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Cross-ratio inequality with infinite type of singularities

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Abstract

Let f be a preserving orientation circle homeomorphism with infinite number of break points, *i.e.*, the points at which the derivative of f has jumps, and finite number of singular points, *i.e.*, the points x_i, $i = 1, 2, \ldots, n$, such that $f'(x_i) = \infty$, $i = 1, 2, \ldots, n$. Then the cross-ratio inequality with respect to f holds.

MSC: 37E10; 37C15

Keywords: cross-ratio inequality; critical circle homeomorphism; break point; rotation number

1 Introduction

Let $S^1 = \mathbf{R}/\mathbf{Z}$ with clearly defined orientation, metric, Lebesgue measure and the operation of addition be the unit circle. Let $\pi : \mathbf{R} \to S^1$ denote the corresponding projection mapping that 'winds' a straight line on the circle. An arbitrary homeomorphism f that preserves the orientation of the unit circle S^1 can be 'lifted' on the straight line $\mathbf{R}$ in the form of the homeomorphism $F : \mathbf{R} \to \mathbf{R}$ with property $F(x + 1) = F(x) + 1$ that is connected with f by relation $\pi \circ F = f \circ \pi$. This homeomorphism F is called the *lift* of the homeomorphism f and is defined up to an integer term. The most important arithmetic characteristic of the homeomorphism f of the unit circle S^1 is the *rotation number*

$$\rho(f) = \lim_{n \to \infty} \frac{F^n(x)}{n} \bmod 1,$$

where F is the lift of f with S^1 to $\mathbf{R}$. Here and below, for a given map F, F^n denotes its nth iterate. The rotation number is rational if and only if f has periodic points (see [1]). We say two circle maps f and g are *topologically conjugate* if there exists a homeomorphism $\varphi : S^1 \to S^1$ such that $f = \varphi^{-1} \circ g \circ \varphi$. The map φ is called *conjugating map* or just conjugation. If f and g are conjugate, many of their properties are the same, *e.g.*, the possible types of the orbits of the points of S^1 under f and under g are the same, the rotation number of f is equal to that of g, *etc.*

Early results about the existence of conjugacy to the linear rotation $f_\rho : x \to x + \rho \bmod 1$ are the following theorems.

Poincaré's theorem [2] *Let f be a circle homeomorphism with irrational rotation number ρ. Then f is semi-conjugate to the rotation f_ρ.*

Denjoy's theorem [3] *A circle diffeomorphism f with irrational rotation number ρ and $\log Df$ of bounded variation is topologically conjugate to the linear rotation f_ρ.*

Denjoy's result has been extended for different classes of circle homeomorphisms. In-depth results have been found, see [4–8]. Two well-known classes of these are the following.

P-homeomorphisms. These are orientation preserving circle homeomorphisms f differentiable except in many countable points called *break points* admitting left and right derivatives (denoted by f'_- and f'_+ respectively) such that

- there exist some constants $0 < \gamma < \zeta < \infty$ such that $\gamma < f'(x) < \zeta$ for all $x \in S^1 \setminus BP(f)$, $\gamma < f'_+(b) < \zeta$ and $\gamma < f'_-(b) < \zeta$ for all $b \in BP(f)$, where $BP(f)$ denotes the set of the break points of f;
- $\log f'$ has bounded variation. In this situation $\log f'$, $\log f'_-$, $\log f'_+$ and $\log(f^{-1})'$, $\log(f^{-1}_-)'$, $\log(f^{-1}_+)'$ have the same total variation denoted by $V = \operatorname{Var} \log f'$.

The ratio $\sigma_f(b) = (f_-(b))'/(f_+(b))'$ is called *jump ratio* of f in b.

Critical circle homeomorphisms. These are orientation preserving C^1 circle maps f such that for each $x_0 \in S^1$ there exist $\alpha \geq 1$, a neighborhood $U(x_0)$ of x_0, and a homeomorphism $\phi : U(x_0) \to R$ such that $\phi(x_0) = 0$, and if $\alpha > 1$ then $f(x) = \pm|\phi(x)|^\alpha + f(x_0)$, $\forall x \in U(x_0)$ if $\alpha = 1$ then $f(x) = |\phi(x)|^\alpha + f(x_0)$, $\forall x \in U(x_0)$.

