Banach space, Salem [20] solved the existence of solutions to the fractional boundary value problems by means of some standard tools of fixed point theory. They investigated the existence results of solutions on finite intervals by classical tools in functional analysis.

For boundary value problems of fractional order on infinite intervals, some excellent results dealing with nonlinear fractional differential equations have appeared (see [21, 22]). In their paper [23], using a fixed point theorem, Zhao and Ge investigated the existence of solutions to the nonlinear fractional differential equation on unbounded domains. By using Darbo's fixed point theorem, Su [24] obtained the existence of solutions to the following fractional differential equation:

$$\begin{cases} D_{0+}^{\alpha}u(t) = f(t,u(t)), & t \in J := [0,\infty), \\ u(0) = 0, & D_{0+}^{\alpha-1}u(1) = u_{\infty}, \end{cases}$$

in a real Banach space. Nyamoradi *et al.* [25] considered an infinite fractional boundary value problem for singular integro-differential equation of mixed type on the half-line. Liu *et al.* [26] established sufficient conditions for the existence of solutions to a boundary value problem of a coupled system of nonlinear fractional differential equations on the half-line given by

$$\begin{cases} -D_{0+}^{\alpha}x(t) = f(t,y(t),D_{0+}^{p}y(t)), & t \in (0,\infty), \\ -D_{0+}^{\beta}y(t) = g(t,x(t),D_{0+}^{q}x(t)), & t \in (0,\infty), \\ a\lim_{t\to 0} t^{2-\alpha}x(t) - b\lim_{t\to 0} D_{0+}^{\alpha-1}x(t) = x_0, \\ c\lim_{t\to 0} t^{2-\beta}y(t) - d\lim_{t\to 0} D_{0+}^{\beta-1}x(t) = y_0, \\ \lim_{t\to\infty} D_{0+}^{\alpha-1}x(t) = x_1, & \lim_{t\to\infty} D_{0+}^{\beta-1}x(t) = y_1. \end{cases}$$

Motivated by the results mentioned above, we discuss the following boundary value problem (BVP for short):

$$\begin{cases} -D_{0+}^{\alpha}x(t) + f(t,x(t),x'(t),y(t),y'(t)) = \theta, & t \in J, \\ -D_{0+}^{\alpha}y(t) + g(t,x(t),x'(t),y(t),y'(t)) = \theta, & t \in J, \\ x(0) = x'(0) = \theta, & D_{0+}^{\alpha-1}x(\infty) = x_{\infty}, \\ y(0) = y'(0) = \theta, & D_{0+}^{\alpha-1}y(\infty) = y_{\infty}, \end{cases} \tag{1}$$

in a Banach space $E$, where $2 < \alpha \leq 3$ is a real number, $J_+ = (0,\infty)$, $x_\infty, y_\infty \in E$, $f \in C[J \times E \times E \times E \times E, E]$, $g \in C[J \times E \times E \times E \times E, E]$, $D_{0+}^{\alpha-1}x(\infty) := \lim_{t\to\infty} D_{0+}^{\alpha-1}x(t)$ and $D_{0+}^{\alpha-1}y(\infty) := \lim_{t\to\infty} D_{0+}^{\alpha-1}y(t)$. We establish some existence results of solutions to BVP (1) in the Banach space. The technique relies on the properties of the Kuratowski noncompactness measure and the Sadovskii fixed point theorem. The method used in this paper is different from ones in the papers mentioned above.

Obviously, problem (1) is more general than the problems discussed in some recent literature, such as ones in [23, 24]. Problem (1) is a system that contains two unknown functions; the nonlinear terms contain the derivatives $x'(t)$ and $y'(t)$; the basic space is a Banach space; and the boundary conditions are given on an infinite interval.

This paper is organized as follows. In Section 2, we recall some definitions and facts. In Section 3, the existence results of solutions to BVP (1) are discussed by using the properties of the Kuratowski noncompactness measure and the Sadovskii fixed point theorem. Finally, in Section 4, we provide an example as an application of our main result.

## 2 Preliminaries

In this section, we recall some definitions and facts which will be used in the later analysis.

**Definition 2.1** (see [27]) The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \to R$ is given by

$$I_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\, dt,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$, where $\Gamma(\alpha)$ is the Euler gamma function defined by

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t}\, dt.$$

**Definition 2.2** (see [27]) The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, \infty) \to R$ is given by

$$D_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_0^x \frac{f(t)}{(x-t)^{\alpha-n-1}}\, dt,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number $\alpha$, provided that the right-hand side is pointwise defined on $(0, \infty)$.

**Lemma 2.1** (see [24]) *If $f$ is a suitable function* (*see* [28]), *we have the composition relations* $D_{0+}^{\alpha} I_{0+}^{\alpha} = f(t)$, $\alpha > 0$, *and* $D_{0+}^{\alpha} I_{0+}^{\gamma} = I_{0+}^{\gamma-\alpha} f(t)$, $\gamma > \alpha > 0$, $t \in (0, \infty)$.

**Lemma 2.2** (see [28]) *Let* $\alpha > 0$. *Then the fractional differential equation*

$$D_{0+}^{\alpha} u(t) = 0$$

*has a unique solution* $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_N t^{\alpha-N}$, $c_i \in R$, $i = 1, 2, 3, \ldots, N$, *where* $N = [\alpha] + 1$.

In view of Lemma 2.1 and Lemma 2.2, it is easy to deduce that

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_N t^{\alpha-N}$$

for some $c_i \in R$, $i = 1, 2, 3, \ldots, N$, $N = [\alpha] + 1$.

**Remark 2.1** The Riemann-Liouville fractional derivative and integral of order $\alpha$ ($\alpha > 0$) have the following properties:

(1) $D_{0+}^{\alpha} I_{0+}^{\alpha} f(t) = f(t)$, $\alpha > 0$;
(2) $I_{0+}^{\alpha} I_{0+}^{\beta} f(t) = I_{0+}^{\alpha+\beta} f(t)$, $\alpha, \beta > 0$;
(3) $D_{0+}^{\alpha} I_{0+}^{\beta} f(t) = I_{0+}^{\beta-\alpha} f(t)$, $\beta > \alpha > 0$.

**Definition 2.3** (Kuratowski noncompactness measure) Let $E$ be a real Banach space, $S$ be a bounded subset of $E$. Denote

$$\alpha(S) = \inf\left\{ \delta > 0 : S = \bigcup_{i=1}^{m} S_i, \operatorname{diam}(S_i) < \delta, i = 1, 2, \ldots, m \right\}.$$

$\alpha(S)$ is called Kuratowski noncompactness measure of $S$, where $\operatorname{diam}(S_i)$ denote the diameters of $S_i$. Obviously $0 \le \alpha(S) < \infty$.

**Definition 2.4** Let $E_1$ and $E_2$ be real Banach spaces, $D \subset E_1$, $A : D \to E_2$ be a continuous and bounded operator. If there exists a constant $k \geq 0$ such that $\alpha(A(S)) \leq k\alpha(S)$ for any bounded set $S$ in $D$, then $A$ is called a $k$-set contraction operator. When $k < 1$, $A$ is called a strict set contraction operator.

**Remark 2.2** A strict set contraction operator is a condensation.

Now, we denote

$$FC[J,E] = \left\{ x \in C[J,E] : \sup_{t\in J} \frac{\|x(t)\|}{1 + t^{\alpha-1}} < +\infty \right\},$$

$$DC^1[J,E] = \left\{ x \in C^1[J,E] : \sup_{t\in J} \frac{\|x(t)\|}{1 + t^{\alpha-1}} < +\infty \text{ and } \sup_{t\in J} \frac{\|x'(t)\|}{1 + t^{\alpha-1}} < +\infty \right\}.$$

Obviously, $C^1[J,E] \subset C[J,E]$ and $DC^1[J,E] \subset FC[J,E]$. It is easy to see that $FC[J,E]$ is a Banach space with norm

$$\|x\|_F = \sup_{t\in J} \frac{\|x(t)\|}{1 + t^{\alpha-1}},$$

and $DC^1[J,E]$ is a Banach space with norm

$$\|x\|_D = \max\left\{ \|x\|_F, \|x'\|_1 \right\},$$

where

$$\|x'\|_1 = \sup_{t\in J} \frac{\|x'(t)\|}{1 + t^{\alpha-1}}.$$

Let $X = DC^1[J,E] \times DC^1[J,E]$ with norm $\|(x,y)\|_X = \max\{\|x\|_D, \|y\|_D\}$ for $(x,y) \in X$. Then $(X, \|\cdot,\cdot\|_X)$ is a Banach space. The basic space used in this paper is $(X, \|\cdot,\cdot\|_X)$. A map $x \in X$ is called a solution of BVP (1) if it satisfies all the equations of (1). For a bounded subset $D$ of the Banach space $E$, let $\alpha(D)$ denote the Kuratowski noncompactness measure of $D$. In this paper, the Kuratowski noncompactness measure in $E$, $C[J,E]$, $FC[J,E]$, $DC[J,E]$ and $X$ are denoted by $\alpha_E(\cdot)$, $\alpha_C(\cdot)$, $\alpha_F(\cdot)$, $\alpha_D(\cdot)$ and $\alpha_X(\cdot)$, respectively. The following properties of the Kuratowski noncompactness measure and Sadovskii fixed point theorem are needed for our discussion.

