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Particular solutions of a certain class of associated Cauchy-Euler fractional partial differential equations via fractional calculus

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Abstract

In recent years, various operators of fractional calculus (that is, calculus of integrals and derivatives of arbitrary real or complex orders) have been investigated and applied in many remarkably diverse fields of science and engineering. Many authors have demonstrated the usefulness of fractional calculus in the derivation of particular solutions of a number of linear ordinary and partial differential equations of the second and higher orders. The purpose of this paper is to present a certain class of the explicit particular solutions of the associated Cauchy-Euler fractional partial differential equation of arbitrary real or complex orders and their applications as follows:

$$Ax^2\frac{\partial^2 u}{\partial x^2}+Bx\frac{\partial u}{\partial x}+Cu=M\frac{\partial^\alpha u}{\partial t^\alpha}+N\frac{\partial^\beta u}{\partial t^\beta}, \tag{P_1}$$

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial u}{\partial x}+Cu=M\frac{\partial^\alpha u}{\partial t^\alpha}+N\frac{\partial^\beta u}{\partial t^\beta}, \tag{P_2}$$

where $u=u(x,t)$; A, B, C, M, N, α and β are arbitrary constants.

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1 Introduction, definitions and preliminaries

The subject of fractional calculus (that is, derivatives and integrals of any real or complex order) has gained importance and popularity during the past two decades or so, due mainly to its demonstrated applications in numerous seemingly diverse fields of science and engineering (*cf.* [1–15]). By applying the following definition of a *fractional differential* (that is, *fractional derivative* and *fractional integral*) of order $\nu \in \mathbb{R}$, many authors have obtained particular solutions of a number of families of homogeneous (as well as nonhomogeneous) linear fractional differ-integral equations.

In this paper, we present a direct way to obtain explicit solutions of such types of the associated Cauchy-Euler fractional partial differential equation with initial and boundary values. The results are a coincidence that the solutions are obtained by the methods applying the Laplace transform with the residue theorem. In this paper, we present some useful definitions and preliminaries for the paper as follows.

Definitions 1.1 (*cf.* [6–10]) If the function $f(z)$ is analytic and has no branch point inside and on $\mathcal{C}$, where

$$\mathcal{C} := \{\mathcal{C}_-, \mathcal{C}_+\} \tag{1.1}$$

$\mathcal{C}_-$ is an integral curve along the cut joining the points z and $-\infty + i\mathcal{T}(z)$, $\mathcal{C}_+$ is an integral curve along the cut joining the points z and $\infty + i\mathcal{T}(z)$,

$$f_\nu(z) = {}_{\mathcal{C}}f_\nu(z) := \frac{\Gamma(\nu+1)}{2\pi i}\int_{\mathcal{C}} \frac{f(\zeta)\,d\zeta}{(\zeta - z)^{\nu+1}} \quad \bigl(\nu \in \mathbb{R}/\mathbb{Z}^-; \mathbb{Z}^- := \{-1,-2,-3,\ldots\}\bigr) \tag{1.2}$$

and

$$f_{-n}(z) := \lim_{\nu \to -n} \bigl\{f_\nu(z)\bigr\} \quad \bigl(n \in \mathbb{N} := \{1,2,3,\ldots\}\bigr), \tag{1.3}$$

where $\zeta \neq z$,

$$-\pi \leq \arg(\zeta - z) \leq \pi \quad \text{for } \mathcal{C}_-, \tag{1.4}$$

and

$$0 \leq \arg(\zeta - z) \leq 2\pi \quad \text{for } \mathcal{C}_+, \tag{1.5}$$

then $f_\nu(z)$ $(\nu > 0)$ is said to be the *fractional derivative* of $f(z)$ of order ν and $f_\nu(z)$ $(\nu < 0)$ is said to be the *fractional integral* of $f(z)$ of order $-\nu$, provided that

$$\bigl|f_\nu(z)\bigr| < \infty \quad (\nu \in \mathbb{R}). \tag{1.6}$$

First of all, we find it is worthwhile to recall here the following useful lemmas and properties associated with the fractional differ-integration which is defined above.