For the class of P-homeomorphisms, the classical result of Denjoy can be easily extended, the existence of the conjugating map for this class was proved by Herman in [5]. The existence of the conjugating map for the class of critical real analytic circle maps was proved by Yoccoz [8] and extended by Świątek [6, 7]. In these works the existence of conjugation was shown by estimating cross-ratio distortions (*i.e.*, proving cross-ratio inequality). Note that the cross-ratio distortions were used in dynamical systems for the first time by Yoccoz [8]. The cross-ratio distortions are the most powerful tools to study the existence and smoothness of a conjugating map for the critical circle homeomorphisms.

Our aim in this work is to prove the cross-ratio inequality for a new class of circle homeomorphisms, which will be defined below with the aid of the above two classes and to show the existence of a conjugating map for this new class. We shall talk about the cross-ratio distortions in the next section.

Now we define the new class of circle homeomorphisms as follows.

(i) f is a preserving orientation circle homeomorphism on S^1.

(ii) There are points $x_i \in S^1$ and $\alpha_i \in (0, 1)$, $i = 1, 2, \ldots, n$, such that
$$f(x) = (x - x_i)^{\alpha_i} + f(x_i) \text{ for some } \epsilon_i\text{-neighborhoods of each } x_i.$$

(iii) f is a P-homeomorphism on $S^1 \setminus \bigcup_{i=1}^n U_{\epsilon_i}(x_i)$.

Note that it follows from condition (ii) that $f'(x_i) = \infty$, $i = 1, 2, \ldots, n$.

2 Cross-ratio inequality

Now we equip S^1 with the usual metric $|x - y| = \inf\{|\widetilde{x} - \widetilde{y}|$, where $\widetilde{x}, \widetilde{y}$ range over the lifts of $x, y \in S^1$ respectively$\}$. Our main analytic tools are ratio and cross-ratio distortions (see [9]). Let $a, b, c, d \in S^1$ be the four points of the circle which preserve orientation; that is, $a \prec b \prec c \prec d \prec a$ on the circle. We define the cross-ratio of these points as

$$Cr(a, b, c, d) = \frac{|c - b||d - a|}{|b - a||d - c|}.$$

If f is a continuous and monotone function on S^1, then the distortion of the cross-ratio by this function is defined by

$$Dcr(a,b,c,d;f) = \frac{Cr(f(a),f(b),f(c),f(d))}{Cr(a,b,c,d)}. \tag{1}$$

Moreover, if f is a C^1 diffeomorphism, then taking limit from (1) when $b \to a$ and $c \to d$, we get the following distortion:

$$R(a,d;f) = \frac{(f(d)-f(a))^2}{(d-a)^2} \cdot \frac{1}{f'(a)f'(d)}. \tag{2}$$

Notice that the cross-ratio distortion is multiplicative with respect to composition for two functions f and g on S^1, that is,

$$R(a,d;f \circ g) = R\big(g(a),g(d);f\big) \cdot R(a,d;g).$$

The main results of this paper are the following theorems.

Theorem 2.1 *Let f be a circle homeomorphism with irrational rotation number satisfying conditions* (i)-(iii). *Consider a system of intervals $\{(a_i,b_i) \subset S^1, i = 1,2,\ldots,k\}$. Suppose that this system of intervals covers each point of the circle S^1 at most p times. Then there exists a constant $C_1 = C_1(p,f)$ which depends on p and f such that the following inequality*

$$\prod_{i=1}^{k} R(a_i,b_i;f) \leq C_1 \tag{3}$$

holds.

Inequality (3) is called cross-ratio inequality with respect to f. As an application of this theorem, we provide the following second result of this paper.

Theorem 2.2 *Let f be a circle homeomorphism with irrational rotation number satisfying conditions* (i)-(iii) *without break points. Then there exists a circle homeomorphism φ such that the following equality*

$$\varphi \circ f = f_\rho \circ \varphi$$

holds.

In the proof of the second main theorem, we use the properties of a dynamical partition, and therefore we introduce this concept.