**Lemma 2.3** (see [29]) *If $H \subset C[I,E]$ is bounded and equicontinuous, then $\alpha_E(H(t))$ is continuous on $I$ and $\alpha_C(H) = \max_{t\in I} \alpha_E(H(t))$, $\alpha_E(\int_I x(t)\,dt : x \in H) \leq \int_I \alpha_E(H(t))\,dt$, where $H(t) = \{x(t) : x \in H\}$ for any $t \in I$.*

**Lemma 2.4** (see [30]) *Let $D$ and $F$ be bounded sets in $E$. Then*

$$\tilde{\alpha}(D \times F) = \max\left\{\alpha(D), \alpha(F)\right\},$$

*where $\tilde{\alpha}$ and $\alpha$ denote the Kuratowski noncompactness measure in $E \times E$ and $E$, respectively.*

**Lemma 2.5** (Sadovskii) *Let $D$ be a bounded, closed and convex subset of the Banach space $E$. If the operator $A : D \to D$ is condensing, then $A$ has a fixed point in $D$.*

## 3 Main results

For convenience, let us list some conditions.

($H_1$) $f, g \in C[J_+ \times E \times E \times E \times E, E]$ and there exist nonnegative functions $a_i, b_i, c_i \in C[0,\infty)$ and $z_i \in C[J \times J \times J \times J, J]$ $(i = 0,1)$ such that

$$\left\|f(t,x_0,x_1,y_0,y_1)\right\| \le a_0(t) + b_0(t)z_0\left(\|x_0\|,\|x_1\|,\|y_0\|,\|y_1\|\right)$$

for all $t \in J_+$, $x_i, y_i \in DC^1[J,E]$ $(i = 0,1)$;

$$\left\|g(t,x_0,x_1,y_0,y_1)\right\| \le a_1(t) + b_1(t)z_1\left(\|x_0\|,\|x_1\|,\|y_0\|,\|y_1\|\right)$$

for all $t \in J_+$, $x_i, y_i \in DC^1[J,E]$ $(i = 0,1)$; and

$$\frac{\|f(t,x_0,x_1,y_0,y_1)\|}{c_0(t)(\|x_0\| + \|x_1\| + \|y_0\| + \|y_1\|)} \to 0,$$
$$\frac{\|g(t,x_0,x_1,y_0,y_1)\|}{c_1(t)(\|x_0\| + \|x_1\| + \|y_0\| + \|y_1\|)} \to 0,$$

as $x_i, y_i \in DC^1[J,E]$ $(i = 0,1)$, $\|x_0\| + \|x_1\| + \|y_0\| + \|y_1\| \to \infty$, uniformly for $t \in J_+$; and for $i = 0,1$,

$$\int_0^\infty a_i(t)\,dt = a_i^* < +\infty,$$
$$\int_0^\infty b_i(t)\,dt = b_i^* < +\infty,$$
$$\int_0^\infty \left(1 + t^{\alpha-1}\right)c_i(t)\,dt = c_i^* < +\infty.$$

($H_2$) For any $r > 0$, $[\alpha,\beta] \subset J$, $f(t,x_0,x_1,y_0,y_1)$ and $g(t,x_0,x_1,y_0,y_1)$ are uniformly continuous on $[\alpha,\beta] \times B_E[\theta,r] \times B_E[\theta,r] \times B_E[\theta,r] \times B_E[\theta,r]$, where $\theta$ is the zero element of $E$ and $B_E[\theta,r] = \{x \in E : \|x\| \le r\}$.

($H_3$) For any $t \in J_+$ and countable bounded set $V_i, W_i \subset DC^1[J,E]$ $(i = 0,1)$, there exist $L_{ij}(t), K_{ij}(t) \in L[J,J]$ $(i = 0,1)$ such that

$$\alpha_E\left(f\left(t, V_0(t), V_1(t), W_0(t), W_1(t)\right)\right) \le \sum_{i=0}^{1}\left(L_{0i}(t)\alpha_E\left(V_i(t)\right) + K_{0i}(t)\alpha_E\left(W_i(t)\right)\right),$$
$$\alpha_E\left(g\left(t, V_0(t), V_1(t), W_0(t), W_1(t)\right)\right) \le \sum_{i=0}^{1}\left(L_{1i}(t)\alpha_E\left(V_i(t)\right) + K_{1i}(t)\alpha_E\left(W_i(t)\right)\right),$$

with

$$G_i^* = \int_0^\infty \left[\left(1 + s^{\alpha-1}\right)\left(L_{i0}(s) + K_{i0}(s)\right) + L_{i1}(s) + K_{i1}(s)\right]ds < 1 \quad (i = 0,1),$$
$$\lambda = \max\left\{G_0^*, G_1^*\right\}.$$

We shall reduce BVP (1) to a system of integral equations in $E$. To this end, we first consider operator $A$ defined by

$$A(x,y)(t) = \left(A_1(x,y)(t), A_2(x,y)(t)\right), \tag{2}$$

where

$$\begin{aligned} &A_1(x,y)(t) \\ &\quad = \frac{x_\infty}{\Gamma(\alpha)} t^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \int_0^t \left[t^{\alpha-1} - (t-s)^{\alpha-1}\right] f\left(s, x(s), x'(s), y(s), y'(s)\right) ds \\ &\qquad - \frac{1}{\Gamma(\alpha)} \int_t^\infty t^{\alpha-1} f\left(s, x(s), x'(s), y(s), y'(s)\right) ds \end{aligned} \tag{3}$$

and

$$\begin{aligned} &A_2(x,y)(t) \\ &\quad = \frac{y_\infty}{\Gamma(\alpha)} t^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \int_0^t \left[t^{\alpha-1} - (t-s)^{\alpha-1}\right] g\left(s, x(s), x'(s), y(s), y'(s)\right) ds \\ &\qquad - \frac{1}{\Gamma(\alpha)} \int_t^\infty t^{\alpha-1} g\left(s, x(s), x'(s), y(s), y'(s)\right) ds. \end{aligned}$$

**Lemma 3.1** *If* ($H_1$) *is satisfied, then the operator $A$ defined by* (2) *is a continuous and bounded operator from $X$ to $X$.*

*Proof* Let

$$\varepsilon_0 = \min\left\{\frac{\Gamma(\alpha)}{32c_0^*}, \frac{\Gamma(\alpha)}{32c_1^*}\right\}.$$

From ($H_1$) there exists $R > 0$ such that

$$\left\|f(t, x_0, x_1, y_0, y_1)\right\| \le \varepsilon_0 c_0(t)\left(\|x_0\| + \|x_1\| + \|y_0\| + \|y_1\|\right)$$

for all $t \in J_+$, $x_i, y_i \in DC^1[J,E]$ $(i = 0,1)$, $\|x_0\| + \|x_1\| + \|y_0\| + \|y_1\| > R$; and

$$\left\|f(t, x_0, x_1, y_0, y_1)\right\| \le a_0(t) + M_0 b_0(t)$$

for all $t \in J_+$, $x_i, y_i \in DC^1[J,E]$ $(i = 0,1)$, $\|x_0\| + \|x_1\| + \|y_0\| + \|y_1\| \le R$, where

$$M_0 = \max\left\{z_0\left(\|x_0\|, \|x_1\|, \|y_0\|, \|y_1\|\right) : 0 \le \|x_i\|, \|y_i\| \le R \ (i = 0,1)\right\}.$$