Lemma 1.1 (Linearity property) *If the functions $f(z)$ and $g(z)$ are single-valued and analytic in some domain $\Omega \subseteq \mathbb{C}$, then*

$$(k_1 f + k_2 g)_\nu = k_1 f_\nu + k_2 g_\nu \quad (\nu \in \mathbb{R}, z \in \Omega) \tag{1.7}$$

for any constants k_1 and k_2.

Lemma 1.2 (Index law) *If the function $f(z)$ is single-valued and analytic in some domain $\Omega \subseteq \mathbb{C}$, then*

$$(f_\mu)_\nu = f_{\mu+\nu} = (f_\nu)_\mu \quad (f_\mu \neq 0, f_\nu \neq 0, \mu, \nu \in \mathbb{R}, z \in \Omega). \tag{1.8}$$

Lemma 1.3 (Generalized Leibniz rule) *If the functions $f(z)$ and $g(z)$ are single-valued and analytic in some domain $\Omega \subseteq \mathbb{C}$, then*

$$(f \cdot g)_\nu = \sum_{n=0}^{\infty} \binom{\nu}{n} f_{\nu-n} \cdot g_n \quad (\nu \in \mathbb{R}, z \in \Omega), \tag{1.9}$$

where g_n is the ordinary derivative of $g(z)$ of order n ($n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$), it being tacitly assumed (for simplicity) that $g(z)$ is the polynomial part (if any) of the product $f(z)g(z)$.

Lemma 1.4 (Cauchy's residue theorem) *Let Ω be a simple connected domain, and let C be a simple closed positively oriented contour that lies in Ω. If f is analytic inside C and on C, expect at the point $z_1, z_2, \ldots, z_n$ that lie inside C, then*

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}[f, z_k].$$

(I) *If f has a simple pole at z_0, then*

$$\operatorname{Res}[f, z_0] = \lim_{z \to z_0} (z - z_0) f(z);$$

(II) *If f has a pole of order k at z_0, then*

$$\operatorname{Res}[f, z_0] = \frac{1}{(k-1)!} \lim_{z \to z_0} \frac{d^{k-1}}{dz^{k-1}} (z - z_0)^k f(z).$$

Property 1.1 (*cf.* [6–10]) *For a constant a,*

$$\left(e^{az}\right)_\nu = a^\nu e^{az} \quad (a \neq 0, \nu \in \mathbb{R}, z \in \mathbb{C}). \tag{1.10}$$

Proof The proofs between 'ν is not an integer' and 'ν is an integer' are not coincident, so we mention the proof as follows.

In case of $|\arg a| < \pi/2$, we have

$$\begin{aligned}
\left(e^{az}\right)_\nu &= \mathcal{C}_-\left(e^{az}\right)_\nu = \frac{\Gamma(\nu+1)}{2\pi i} \int_{\mathcal{C}_-} \frac{e^{a\zeta}}{(\zeta - z)^{\nu+1}}\,d\zeta \quad (\text{put } \zeta - z = \eta \text{ and } a\eta = \xi, |\arg \eta| \leq \pi) \\
&= a^\nu e^{az} \frac{\Gamma(\nu+1)}{2\pi i} \int_{-\infty e^{i\phi}}^{(0+)} \xi^{-(\nu+1)} e^{\xi}\,d\xi \quad (\phi = \arg a) \\
&= a^\nu e^{az} \frac{\Gamma(\nu+1)}{2\pi i} \int_{-\infty}^{(0+)} \xi^{-(\nu+1)} e^{\xi}\,d\xi \quad \left(\text{for } |\phi| < \frac{\pi}{2}\right) = a^\nu e^{az}
\end{aligned}$$

for $|\arg a| < \pi/2$ since $\int_{-\infty}^{(0+)} \xi^{-(\nu+1)} e^{\xi}\,d\xi = \frac{2\pi i}{\Gamma(\nu+1)}$.