3 Dynamical partition of the circle

Given a circle homeomorphism f with irrational rotation number ρ, one may consider a *positive marked trajectory* (*i.e.*, the positive trajectory of a marked point) $\xi_i = f^i(\xi_0) \in S^1$, where $i \geq 0$, and pick out of it the sequence of the *dynamical convergents* ξ_{q_n}, $n \geq 0$. We will also conventionally use $\xi_{q_{-1}} = \xi_0 - 1$. The well-understood arithmetical properties of

rational convergents and the combinatorial equivalence between f and linear rotation f_ρ imply that the dynamical convergents approach the marked point alternating their order in the following way:

$$\xi_{q_{-1}} < \xi_{q_1} < \xi_{q_3} < \cdots < \xi_{q_{2m+1}} < \cdots < \xi_0 < \cdots < \xi_{q_{2m}} < \cdots < \xi_{q_2} < \xi_{q_0}.$$

For the marked trajectory, we use the notations $I_0^n = I_0^n(\xi_0)$ and $I_i^n = f^i(I_0^n)$, where $I_0^n(\xi_0)$ is the nth generator interval. It is well known that the set of intervals $\mathbf{P}_n = \mathbf{P}_n(\xi_0, f)$ with mutually disjoint interiors defined as

$$\mathbf{P}_n = \left\{ I_i^{n-1}, 0 \le i < q_n; I_j^n, 0 \le j < q_{n-1} \right\} \tag{4}$$

determines a partition of the circle for any n. The partition $\mathbf{P}_n$ is called the nth *dynamical partition* of S^1. Obviously the partition $\mathbf{P}_{n+1}$ is a refinement of the partition $\mathbf{P}_n$: indeed the intervals of order n are members of $\mathbf{P}_{n+1}$ and each interval $I_i^{n-1} \in \mathbf{P}_n$, $0 \le i < q_n$, is partitioned into $k_{n+1} + 1$ intervals belonging to $\mathbf{P}_{n+1}$ such that

$$I_i^{n-1} = I_i^{n+1} \cup \bigcup_{s=0}^{k_{n+1}-1} I_{i+q_{n-1}+sq_n}^n. \tag{5}$$

4 Proof of the main results

In this section, first we prove few lemmas, and then using these lemmas we give the proof of the main results. Note that the first main theorem is proved in a similar way as that of [10].

Lemma 4.1 *Let f be a circle homeomorphism with irrational rotation number satisfying conditions* (i)-(iii). *Consider a system of intervals $\{(a_i, b_i) \subset S^1, i = 1, 2, \ldots, k\}$. Suppose that this system of intervals covers each point of the set $S^1 \setminus \bigcup_{i=1}^n U_{\epsilon_i}(x_i)$ at most p times. Then there exists a constant $C_2 = C_2(p, f)$ which depends on p and f such that the following inequality*

$$\prod_{i=1}^k R(a_i, b_i; f) \le C_2$$

holds.

Proof Since f is a P-homeomorphism on the set $S^1 \setminus \bigcup_{i=1}^n U_{\epsilon_i}(x_i)$ for any interval (a_i, b_i), we have

$$\left| \log R(a_i, b_i; f) \right| \le 2 \sup_{x,y \in [a_i, b_i]} \left| \log f'(x) - \log f'(y) \right| \le 2 \operatorname*{Var}_{[a_i, b_i]} \log f'.$$

Using this inequality, we get

$$\left| \log \prod_{i=1}^k R(a_i, b_i; f) \right| \le \sum_{i=1}^k \left| \log R(a_i, b_i; f) \right| \le 2 \sum_{i=1}^k \operatorname*{Var}_{[a_i, b_i]} \log f'.$$

By assumption, the system of intervals $\mathcal{F} := \{[a_i, b_i], i = 1, 2, \ldots, k\}$ covers each point of the set $S^1 \setminus \bigcup_{i=1}^n U_{\epsilon_i}(x_i)$ at most p times. Now we describe this system of intervals as a union of subsystems of $\mathcal{F}_j, j \le p$, in the following way: first we take $[a_1, b_1]$ as an element of $\mathcal{F}_1$, then consider the intersection $[a_1, b_1] \cap [a_2, b_2]$; if this intersection is empty, then we count the interval $[a_2, b_2]$ is an element of $\mathcal{F}_1$, otherwise we count an element of $\mathcal{F}_2$. Next, consider $\mathcal{F}_1 \cap [a_3, b_3]$ (here and below the intersection with each element of $\mathcal{F}_1$ is considered); if it is empty, we count $[a_3, b_3]$ is an element of $\mathcal{F}_1$, otherwise we check the intersection $\mathcal{F}_2 \cap [a_3, b_3]$. Again if $\mathcal{F}_2 \cap [a_3, b_3]$ is empty, we count $[a_3, b_3]$ is an element of $\mathcal{F}_2$, otherwise we count $[a_3, b_3]$ is an element of $\mathcal{F}_3$. Continuing this process we get all $\mathcal{F}_j, j \le p$. By construction of subsystems $\mathcal{F}_j, j \le p$ of $\mathcal{F}$, the elements of each subsystem do not intersect with each other and $\mathcal{F} = \bigcup_j \mathcal{F}_j, j \le p$. Therefore