Hence

$$\left\|f(t, x_0, x_1, y_0, y_1)\right\| \le \varepsilon_0 c_0(t)\left(\|x_0\| + \|x_1\| + \|y_0\| + \|y_1\|\right) + a_0(t) + M_0 b_0(t) \tag{4}$$

for all $t \in J_+$, $x_i, y_i \in DC^1[J,E]$ $(i = 0,1)$. Let $(x,y) \in X$. From (4) we have

$$\begin{aligned} &\left\|f\left(t, x, x', y, y'\right)\right\| \\ &\quad \le \varepsilon_0 c_0(t)\left(1 + t^{\alpha-1}\right)\left(\frac{\|x(t)\|}{1 + t^{\alpha-1}} + \frac{\|x'(t)\|}{1 + t^{\alpha-1}} + \frac{\|y(t)\|}{1 + t^{\alpha-1}} + \frac{\|y'(t)\|}{1 + t^{\alpha-1}}\right) + a_0(t) + M_0 b_0(t) \\ &\quad \le \varepsilon_0 c_0(t)\left(1 + t^{\alpha-1}\right)\left(\|x\|_F + \left\|x'\right\|_1 + \|y\|_F + \left\|y'\right\|_1\right) + a_0(t) + M_0 b_0(t) \end{aligned}$$

$$
\begin{aligned}
&\leq 2\varepsilon_0 c_0(t)\bigl(1+t^{\alpha-1}\bigr)\bigl(\|x\|_D+\|y\|_D\bigr)+a_0(t)+M_0 b_0(t)\\
&\leq 4\varepsilon_0 c_0(t)\bigl(1+t^{\alpha-1}\bigr)\bigl\|(x,y)\bigr\|_X+a_0(t)+M_0 b_0(t),\quad \forall t\in J_+.
\end{aligned}
\tag{5}
$$

It follows from ($H_1$) and (5) that the infinite integral

$$
\int_0^\infty \bigl\|f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\bigr\|\,ds
$$

is convergent. From (3) and ($H_1$) we have

$$
\begin{aligned}
&\bigl\|A_1(x,y)(t)\bigr\|\\
&\quad=\biggl\|\frac{x_\infty}{\Gamma(\alpha)}t^{\alpha-1}-\frac{1}{\Gamma(\alpha)}\int_0^t\bigl[t^{\alpha-1}-(t-s)^{\alpha-1}\bigr]f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\,ds\\
&\qquad-\frac{1}{\Gamma(\alpha)}\int_t^\infty t^{\alpha-1}f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\,ds\biggr\|\\
&\quad\leq\frac{t^{\alpha-1}}{\Gamma(\alpha)}\|x_\infty\|+\frac{1}{\Gamma(\alpha)}\int_0^t\bigl[t^{\alpha-1}-(t-s)^{\alpha-1}\bigr]\bigl\|f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\bigr\|\,ds\\
&\qquad+\frac{1}{\Gamma(\alpha)}\int_t^\infty t^{\alpha-1}\bigl\|f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\bigr\|\,ds.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{\|A_1(x,y)(t)\|}{1+t^{\alpha-1}}\\
&\quad\leq\frac{1}{\Gamma(\alpha)}\|x_\infty\|+\frac{1}{\Gamma(\alpha)}\int_0^\infty\bigl\|f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\bigr\|\,ds\\
&\qquad+\frac{1}{\Gamma(\alpha)}\int_0^\infty\bigl\|f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\bigr\|\,ds\\
&\quad\leq\frac{1}{\Gamma(\alpha)}\|x_\infty\|+\frac{2}{\Gamma(\alpha)}\int_0^\infty\bigl\|f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\bigr\|\,ds\\
&\quad\leq\frac{1}{\Gamma(\alpha)}\|x_\infty\|+\frac{2}{\Gamma(\alpha)}\int_0^\infty\bigl[4\varepsilon_0 c_0(t)\bigl(1+t^{\alpha-1}\bigr)\bigl\|(x,y)\bigr\|_X\\
&\qquad+a_0(t)+M_0 b_0(t)\bigr]\,dt\\
&\quad\leq\frac{1}{\Gamma(\alpha)}\|x_\infty\|+\frac{2}{\Gamma(\alpha)}4\varepsilon_0 c_0^*\bigl\|(x,y)\bigr\|_X+\bigl[a_0^*+M_0 b_0^*\bigr]\\
&\quad\leq\frac{1}{\Gamma(\alpha)}\|x_\infty\|+\frac{1}{4}\bigl\|(x,y)\bigr\|_X+\bigl[a_0^*+M_0 b_0^*\bigr]\\
&\quad<+\infty.
\end{aligned}
\tag{6}
$$

Differentiating (3), we have

$$
\begin{aligned}
&A_1'(x,y)(t)\\
&\quad=\frac{\alpha-1}{\Gamma(\alpha)}t^{\alpha-2}x_\infty-\frac{\alpha-1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-2}f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\,ds\\
&\qquad+\frac{\alpha-1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-2}f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\,ds.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\frac{\|A_1'(x,y)(t)\|}{1+t^{\alpha-1}} \\
&\quad\leq \frac{\alpha-1}{\Gamma(\alpha)}\frac{t^{\alpha-2}}{1+t^{\alpha-1}}\|x_\infty\| + \frac{\alpha-1}{\Gamma(\alpha)}\int_0^\infty \frac{t^{\alpha-2}}{1+t^{\alpha-1}}\big\|f\big(s,x(s),x'(s),y(s),y'(s)\big)\big\|\,ds \\
&\qquad + \frac{\alpha-1}{\Gamma(\alpha)}\int_0^t \frac{(t-s)^{\alpha-2}}{1+t^{\alpha-1}}\big\|f\big(s,x(s),x'(s),y(s),y'(s)\big)\big\|\,ds \\
&\quad\leq \frac{2}{\Gamma(\alpha)}\frac{t^{\alpha-2}}{1+t^{\alpha-1}}\|x_\infty\| + \frac{4}{\Gamma(\alpha)}\int_0^\infty \frac{t^{\alpha-2}}{1+t^{\alpha-1}}\big\|f\big(s,x(s),x'(s),y(s),y'(s)\big)\big\|\,dt \\
&\quad< \frac{2}{\Gamma(\alpha)}\|x_\infty\| + \frac{4}{\Gamma(\alpha)}4\varepsilon_0 c_0^*\big\|(x,y)\big\|_X + \big[a_0^* + M_0 b_0^*\big] \\
&\quad\leq \frac{2}{\Gamma(\alpha)}\|x_\infty\| + \frac{1}{2}\big\|(x,y)\big\|_X + \big[a_0^* + M_0 b_0^*\big] \\
&\quad< +\infty.
\end{aligned}
\tag{7}
$$

It follows from (6) and (7) that

$$
\big\|A_1(x,y)\big\|_D \leq \frac{2}{\Gamma(\alpha)}\|x_\infty\| + \frac{1}{2}\big\|(x,y)\big\|_X + \big[a_0^* + M_0 b_0^*\big]. \tag{8}
$$

Thus, $A_1(x,y) \in DC^1[J,E]$ for any $(x,y) \in X$. In the same way, we can obtain

$$
\big\|A_2(x,y)\big\|_D \leq \frac{2}{\Gamma(\alpha)}\|y_\infty\| + \frac{1}{2}\big\|(x,y)\big\|_X + \big[a_1^* + M_1 b_1^*\big], \tag{9}
$$

where $M_1 = \max\{z_1(\|x_0\|,\|x_1\|,\|y_0\|,\|y_1\|) : 0 \leq \|x_i\|,\|y_i\| \leq R\ (i = 0,1)\}$. Thus, $A_2(x,y) \in DC^1[J,E]$ for any $(x,y) \in X$. Thus, $A$ maps $X$ to $X$ and $A$ is well defined.