In case of $\pi/2 < |\arg a| \leq \pi$, we have

$$\left(e^{az}\right)_\nu = \mathcal{C}_+\left(e^{az}\right)_\nu = \frac{\Gamma(\nu+1)}{2\pi i} \int_{\mathcal{C}_+} \frac{e^{a\zeta}}{(\zeta - z)^{\nu+1}}\,d\zeta$$

$$(\text{put } \zeta - z = \eta \text{ and } a\eta = \xi, 0 \leq \arg \eta \leq 2\pi)$$

$$= a^{\nu} e^{az} \frac{\Gamma(\nu+1)}{2\pi i} \int_{\infty e^{i\phi}}^{(0+)} \xi^{-(\nu+1)} e^{\xi}\, d\xi \quad (\phi = \arg a)$$
$$= a^{\nu} e^{az} \frac{\Gamma(\nu+1)}{2\pi i} \int_{-\infty}^{(0+)} \xi^{-(\nu+1)} e^{\xi}\, d\xi \quad \left(\text{for } \frac{\pi}{2} < |\phi| \le \pi\right)$$
$$= a^{\nu} e^{az}.$$

Therefore we have Property 1.1 for arbitrary $a \neq 0$. □

Property 1.2 *For a constant a,*

$$\left(e^{-az}\right)_{\nu} = e^{-i\pi\nu} a^{\nu} e^{-az} \quad (a \neq 0, \nu \in \mathbb{R}, z \in \mathbb{C}). \tag{1.11}$$

Property 1.3 *For a constant a,*

$$\left(z^{a}\right)_{\nu} = e^{-i\pi\nu} \frac{\Gamma(\nu - a)}{\Gamma(-a)} z^{a-\nu} \quad \left(\nu \in \mathbb{R}, z \in \mathbb{C}, \left|\frac{\Gamma(\nu - a)}{\Gamma(-a)}\right| < \infty\right). \tag{1.12}$$

Property 1.4 (*cf.* [2, 16, 17]) *The fractional derivative of a causal function $f(t)$ is defined by*

$$\frac{d^{\alpha}}{dt^{\alpha}} f(t) := \begin{cases} f^{(n)}(t) & \text{if } \alpha = n \in \mathbb{N}, \\ \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}\, d\tau & \text{if } n-1 < \alpha < n, \end{cases} \tag{1.13}$$

where $f^{(n)}(t)$ denotes the ordinary derivative of order n and Γ is the gamma function.

The Laplace transform of a function $f(t)$ is denoted as

$$\mathcal{L}\{f(t)\}(s) = \int_0^{+\infty} e^{-st} f(t)\, dt, \tag{1.14}$$

where s is the Laplace complex parameter. We recall from the fundamental formula (*cf.* [16])

$$\mathcal{L}\left\{\frac{d^{\alpha}}{dt^{\alpha}} f(t)\right\}(s) = s^{\alpha} \mathcal{L}\{f(t)\}(s) - \sum_{k=1}^{n} s^{\alpha-k} f^{(k-1)}(0), \quad n-1 < \alpha \le n, n \in \mathbb{N}. \tag{1.15}$$

2 Main results

Theorem 2.1 *The fractional partial differential equation*

$$Ax^2 \frac{\partial^2 u}{\partial x^2} + Bx \frac{\partial u}{\partial x} + Cu = M \frac{\partial^{\alpha} u}{\partial t^{\alpha}} + N \frac{\partial^{\beta} u}{\partial t^{\beta}}, \tag{2.1}$$

with $u = u(x,t)$, $n-1 < \alpha, \beta \le n$, $n \in \mathbb{N}$ and A $(\neq 0)$, B, C, M, N are constants, has its solutions of the form given by

(a)

$$u(x,t) = e^{\lambda t}\left[k_1 x^{\frac{A-B+\sqrt{(A-B)^2-4A(C-M\lambda^{\alpha}-N\lambda^{\beta})}}{2A}} + k_2 x^{\frac{A-B-\sqrt{(A-B)^2-4A(C-M\lambda^{\alpha}-N\lambda^{\beta})}}{2A}}\right], \tag{2.2}$$

when the discriminant $(A-B)^2 - 4A(C - M\lambda^{\alpha} - N\lambda^{\beta}) > 0$;