$$\sum_{i=1}^k \operatorname*{Var}_{[a_i, b_i]} \log f' = \sum_j \sum_{[a_i, b_i] \in \mathcal{F}_j} \operatorname*{Var}_{[a_i, b_i]} \log f' \le pV,$$

where $V = \operatorname*{Var}_{S^1 \setminus \bigcup_{i=1}^n U_{\epsilon_i}(x_i)} \log f'$. □

Lemma 4.2 *Let f be a circle homeomorphism with irrational rotation number satisfying conditions* (i)-(iii) *with one singular point $x_{i*} \in S^1$ and corresponding $\alpha_{i*} \in (0, 1)$. Then, for any interval $(a, b) \subset S^1$, there is a constant $C_3 = C_3(f)$ which depends on f such that*

$$R(a, b; f) = \frac{(f(b) - f(a))^2}{(b - a)^2} \cdot \frac{1}{f'(a)f'(b)} \le C_3.$$

Proof Denote by $E = S^1 \setminus (x_{i*} - \frac{\epsilon_{i*}}{2}, x_{i*} + \frac{\epsilon_{i*}}{2})$ and $V = (x_{i*} - \epsilon_{i*}, x_{i*} + \epsilon_{i*})$. Consider the following three cases:

(1) $(a, b) \subset E$;

(2) $(a, b) \subset V$;

(3) neither (1) nor (2).

If $(a, b) \subset E$ then it is easy to get that

$$R(a, b; f) = \frac{(f(b) - f(a))^2}{(b - a)^2} \cdot \frac{1}{f'(a)f'(b)} \le \frac{L^2}{M^2}, \tag{6}$$

where $L = \sup_{x \in E} f'(x)$, $M = \inf_{x \in E} f'(x)$. Now consider the case $(a, b) \subset V$. We divide this case into two parts:

- the interval (a, b) contains the point x_{i*},
- the interval (a, b) does not contain x_{i*}.

If (a, b) does not contain x_{i*}, then there can be two cases: $x_{i*} \prec a \prec b$ or $a \prec b \prec x_{i*}$. We prove the first case $x_{i*} \prec a \prec b$, the second case can be analogously proved.

$$\begin{aligned} R(a, b; f) &= \frac{(f(b) - f(a))^2}{(b - a)^2} \cdot \frac{1}{f'(a)f'(b)} \\ &= \frac{1}{\alpha_{i*}^2} \cdot \frac{((b - x_{i*})^{\alpha_{i*}} - (a - x_{i*})^{\alpha_{i*}})^2}{((b - x_{i*}) - (a - x_{i*}))^2} \cdot \frac{1}{(b - x_{i*})^{\alpha_{i*} - 1}(a - x_{i*})^{\alpha_{i*} - 1}}. \end{aligned} \tag{7}$$

If we take $y := \frac{(b - x_{i*})^{\frac{1}{2}}}{(a - x_{i*})^{\frac{1}{2}}}$, then it is easy to see that $y > 1$, and using this notation we can write (7) in the form

$$R(y) := R(a, b; f) = \frac{1}{\alpha_{i*}^2} \cdot \frac{(y^{\alpha_{i*}} - y^{-\alpha_{i*}})^2}{(y - y^{-1})^2}, \quad y > 1.$$