Secondly, we show that $A$ maps bounded sets into bounded sets in $X$. It suffices to show that for any $\eta > 0$, there exists a positive constant $M > 0$ such that for each $(x,y) \in B_\eta = \{(x,y) \in X, \|(x,y)\|_X \leq \eta\}$, we have $\|A(x,y)\|_X \leq M$. Let

$$
M = \frac{1}{2}\eta + \gamma,
$$

where

$$
\gamma = \max\left\{\frac{2}{\Gamma(\alpha)}\|x_\infty\| + \big(a_0^* + M_0 b_0^*\big), \frac{2}{\Gamma(\alpha)}\|y_\infty\| + \big(a_1^* + M_1 b_1^*\big)\right\}.
$$

According to (8), (9) and ($H_1$), we have

$$
\big\|A(x,y)\big\|_X \leq \frac{1}{2}\big\|(x,y)\big\|_X + \gamma \leq \frac{1}{2}\eta + \gamma.
$$

It follows from the above inequality that $A$ maps bounded sets into bounded sets of $X$.

Thirdly, we prove that $A$ is continuous on $X$. Let $\{(x_m,y_m)\}_{m=1}^\infty \subset X$ and $(x,y) \in X$ such that $\lim_{m\to\infty}\|(x_m,y_m)-(x,y)\|_X \to 0$. Then $\{(x_m,y_m)\}$ is a bounded subset of $X$. Thus, there exists $r > 0$ such that $\|(x_m, y_m)\|_X < r$ for $m \geq 1$ and $\|(x,y)\|_X \leq r$. It is easy to show that

$$\begin{aligned} &\left\| \frac{A_1(x_m, y_m)}{1+t^{\alpha-1}} - \frac{A_1(x,y)}{1+t^{\alpha-1}} \right\| \\ &\quad \leq \frac{2}{\Gamma(\alpha)} \int_0^\infty \big\| f\big(s, x_m(s), x_m'(s), y_m(s), y_m'(s)\big) \\ &\qquad - f\big(s, x(s), x'(s), y(s), y'(s)\big) \big\| \, ds \end{aligned} \tag{10}$$

and

$$\begin{aligned} &\left\| \frac{A_1'(x_m, y_m)}{1+t^{\alpha-1}} - \frac{A_1'(x,y)}{1+t^{\alpha-1}} \right\| \\ &\quad \leq \frac{2}{\Gamma(\alpha)} \int_0^\infty \big\| f\big(s, x_m(s), x_m'(s), y_m(s), y_m'(s)\big) \\ &\qquad - f\big(s, x(s), x'(s), y(s), y'(s)\big) \big\| \, ds. \end{aligned} \tag{11}$$

It is clear that

$$f\big(s, x_m(s), x_m'(s), y_m(s), y_m'(s)\big) \to f\big(s, x(s), x'(s), y(s), y'(s)\big), \tag{12}$$

as $m \to \infty$, for all $s \in J_+$. From (5) we have

$$\begin{aligned} &\big\| f\big(s, x_m(s), x_m'(s), y_m(s), y_m'(s)\big) - f\big(s, x(s), x'(s), y(s), y'(s)\big) \big\| \\ &\quad \leq 8\varepsilon_0 c_0(s)\big(1+s^{\alpha-1}\big)r + 2a_0(s) + 2M_0(s)b_0(s) \\ &\quad = \sigma_0(s), \quad \sigma_0(s) \in L[J,J], m = 1,2,3,\ldots, \forall s \in J_+. \end{aligned} \tag{13}$$

From (12), (13) and the dominated convergence theorem we have

$$\begin{aligned} &\lim_{m\to\infty} \int_0^\infty \big\| f\big(s, x_m(s), x_m'(s), y_m(s), y_m'(s)\big) \\ &\quad - f\big(s, x(s), x'(s), y(s), y'(s)\big) \big\| \, ds = 0. \end{aligned} \tag{14}$$

It follows from (10), (11) and (14) that $\|A_1(x_m, y_m) - A_1(x,y)\|_D \to 0$ as $m \to \infty$. By the same method, we have $\|A_2(x_m, y_m) - A_2(x,y)\|_D \to 0$ as $m \to \infty$. Therefore, the continuity of $A$ is proved. □

**Lemma 3.2** *Under assumption* ($H_1$), $(x,y) \in X$ *is a solution of BVP* (1) *if and only if* $(x,y) \in X$ *is a fixed point of* $A$.

*Proof* Suppose that $(x,y) \in X$ is a solution of BVP (1). By Lemma 2.2, the solution of BVP (1) can be written as

$$x(t) = c_{11}t^{\alpha-1} + c_{12}t^{\alpha-2} + c_{13}t^{\alpha-3} - I_{0+}^{\alpha} f\big(t, x(t), x'(t), y(t), y'(t)\big) \tag{15}$$

and

$$y(t) = c_{21}t^{\alpha-1} + c_{22}t^{\alpha-2} + c_{23}t^{\alpha-3} - I_{0+}^{\alpha} g\big(t, x(t), x'(t), y(t), y'(t)\big). \tag{16}$$

From $x(0) = x'(0) = 0$ and $y(0) = y'(0) = 0$, we know that $c_{12} = c_{13} = c_{22} = c_{23} = 0$. Together with $D_{0+}^{\alpha-1}x(\infty) = x_\infty$ and $D_{0+}^{\alpha-1}y(\infty) = y_\infty$, we have

$$c_{11} = \frac{x_\infty}{\Gamma(\alpha)} - \frac{1}{\Gamma(\alpha)}\int_0^\infty f\bigl(t, x(t), x'(t), y(t), y'(t)\bigr)\,dt \tag{17}$$

and

$$c_{21} = \frac{y_\infty}{\Gamma(\alpha)} - \frac{1}{\Gamma(\alpha)}\int_0^\infty g\bigl(t, x(t), x'(t), y(t), y'(t)\bigr)\,dt. \tag{18}$$

By substituting (17) and (18) into (15) and (16) respectively, we obtain

$$\begin{aligned} x(t) = {}& \frac{x_\infty}{\Gamma(\alpha)}t^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_0^t \bigl[t^{\alpha-1} - (t-s)^{\alpha-1}\bigr] f\bigl(s, x(s), x'(s), y(s), y'(s)\bigr)\,ds \\ & - \frac{1}{\Gamma(\alpha)}\int_t^\infty t^{\alpha-1} f\bigl(s, x(s), x'(s), y(s), y'(s)\bigr)\,ds \end{aligned}$$

and

$$\begin{aligned} y(t) = {}& \frac{y_\infty}{\Gamma(\alpha)}t^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_0^t \bigl[t^{\alpha-1} - (t-s)^{\alpha-1}\bigr] g\bigl(s, x(s), x'(s), y(s), y'(s)\bigr)\,ds \\ & - \frac{1}{\Gamma(\alpha)}\int_t^\infty t^{\alpha-1} g\bigl(s, x(s), x'(s), y(s), y'(s)\bigr)\,ds. \end{aligned}$$

Obviously, the integrals $\int_0^\infty f(s, x(s), x'(s), y(s), y'(s))\,ds$ and $\int_0^\infty g(s, x(s), x'(s), y(s), y'(s))\,ds$ are convergent. Therefore, $(x, y)$ is a fixed point of operator $A$. Conversely, if $(x, y)$ is the fixed point of operator $A$, then direct differentiation gives the proof. □

**Lemma 3.3** *Let condition* $(\mathrm{H}_1)$ *be satisfied and* $V$ *be a bounded subset of* $X$. *Then* $\frac{(AV)(t)}{1+t^{\alpha-1}}$ *and* $\frac{(AV)'(t)}{1+t^{\alpha-1}}$ *are equicontinuous on any finite subinterval of* $J$; *and for any* $\varepsilon > 0$, *there exists* $N > 0$ *such that*

$$\left\| \frac{A_i(x,y)(t_1)}{1+t_1^{\alpha-1}} - \frac{A_i(x,y)(t_2)}{1+t_2^{\alpha-1}} \right\| < \varepsilon, \qquad \left\| \frac{A_i'(x,y)(t_1)}{1+t_1^{\alpha-1}} - \frac{A_i'(x,y)(t_2)}{1+t_2^{\alpha-1}} \right\| < \varepsilon, \quad i = 1, 2,$$

*uniformly with respect to* $(x, y) \in V$ *as* $t_1, t_2 \geq N$.

*Proof* We only give the proof for operator $A_1$. Rewrite

$$\begin{aligned} A_1(x,y)(t) = {}& \frac{x_\infty}{\Gamma(\alpha)}t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\bigl(s, x(s), x'(s), y(s), y'(s)\bigr)\,ds \\ & - \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1} f\bigl(s, x(s), x'(s), y(s), y'(s)\bigr)\,ds. \end{aligned}$$

In view of condition $(\mathrm{H}_1)$ and the boundedness of $V$, there exists $M > 0$ such that

$$\int_0^\infty \bigl\| f\bigl(s, x(s), x'(s), y(s), y'(s)\bigr)\bigr\|\,ds \leq M \quad \text{for any } (x, y) \in V. \tag{19}$$

Let the constant $R$ be such that $\|(x,y)\|_X \le R$ for any $(x,y) \in V$ and $[a,b] \subset J$ be a finite interval and $t_1, t_2 \in [a,b]$ with $t_1 < t_2$. Using (19) and the monotonicity of $\frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}}$ and $\frac{(t-s)^{\alpha-2}}{1+t^{\alpha-1}}$ in $t$ for $s < t$, we have

$$\begin{aligned}
&\left\| \frac{A_1(x,y)(t_2)}{1+t_2^{\alpha-1}} - \frac{A_1(x,y)(t_1)}{1+t_1^{\alpha-1}} \right\| \\
&\quad \le \left\| \frac{x_\infty t_2^{\alpha-1}}{\Gamma(\alpha)(1+t_2^{\alpha-1})} - \frac{x_\infty t_1^{\alpha-1}}{\Gamma(\alpha)(1+t_1^{\alpha-1})} \right\| \\
&\qquad + \left\| \frac{1}{\Gamma(\alpha)} \int_0^\infty \left( \frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}} - \frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}} \right) f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\,ds \right\| \\
&\qquad + \left\| \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{(1+t_2^{\alpha-1})\Gamma(\alpha)} f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\,ds \right. \\
&\qquad \left. - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{(1+t_1^{\alpha-1})\Gamma(\alpha)} f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\,ds \right\| \\
&\quad \le \frac{\|x_\infty\|}{\Gamma(\alpha)} \left( \frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}} - \frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}} \right) \\
&\qquad + \frac{1}{\Gamma(\alpha)} \left( \frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}} - \frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}} \right) \int_0^\infty \bigl\| f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr) \bigr\|\,ds \\
&\qquad + \frac{1}{\Gamma(\alpha)} \left\| \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}} f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\,ds \right. \\
&\qquad \left. - \int_0^{t_1} \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}} f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\,ds \right\| \\
&\qquad + \frac{1}{\Gamma(\alpha)} \left\| \int_0^{t_1} \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}} f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\,ds \right. \\
&\qquad \left. - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}} f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\,ds \right\| \\
&\quad \le \frac{\|x_\infty\|}{\Gamma(\alpha)} \left( \frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}} - \frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}} \right) \\
&\qquad + \frac{1}{\Gamma(\alpha)} \left( \frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}} - \frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}} \right) \int_0^\infty \bigl\| f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr) \bigr\|\,ds \\
&\qquad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}} \bigl\| f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr) \bigr\|\,ds \\
&\qquad + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left[ \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}} - \frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}} \right] \bigl\| f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr) \bigr\|\,ds
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
&\left\| \frac{A_1'(x,y)(t_2)}{1+t_2^{\alpha-1}} - \frac{A_1'(x,y)(t_1)}{1+t_1^{\alpha-1}} \right\| \\
&\quad \le \left\| \frac{\alpha-1}{\Gamma(\alpha)} x_\infty \right\| \left( \frac{t_2^{\alpha-2}}{1+t_2^{\alpha-1}} - \frac{t_1^{\alpha-2}}{1+t_1^{\alpha-1}} \right) \\
&\qquad + \frac{\alpha-1}{\Gamma(\alpha)} \int_0^\infty \left( \frac{t_2^{\alpha-2}}{1+t_2^{\alpha-1}} - \frac{t_1^{\alpha-2}}{1+t_1^{\alpha-1}} \right) \bigl\| f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr) \bigr\|\,ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\alpha-1}{\Gamma(\alpha)}\left\|\int_0^{t_2}\frac{(t_2-s)^{\alpha-2}}{1+t_2^{\alpha-1}}f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\,ds\right.\\
&\left.-\int_0^{t_1}\frac{(t_1-s)^{\alpha-2}}{1+t_1^{\alpha-1}}f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\,ds\right\|\\
\le{}&\frac{\alpha-1}{\Gamma(\alpha)}\|x_\infty\|\left(\frac{t_2^{\alpha-2}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-2}}{1+t_1^{\alpha-1}}\right)\\
&+\frac{\alpha-1}{\Gamma(\alpha)}\int_0^{\infty}\left(\frac{t_2^{\alpha-2}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-2}}{1+t_1^{\alpha-1}}\right)\bigl\|f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\bigr\|\,ds\\
&+\frac{\alpha-1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-2}}{1+t_2^{\alpha-1}}\bigl\|f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\bigr\|\,ds\\
&+\frac{\alpha-1}{\Gamma(\alpha)}\int_0^{t_1}\left[\frac{(t_2-s)^{\alpha-2}}{1+t_2^{\alpha-1}}-\frac{(t_1-s)^{\alpha-2}}{1+t_1^{\alpha-1}}\right]\bigl\|f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\bigr\|\,ds.
\end{aligned}
\tag{21}
$$

It follows from (20), (21) and ($H_1$) that $\frac{(A_1V)(t)}{1+t^{\alpha-1}}$ and $\frac{(A_1V)'(t)}{1+t^{\alpha-1}}$ are equicontinuous on any finite subinterval of $J$. We are in a position to show that for any $\varepsilon>0$, there exists $N'>0$ such that

$$\left\|\frac{A_1(x,y)(t_2)}{1+t_2^{\alpha-1}}-\frac{A_1(x,y)(t_1)}{1+t_1^{\alpha-1}}\right\|<\varepsilon$$

and

$$\left\|\frac{A_1'(x,y)(t_2)}{1+t_2^{\alpha-1}}-\frac{A_1'(x,y)(t_1)}{1+t_1^{\alpha-1}}\right\|<\varepsilon$$

uniformly with respect to $x\in V$ as $t_1,t_2\ge N'$.

According to (20) and (21), we only need to prove the following conclusions:

(i) $\forall\varepsilon>0$, there exists $N_1$ such that for any $(x,y)\in X$, $t_1,t_2>N_1$,

$$
\begin{aligned}
&\left\|\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\,ds\right.\\
&\left.\quad-\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\,ds\right\|\\
&<\frac{\varepsilon}{3}.
\end{aligned}
$$

(ii) $\forall\varepsilon>0$, there exists $N_2$ such that for any $(x,y)\in X$, $t_1,t_2>N_2$,

$$
\begin{aligned}
&\left\|\int_0^{t_2}\frac{(t_2-s)^{\alpha-2}}{1+t_2^{\alpha-1}}f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\,ds\right.\\
&\left.\quad-\int_0^{t_1}\frac{(t_1-s)^{\alpha-2}}{1+t_1^{\alpha-1}}f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\,ds\right\|\\
&<\frac{\varepsilon}{3}.
\end{aligned}
$$

It follows from (5) that there exists $N_0>0$ such that

$$\int_{N_0}^{\infty}\bigl\|f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\bigr\|\,ds\le\frac{\varepsilon}{9}\tag{22}$$

uniformly with respect to $(x,y)\in V$.