Hence, the function $R(y) = \frac{1}{\alpha_{i*}^2} \cdot \frac{(y^{\alpha_{i*}} - y^{-\alpha_{i*}})^2}{(y - y^{-1})^2}$ is decreasing on $(1, \infty)$ and

$$\sup_{y > 1} R(y) = 1. \tag{8}$$

Let $x_{i*} \in (a, b)$, then

$$
\begin{aligned}
R(a, b; f) &= \frac{(f(b) - f(a))^2}{(b - a)^2} \cdot \frac{1}{f'(a)f'(b)} \\
&= \frac{1}{\alpha_{i*}^2} \cdot \frac{((b - x_{i*})^{\alpha_{i*}} - (x_{i*} - a)^{\alpha_{i*}})^2}{((b - x_{i*}) - (x_{i*} - a))^2} \cdot \frac{1}{(b - x_{i*})^{\alpha_{i*} - 1}(x_{i*} - a)^{\alpha_{i*} - 1}}.
\end{aligned}
\tag{9}
$$

Again if we take $y := \frac{(b - x_{i*})^{\frac{1}{2}}}{(x_{i*} - a)^{\frac{1}{2}}}$, then it is easy to see that $0 < y < \infty$, and using this notation we can write (9) in the form

$$R(y) := R(a, b; f) = \frac{1}{\alpha_{i*}^2} \cdot \frac{(y^{\alpha_{i*}} + y^{-\alpha_{i*}})^2}{(y + y^{-1})^2}, \quad y > 0.$$

Hence, the function $R(y) = \frac{1}{\alpha_{i*}^2} \frac{(y^{\alpha} + y^{-\alpha_{i*}})^2}{(y + y^{-1})^2}$ is increasing on $(0, 1)$ and decreasing on $(1, \infty)$, so

$$\sup_{0 < y < \infty} R(y) = \frac{1}{\alpha_{i*}^2}. \tag{10}$$

Now we consider case (3). It is clear that $b - a > \frac{\epsilon_{i*}}{2}$, and since f is strictly increasing we have that $\frac{f(b) - f(a)}{b - a} \leq \text{const}$. Besides that $\frac{1}{f'(a)f'(b)}$ cannot be sufficiently large. Hence, from this and from (6), (8), (10) the proof of the lemma follows. □

Proof of Theorem 2.1 Denote by $E^* = S^1 \setminus \bigcup_{i=1}^{n}(x_i - \epsilon_i, x_i + \epsilon_i)$, $V^* = \bigcup_{i=1}^{n}(x_i - \frac{\epsilon_i}{2}, x_i + \frac{\epsilon_i}{2})$ and $A = \{1, 2, \ldots, k\}$. We separate the set A into three disjoint subsets A_1, A_2, A_3 as follows:

(a) $A_1 = \{i \in A : (a_i, b_i) \subset V^*\}$,
(b) $A_2 = \{i \in A : (a_i, b_i) \subset E^*\}$,
(c) $A_3 = A \setminus (A_1 \cup A_2)$.

Using these subsets we write the left site of (3) in the following form:

$$\prod_{i=1}^{k} R(a_i, b_i; f) = \prod_{i \in A_1} R(a_i, b_i; f) \times \prod_{i \in A_2} R(a_i, b_i; f) \times \prod_{i \in A_3} R(a_i, b_i; f). \tag{11}$$

Now we estimate every product of (11) separately.

(a) Let $i \in A_1$. In this case we consider two subsets of A_1: $A_1^1 = \{i \in A_1 : x_j \in (a_i, b_i), j = 1, 2, \ldots, n\}$ and $A_1^2 = A_1 \setminus A_1^1$. Let $i \in A_1^1$, then by Lemma 4.2 the cross-ratio distortion

$R(a_i, b_i; f)$ is bounded by $\frac{1}{\alpha_i^2}$ for fixed i in the neighborhood of each singular point x_j, $j = 1, 2, \ldots, n$, and since the number of intervals covering every singular point x_j does not exceed p, we have

$$\prod_{i \in A_1^1} R(a_i, b_i; f) \leq \frac{1}{\alpha_1^{2p}} \cdot \frac{1}{\alpha_2^{2p}} \cdots \frac{1}{\alpha_n^{2p}}. \tag{12}$$

Let $i \in A_1^2$, then by Lemma 4.2 we have

$$\prod_{i \in A_1^2} R(a_i, b_i; f) \leq 1. \tag{13}$$

(b) Let $i \in A_2$, then by Lemma 4.1 we have

$$\prod_{i \in A_2} R(a_i, b_i; f) \leq C_2. \tag{14}$$

(c) Let $i \in A_3$, then similarly to item (a), the number of intervals covering the end points of V^* and E^* does not exceed $2p$, and by Lemma 4.2 we have

$$\prod_{i \in A_3} R(a_i, b_i; f) \leq C_3^{2pn}. \tag{15}$$

Hence, the proof of Theorem 2.1 follows from (11)-(15). □

Before we prove the next lemma, we introduce a concept of wandering interval.