On the other hand, since

$$\lim_{t\to\infty}\frac{(t-N_0)^{\alpha-1}}{1+t^{\alpha-1}}=1,$$

there exists $N_1 > N_0$ such that for any $t_1, t_2 \geq N_1$ and $s \in [0, N_0]$,

$$\begin{aligned}&\left|\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\\&\quad\leq\left[1-\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}\right]+\left[1-\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right]\\&\quad\leq\left[1-\frac{(t_1-N_0)^{\alpha-1}}{1+t_1^{\alpha-1}}\right]+\left[1-\frac{(t_2-N_0)^{\alpha-1}}{1+t_2^{\alpha-1}}\right]\\&\quad<\frac{\varepsilon}{9M}.\end{aligned}\tag{23}$$

Thus, for any $\varepsilon > 0$, $(x, y) \in V$, when $t_1, t_2 \geq N_1$, from (19), (22) and (23), we can arrive at

$$\begin{aligned}&\left\|\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\,ds\right.\\&\qquad\left.-\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\,ds\right\|\\&\quad\leq\int_0^{N_1}\left|\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\bigl\|f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\bigr\|\,ds\\&\qquad+\int_{N_1}^{t_1}\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}\bigl\|f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\bigr\|\,ds\\&\qquad+\int_{N_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\bigl\|f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\bigr\|\,ds\\&\quad<\frac{\varepsilon}{9M}\int_0^{\infty}\bigl\|f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\bigr\|\,ds+2\int_{N_1}^{\infty}\bigl\|f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\bigr\|\,ds\\&\quad\leq\frac{\varepsilon}{3}.\end{aligned}$$

Since

$$\lim_{t\to\infty}\frac{(t-N_0)^{\alpha-2}}{1+t^{\alpha-1}}=0,$$

there exists $N_2 > N_0$ such that for any $t_1, t_2 > N_2$ and $s \in [0, N_0]$,

$$\frac{(t_1-N_0)^{\alpha-2}}{1+t_1^{\alpha-1}}<\frac{\varepsilon}{18M}<1,\qquad\frac{(t_2-N_0)^{\alpha-2}}{1+t_2^{\alpha-1}}<\frac{\varepsilon}{18M}<1.$$

Thus, for any $\varepsilon > 0$, $(x, y) \in V$, when $t_1, t_2 \geq N_2$, from (19), (22) and (23) we have

$$\begin{aligned}&\left\|\int_0^{t_2}\frac{(t_2-s)^{\alpha-2}}{1+t_2^{\alpha-1}}f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\,ds\right.\\&\qquad\left.-\int_0^{t_1}\frac{(t_1-s)^{\alpha-2}}{1+t_1^{\alpha-1}}f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\,ds\right\|\end{aligned}$$

$$\begin{aligned}
&\leq \int_0^{N_2}\left|\frac{(t_2-s)^{\alpha-2}}{1+t_2^{\alpha-1}}-\frac{(t_1-s)^{\alpha-2}}{1+t_1^{\alpha-1}}\right|\left\|f\big(s,x(s),x'(s),y(s),y'(s)\big)\right\|ds \\
&\quad+\int_{N_2}^{t_1}\frac{(t_1-s)^{\alpha-2}}{1+t_1^{\alpha-1}}\left\|f\big(s,x(s),x'(s),y(s),y'(s)\big)\right\|ds \\
&\quad+\int_{N_2}^{t_2}\frac{(t_2-s)^{\alpha-2}}{1+t_2^{\alpha-1}}\left\|f\big(s,x(s),x'(s),y(s),y'(s)\big)\right\|ds \\
&< \frac{\varepsilon}{9M}\int_0^{\infty}\left\|f\big(s,x(s),x'(s),y(s),y'(s)\big)\right\|ds+2\int_{N_2}^{\infty}\left\|f\big(s,x(s),x'(s),y(s),y'(s)\big)\right\|ds \\
&\leq \frac{\varepsilon}{3}.
\end{aligned}$$

The proof for operator $A_2$ can be given in a similar way and the proof is complete. □

**Lemma 3.4** (see [16]) *Let* $(H_1)$ *be satisfied, $V$ be a bounded set in $DC^1[J,E]\times DC^1[J,E]$. Then*

$$\alpha_D(A_iV)=\max\left\{\sup_{t\in J}\alpha_E\left(\frac{A_iV(t)}{1+t^{\alpha-1}}\right),\sup_{t\in J}\alpha_E\left(\frac{A_i'V(t)}{1+t^{\alpha-1}}\right)\right\}\quad(i=0,1).$$

The main result of this paper is as follows.

**Theorem 3.1** *Let conditions* $(H_1)$, $(H_2)$ *and* $(H_3)$ *be satisfied. Then BVP* (1) *has at least one solution belonging to $X$.*

*Proof* We only need to prove the existence of a fixed point of operator $A$ in $X$. By condition $(H_1)$, we can choose a real number $R$ such that

$$R>\max\left\{2\left[\frac{\|x_\infty\|+\|y_\infty\|}{\Gamma(\alpha)}+\big(a_0^*+M_0b_0^*\big)\right],2\left[\frac{2(\|x_\infty\|+\|y_\infty\|)}{\Gamma(\alpha)}+\big(a_1^*+M_1b_1^*\big)\right]\right\}.$$

Let

$$B=:B_X(\theta,R)=\big\{(x,y)\in X:\big\|(x,y)\big\|_X\leq R\big\}.$$

In the following, we proceed to show $AB\subset B$. In fact, for any $(x,y)\in B$, from (6) and (7) we have

$$\begin{aligned}
\left\|\frac{A_1(x,y)(t)}{1+t^{\alpha-1}}\right\| &\leq \frac{\|x_\infty\|}{\Gamma(\alpha)}+\frac{1}{4}\big\|(x,y)\big\|_X+\big(a_0^*+M_0b_0^*\big) \\
&< \frac{\|x_\infty\|}{\Gamma(\alpha)}+\frac{R}{4}+\left[\frac{R}{2}-\frac{\|x_\infty\|}{\Gamma(\alpha)}\right] \\
&< R\quad(\forall t\in J)
\end{aligned}$$

and

$$\begin{aligned}
\left\|\frac{A_1'(x,y)(t)}{1+t^{\alpha-1}}\right\| &\leq \frac{2\|x_\infty\|}{\Gamma(\alpha)}+\frac{1}{2}\big\|(x,y)\big\|_X+\big(a_1^*+M_1b_1^*\big) \\
&< \frac{2\|x_\infty\|}{\Gamma(\alpha)}+\frac{R}{2}+\left[\frac{R}{2}-\frac{2\|x_\infty\|}{\Gamma(\alpha)}\right] \\
&\leq R\quad(\forall t\in I).
\end{aligned}$$

Similarly, we can get $\|\frac{A_2(x,y)(t)}{1+t^{\alpha-1}}\| < R$ and $\|\frac{A_2'(x,y)(t)}{1+t^{\alpha-1}}\| < R$. Thus, taking Lemma 3.1 into consideration, we obtain $AB \subset B$.

Let $D = \bar{\mathrm{co}}_X(AB)$, *i.e.*, $D$ is the convex closure of $AB$ in $X$. Clearly, $D$ is a nonempty, bounded, convex and closed subset of $B$. It follows from Lemma 3.3 that $\frac{(AB)(t)}{1+t^{\alpha-1}}$ and $\frac{(AB)'(t)}{1+t^{\alpha-1}}$ are equicontinuous on $J$. Together with the definition of $D$, we know that $\frac{D(t)}{1+t^{\alpha-1}}$ and $\frac{D'(t)}{1+t^{\alpha-1}}$ are equicontinuous on $J$.

Now we are in a position to show that $A$ is a strict set contraction operator from $D$ to $D$. Observing that $D \subset B$ and $AB \subset D$, together with Lemma 3.1 we know that $A : D \to D$ is bounded and continuous.

Finally, we prove that there exists a constant $\lambda \in [0,1)$ such that $\alpha_X(AV) \le \lambda\alpha_X(V)$ for $V \subset D$. Further, due to Lemma 2.4,

$$\alpha_X(AV) = \max\bigl\{\alpha_D(A_1V), \alpha_D(A_2V)\bigr\}.$$

Thanks to Lemma 3.4, we have

$$\alpha_D(A_1V) = \max\left\{\sup_{t\in J}\alpha_E\left(\frac{A_1V(t)}{1+t^{\alpha-1}}\right), \sup_{t\in J}\alpha_E\left(\frac{A_1'V(t)}{1+t^{\alpha-1}}\right)\right\}.$$

It is enough to verify that

$$\sup_{t\in J}\alpha_E\left(\frac{A_1V(t)}{1+t^{\alpha-1}}\right) \le \lambda\alpha_X(V), \qquad \sup_{t\in J}\alpha_E\left(\frac{A_1'V(t)}{1+t^{\alpha-1}}\right) \le \lambda\alpha_X(V) \tag{24}$$

and

$$\sup_{t\in J}\alpha_E\left(\frac{A_2V(t)}{1+t^{\alpha-1}}\right) \le \lambda\alpha_X(V), \qquad \sup_{t\in J}\alpha_E\left(\frac{A_2'V(t)}{1+t^{\alpha-1}}\right) \le \lambda\alpha_X(V). \tag{25}$$

Define

$$\begin{aligned} A_1^n(x,y)(t) := {}& \frac{x_\infty}{\Gamma(\alpha)}t^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_0^t \bigl[t^{\alpha-1} - (t-s)^{\alpha-1}\bigr]f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\,ds \\ & - \frac{1}{\Gamma(\alpha)}\int_t^n t^{\alpha-1}f\bigl(s,x(s),x'(s),y(s),y'(s)\bigr)\,ds. \end{aligned}$$

From ($H_1$) and (5) we have

$$\begin{aligned} &\left\|\frac{A_1^n(x,y)(t)}{1+t^{\alpha-1}} - \frac{A_1(x,y)(t)}{1+t^{\alpha-1}}\right\| \\ &\quad\le \frac{1}{\Gamma(\alpha)}\int_n^\infty \bigl\|f\bigl(t,x(t),x'(t),y(t),y'(t)\bigr)\bigr\|\,dt \\ &\quad\le \frac{2}{\Gamma(\alpha)}\left\{R\int_n^\infty 4\varepsilon_0 c_0(t)\bigl(1+t^{\alpha-1}\bigr)\,dt + \int_n^\infty \bigl[a_0(t) + M_0 b_0(t)\bigr]\,dt\right\}. \end{aligned}$$

It follows that $d(\frac{A_1^nV(t)}{1+t^{\alpha-1}}, \frac{A_1V(t)}{1+t^{\alpha-1}}) \to 0$ as $n \to \infty$, $t \in J$, where $d(\cdot)$ denotes the Hausdorff metric in the space $E$. Thus, due to the property of noncompactness measure, we get

$$\lim_{n\to\infty}\alpha_E\left(\frac{A_1^nV(t)}{1+t^{\alpha-1}}\right) = \alpha_E\left(\frac{A_1V(t)}{1+t^{\alpha-1}}\right), \quad t\in J. \tag{26}$$

Next, we estimate $\alpha_E(\frac{A_1^n V(t)}{1+t^{\alpha-1}})$. Lemma 3.3 implies that $\{\frac{D(t)}{1+t^{\alpha-1}}\}$ is equicontinuous on any finite interval of $J$, and hence $\{\frac{V(t)}{1+t^{\alpha-1}}\}$ is equicontinuous on any finite interval of $J$. By condition ($H_2$), it is easy to know that $\{f(t,x(t),x'(t),y(t),y'(t)):(x,y)\in V\}$ is equicontinuous on $[0,n]$. Moreover, $\{f(t,x(t),x'(t),y(t),y'(t)):(x,y)\in V\}$ is bounded on $[0,n]$ by ($H_1$). From Lemma 2.3 and condition ($H_3$) we have

$$\begin{aligned}
&\alpha_E\left(\frac{A_1^n V(t)}{1+t^{\alpha-1}}\right)\\
&\quad\leq \frac{1}{\Gamma(\alpha)}\int_0^t \alpha_E\bigl(\bigl\{f\bigl(t,x(t),x'(t),y(t),y'(t)\bigr):(x,y)\in V\bigr\}\bigr)\,dt\\
&\qquad+\frac{1}{\Gamma(\alpha)}\int_t^n \alpha_E\bigl(\bigl\{f\bigl(t,x(t),x'(t),y(t),y'(t)\bigr):(x,y)\in V\bigr\}\bigr)\,dt\\
&\quad\leq \frac{1}{\Gamma(\alpha)}\int_0^n \alpha_E\bigl(\bigl\{f\bigl(t,x(t),x'(t),y(t),y'(t)\bigr):(x,y)\in V\bigr\}\bigr)\,dt\\
&\quad\leq \frac{1}{\Gamma(\alpha)}\int_0^n \bigl[(1+s)^{\alpha-1}\bigl(L_{00}(s)+K_{00}(s)\bigr)+L_{01}(s)+K_{01}(s)\bigr]\,ds\alpha_X(V).
\end{aligned}$$

Taking (26) into consideration, we have

$$\alpha_E\left(\frac{A_1 V(t)}{1+t^{\alpha-1}}\right)\leq G_0^*\alpha_X(V)\leq\lambda\alpha_X(V),$$

with

$$G_0^*=\int_0^n\bigl[(1+s)^{\alpha-1}\bigl(L_{00}(s)+K_{00}(s)\bigr)+L_{01}(s)+K_{01}(s)\bigr]\,ds\leq\lambda<1.$$

In the same way, we can obtain $\alpha_E(\frac{A_1' V(t)}{1+t^{\alpha-1}})\leq\lambda\alpha_X(V)$, and hence relation (24) is valid. By the same method, we can also obtain relation (25). Thus, we know that $\alpha_X(AV)\leq\lambda\alpha_X(V)$. Obviously, $0\leq\lambda<1$ by ($H_3$). Consequently, $A$ is a strict set contraction operator from $V$ to $V$. Obviously, $A$ is condensing, too. It follows from Lemma 2.5 that $A$ has at least one fixed point in $V$, that is, BVP (1) has at least one solution in $X$. □

As a special case of Theorem 3.1, we obtain the following result.

**Corollary 3.1** *If the following assumptions hold*:

($H_1'$) *$f\in C[J_+\times E\times E,E]$ and there exist $a,b,c\in C[J_+,J]$ and $z\in C[J_+\times J_+,J]$ such that*

(i) $\|f(t,x,y)\|\leq a(t)+b(t)z(x,y)$ *for all* $t\in J_+, x,y\in E$;

(ii) $\dfrac{\|f(t,x,y)\|}{c(t)(\|x\|+\|y\|)}\to 0$, *as* $x,y\in E$, $\|x\|+\|y\|\to\infty$ *uniformly for* $t\in J_+$;

(iii) $\int_0^\infty a(t)\,dt=a^*<\infty$, $\int_0^\infty b(t)\,dt=b^*<\infty$, $\int_0^\infty(1+t)c(t)\,dt=c^*<\infty$.

($H_2'$) *For any $r>0$, $[\alpha,\beta]\subset I$, $f(t,x,y)$ is uniformly continuous on $[\alpha,\beta]\times B_E[\theta,r]\times B_E[\theta,r]$, where $\theta$ is the zero element of $E$ and $B_E[\theta,R]=\{x\in E:\|x\|_D\leq r\}$.*

($H_3'$) *For any* $t \in J_+$ *and a countable bounded set* $V, W \subset DC^1[J,E]$, *there exist* $l_i(t) \in L[J,J]$ $(i = 0,1)$ *such that*

$$\alpha\bigl(f(t,V,W)\bigr) \leq l_1(t)\alpha_E(V) + l_2(t)\alpha_E(W)$$

*and*

$$\int_0^\infty \bigl[(1+t)l_1(t) + l_2(t)\bigr]\,dt < 1.$$

*Then the fractional differential equation boundary value problem*

$$\begin{cases} D_{0+}^\alpha x(t) + f(t,x(t),x'(t)) = 0, \quad 2 < \alpha \leq 3, t \in J_+, \\ x(0) = x'(0) = 0, \qquad D_{0+}^{\alpha-1}x(\infty) = x_\infty \end{cases} \tag{27}$$

*has at least one solution in* $DC^1[I,E]$.