Definition 4.3 We say that I is a wandering interval of the circle map f if:
- the intervals $I, f(I), \ldots, f^n(I), \ldots$ are pairwise disjoint;
- the ω-limit set of I is not equal to a single periodic orbit.

Lemma 4.4 *Suppose that a circle homeomorphism f with irrational rotation number satisfies conditions* (i)-(iii) *without break points. Then f has no wandering interval.*

Proof Assume that $I \subset S^1$ is a wandering interval of f. It is easy to see that by the properties of dynamical partition of S^1 and by the mean value theorem we have

$$\left(f^{q_n}(a)\right)' = \frac{|f^{q_n}(f^{q_{n-1}}(I))|}{|f^{q_{n-1}}(I)|} \quad \text{and} \quad \left(f^{q_n}(b)\right)' = \frac{|f^{q_n}(I)|}{|I|}, \tag{16}$$

where $a \in f^{q_{n-1}}(I)$, $b \in I$ and the sign $|\cdot|$ denotes the length of the given interval. If we choose an interval $J = [a, b]$, then $f^{q_n}(J) = [f^{q_n}(a), f^{q_n}(b)]$. Using (2) and (16) we have

$$\begin{aligned} R\left(J; f^{q_n}\right) &= \frac{|f^{q_n}(J)|^2}{|J|^2} \cdot \frac{1}{(f^{q_n}(a))'(f^{q_n}(b))'} \\ &= \frac{|f^{q_n}(J)|^2}{|J|^2} \cdot \frac{|f^{q_{n-1}}(I)|}{|f^{q_n}(I)|} \cdot \frac{|I|}{|f^{q_n}(f^{q_{n-1}}(I))|}. \end{aligned} \tag{17}$$

Since I is a wandering interval, the sequence $\{|f^{q_n}(I)|\}_{n=1}^{\infty}$ tends to zero when n goes to infinity, and it follows from this that the second fraction $\frac{|f^{q_{n-1}}(I)|}{|f^{q_n}(I)|}$ in the second product of (17) does not tend to zero. Since $I \subset f^{q_n}(J)$, the first fraction also does not tend to zero, but the third fraction tends to infinity when n goes to infinity, *i.e.*, $\frac{|I|}{|f^{q_n}(f^{q_{n-1}}(I))|} \to \infty$, $n \to \infty$. So the last product in (17) tends to infinity. This is a contradiction to (3) and it proves the lemma. $\qquad \square$

Proof of Theorem 2.2 The proof of the theorem easily follows from Lemma 4.4 with the fact (see [9]) that the homeomorphism f without periodic points is conjugate to the linear rotation f_ρ if and only if f does not have a wandering interval. $\qquad \square$

References
1. Cornfeld, IP, Fomin, SV, Sinai, YG: Ergodic Theory. Springer, Berlin (1982)
2. Poincaré, H: Sur les courbes définies par les équations différentielles. J. Math. Pures Appl. (4) **1**, 167-244 (1885)
3. Denjoy, A: Sur les courbes définies par les équations différentielles a la surface du tore. J. Math. Pures Appl. (9) **11**, 333-375 (1932)
4. Dzhalilov, A, Noorani, S, Akhatkulov, A: On critical circle homeomorphisms with infinite number of break points. Abstr. Appl. Anal. **2014**, 378742 (2014)
5. Herman, M: Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations. Publ. Math. Inst. Hautes Études Sci. **49**, 5-234 (1979)
6. Świątek, G: On critical circle homeomorphisms. Bol. Soc. Bras. Mat. **29**(2), 329-351 (1998)
7. Świątek, G: Circle homeomorphisms with flat critical points. Fundam. Math. **138**, 205-217 (1991)
8. Yoccoz, JC: Il n'y a pas de contre-exemple de Denjoy analytique. C. R. Acad. Sci. Paris Sér. I Math. **298**(7), 141-144 (1984)
9. de Melo, W, van Strien, S: One Dimensional Dynamics. Springer, Berlin (1993)
10. Świątek, G: Rational rotation number for maps of the circle. Commun. Math. Phys. **119**(1), 109-128 (1988)