*Proof* Letting $g \equiv 0$ and $y \equiv 0$ in Theorem 3.1, we get the desired result. □

## 4 Example

As an application of Theorem 3.1, we consider the infinite system of nonlinear differential equations of fractional order:

$$\begin{cases} -D_{0+}^{\frac{5}{2}}x_n(t) = \frac{(2+y_n(t)+x'_{2n}(t)+y'_{3n}(t))^{\frac{1}{2}}}{2n^3\sqrt[3]{e^{2t}}(1+t)^{\frac{5}{2}}(1+t^{\frac{3}{2}})} + \frac{x_n(t)}{2\sqrt[6]{t}(1+t)^{\frac{5}{2}}(1+t^{\frac{3}{2}})}, \\ -D_{0+}^{\frac{5}{2}}y_n(t) = \frac{(6+x_{3n}(t)+x'_{4n}(t))^{\frac{1}{3}}}{14n^4\sqrt[3]{e^{2t}}(2+5t)^8} + \frac{\ln[(3+4t)y'_{4n}(t)]}{14\sqrt[6]{t}(3+4t)^3}, \\ x_n(0) = x'_n(0) = \theta, \qquad D_{0+}^{\frac{3}{2}}x_n(\infty) = x_\infty, \\ y_n(0) = y'_n(0) = \theta, \qquad D_{0+}^{\frac{3}{2}}y_n(\infty) = y_\infty \quad (n = 1,2,\ldots). \end{cases} \tag{28}$$

Let $E = \{x = (x_1,\ldots,x_n,\ldots) : x_n \to 0\}$ with the norm $\|x\| = \sup_n |x_n|$. Obviously, $(E, \|\cdot\|)$ is a Banach space. Problem (28) can be regarded as a boundary value problem of form (1) in $E$ with $x_\infty = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$, $y_\infty = (\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots)$. In this situation, $x = (x_1,\ldots,x_n,\ldots)$, $u = (u_1,\ldots,u_n,\ldots)$, $y = (y_1,\ldots,y_n,\ldots)$, $v = (v_1,\ldots,v_n,\ldots)$, $f = (f_1,\ldots,f_n,\ldots)$ with

$$f_n(t,x,u,y,v) = \frac{1}{2n^3\sqrt[3]{e^{2t}}(1+t)^{\frac{5}{2}}(1+t^{\frac{3}{2}})}(2 + y_n + u_{2n} + v_{3n})^{\frac{1}{2}} + \frac{1}{2\sqrt[6]{t}(1+t)^{\frac{5}{2}}(1+t^{\frac{3}{2}})}x_n \tag{29}$$

and

$$g_n(t,x,u,y,v) = \frac{1}{14n^4\sqrt[3]{e^{2t}}(2+5t)^8}(6 + x_{3n} + u_{4n})^{\frac{1}{3}} + \frac{1}{14\sqrt[6]{t}(3+4t)^3}\ln\bigl[(3+4t)v_{4n}\bigr]. \tag{30}$$

Now, we verify that conditions ($H_1$)-($H_3$) are satisfied. Note that $\sqrt[3]{e^{2t}} > \sqrt[6]{t}$ for $t > 0$, it follows from (29) and (30) that

$$\bigl\|f(t,x,u,y,v)\bigr\| \leq \frac{1}{2\sqrt[6]{t}(1+t)^{\frac{5}{2}}}\bigl\{\bigl(2 + \|y\| + \|u\| + \|v\|\bigr)^{\frac{1}{2}} + \|x\|\bigr\}$$

and

$$\|g(t,x,u,y,v)\| \le \frac{1}{14\sqrt[6]{t}(3+4t)^2}\left\{\left(6+\|x\|+\|u\|\right)^{\frac{1}{3}}+\ln\left[(3+4t)\|v\|\right]\right\},$$

which implies $(H_1)$ is satisfied for $a_0(t)=0$, $b_0(t)=c_0(t)=\frac{1}{2\sqrt[6]{t}(1+t)^{\frac{5}{2}}}$, $a_1(t)=0$, $b_1(t)=c_1(t)=\frac{1}{14\sqrt[6]{t}(3+4t)^2}$ and

$$z_0(x,u,y,v)=(2+y+u+v)^{\frac{1}{2}}+x,$$
$$z_1(x,u,y,v)=(6+x+u)^{\frac{1}{3}}+\ln\left[(3+4t)v\right].$$

It is easy to see that $(H_2)$ is satisfied. Finally, we verify condition $(H_3)$. Let $f^1=\{f_n^1,f_n^1,\ldots,f_n^1,\ldots\}$, $f^2=\{f_n^2,f_n^2,\ldots,f_n^2,\ldots\}$, $g^1=\{g_n^1,g_n^1,\ldots,g_n^1,\ldots\}$, $g^2=\{g_n^2,g_n^2,\ldots,g_n^2,\ldots\}$, where

$$f_n^1(t,x,u,y,v)=\frac{1}{2n^3\sqrt[3]{e^{2t}}(1+t)^{\frac{5}{2}}(1+t^{\frac{3}{2}})}(2+y_n+u_{2n}+v_{3n})^{\frac{1}{2}},\tag{31}$$
$$f_n^2(t,x,u,y,v)=\frac{1}{2\sqrt[6]{t}(1+t)^{\frac{5}{2}}(1+t^{\frac{3}{2}})}x_n,\tag{32}$$
$$g_n^1(t,x,u,y,v)=\frac{1}{14n^4\sqrt[3]{e^{2t}}(2+5t)^8}(6+x_{3n}+u_{4n})^{\frac{1}{3}},$$
$$g_n^2(t,x,u,y,v)=\frac{1}{14\sqrt[6]{t}(3+4t)^3}\ln\left[(3+4t)v_{4n}\right].$$

Let $t\in J_+$ and $\{z^{(m)}\}$ be any sequence in $f^1(t,E,E,E,E)$, where $z^{(m)}=(z_1^{(m)},\ldots,z_n^{(m)},\ldots)$, it follows from (31) that

$$0\le z_n^{(m)}\le\frac{1}{2n^3\sqrt[3]{e^{2t}}(1+t)^{\frac{5}{2}}(1+t^{\frac{3}{2}})}(2+3R)^{\frac{1}{2}}\quad(n,m=1,2,3,\ldots).\tag{33}$$

Thus, $\{z^{(m)}\}$ is bounded. By the diagonal method we can choose a subsequence $\{m_i\}\subset\{m\}$ such that

$$z_n^{(m_i)}\to\overline{z}_n,\quad i\to\infty\ (n=1,2,3,\ldots).\tag{34}$$

Taking (33) into consideration, we get

$$0\le\overline{z}\le\frac{1}{2n^3\sqrt[3]{e^{2t}}(1+t)^{\frac{5}{2}}(1+t^{\frac{3}{2}})}(2+3R)^{\frac{1}{2}}\quad(n=1,2,3,\ldots).\tag{35}$$

Hence $\overline{z}=(\overline{z}_1,\ldots,\overline{z}_n,\ldots)\in E$. It is easy to see from (33), (34) and (35) that

$$\left\|z^{(m_i)}-\overline{z}\right\|=\sup_n\left|z_n^{(m_i)}-\overline{z}_n\right|\to 0,\quad i\to\infty.$$

Thus, we have proved that $f^1(t,E,E,E,E)$ is relatively compact in $E$. For any $t\in J_+$, $x,y,\overline{x},\overline{y}\in D\subset E$, from (32) we obtain

$$\left|f_n^2(t,x,u,y,v)-f_n^2(t,\overline{x},\overline{u},\overline{y},\overline{v})\right|=\frac{1}{2\sqrt[6]{t}(1+t)^{\frac{5}{2}}(1+t^{\frac{3}{2}})}|x_n-\overline{x}_n|.$$

Thus,

$$\begin{aligned} &\left\| f^2(t,x,u,y,v) - f^2(t,\overline{x},\overline{u},\overline{y},\overline{v}) \right\| \\ &\quad \le \frac{1}{2\sqrt[6]{t}(1+t)^{\frac{3}{2}}(1+t^{\frac{3}{2}})} \|x_n - \overline{x}_n\|, \quad x,y,\overline{x},\overline{y} \in D. \end{aligned} \tag{36}$$

In the same way, we can prove that $g^1(t,E,E,E,E)$ is relatively compact in $E$. We can obtain

$$\left\| g^2(t,x,u,y,v) - g^2(t,\overline{x},\overline{u},\overline{y},\overline{v}) \right\| \le \frac{1}{14\sqrt[6]{t}(3+4t)^2} \|v - \overline{v}\|, \quad x,y,\overline{x},\overline{y} \in D.$$

From this inequality and (36) we can obtain that ($H_3$) holds for $L_{00}(t) = \frac{1}{2\sqrt[6]{t}(1+t)^{\frac{3}{2}}(1+t^{\frac{3}{2}})}$ and $K_{11}(t) = \frac{1}{14\sqrt[6]{t}(3+4t)^2}$. By a simple computation, we have $G_0^* \approx 0.8623$, $G_1^* \approx 0.0065$ and $\lambda = \max\{G_0^*, G_1^*\} = 0.8623 < 1$. All of the conditions in Theorem 3.1 are satisfied. By using Theorem 3.1, we know that problem (28) has at least one solution.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors typed, read and approved the final manuscript.

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