(b)

$$u(x,t) = e^{\lambda t}\left[k_1 x^m + k_2 x^m \ln x\right],$$

when the discriminant $(A-B)^2 - 4A(C - M\lambda^\alpha - N\lambda^\beta) = 0$, *and the roots* m_1, m_2 *of Equation* (2.5) *are repeated*; *that is*, $m_1 = m_2 = m$;

(c)

$$u(x,t) = e^{\lambda t}\left[k_1 x^{a+bi} + k_2 x^{a-bi}\right],$$

when the discriminant $(A-B)^2 - 4A(C - M\lambda^\alpha - N\lambda^\beta) < 0$, *and* $a+bi$, $a-bi$ *are the conjugate pair roots of Equation* (2.5).

Proof Suppose that $u(x,t) = x^m e^{\lambda t}$. We have

$$\begin{aligned} &\frac{\partial u}{\partial x} = m x^{m-1} e^{\lambda t}, \\ &\frac{\partial^2 u}{\partial x^2} = m(m-1) x^{m-1} e^{\lambda t}, \\ &\frac{\partial^\alpha u}{\partial t^\alpha} = \lambda^\alpha x^m e^{\lambda t}, \quad \text{and} \\ &\frac{\partial^\beta u}{\partial t^\beta} = \lambda^\beta x^m e^{\lambda t}. \end{aligned} \tag{2.3}$$

So that the given equation (2.1) becomes

$$Am(m-1)x^m e^{\lambda t} + Bmx^m e^{\lambda t} + Cx^m e^{\lambda t} - M\lambda^\alpha x^m e^{\lambda t} - N\lambda^\beta x^m e^{\lambda t} = 0. \tag{2.4}$$

Equation (2.4) leads to the auxiliary equation

$$\begin{aligned} &Am(m-1) + Bm + C - M\lambda^\alpha - N\lambda^\beta = 0 \\ &\quad \bigl(\text{or } Am^2 - (A-B)m + \bigl(C - M\lambda^\alpha - N\lambda^\beta\bigr) = 0\bigr). \end{aligned} \tag{2.5}$$

That is,

$$m_1 = \frac{A - B + \sqrt{(A-B)^2 - 4A(C - M\lambda^\alpha - N\lambda^\beta)}}{2A}$$

and

$$m_2 = \frac{A - B - \sqrt{(A-B)^2 - 4A(C - M\lambda^\alpha - N\lambda^\beta)}}{2A}$$

are the two roots of the auxiliary equation (2.5). Thus, $u(x,t) = \sum_{i=1}^{2} k_i x^{m_i} e^{\lambda t}$ is a solution of the fractional partial differential equation (2.1) whenever m_i ($i = 1, 2$) is a solution of the auxiliary equation (2.5).

There are three different cases to be considered, depending on whether the roots of this quadratic equation (2.5) are distinct real roots, equal real roots (repeated real roots), or complex roots (roots appear as a conjugate pair). The three cases are due to the discriminant of the coefficients $\sqrt{(A-B)^2 - 4A(C - M\lambda^\alpha - N\lambda^\beta)}$.

- Case I: Distinct real roots (when $(A-B)^2 - 4A(C - M\lambda^\alpha - N\lambda^\beta) > 0$).

  Let m_1 and m_2 denote the real roots of Equation (2.5) such that $m_1 \neq m_2$. Then the general solution of Equation (2.1) is

  $$u(x,t) = e^{\lambda t}\left(k_1 x^{m_1} + k_2 x^{m_2}\right),$$

  where k_i $(i = 1, 2)$ are constants.
- Case II: Repeated real roots (when $(A-B)^2 - 4A(C - M\lambda^\alpha - N\lambda^\beta) = 0$).

  If the roots of Equation (2.5) are repeated, that is, $m_1 = m_2 = m$, then the general solution of Equation (2.1) is

  $$u(x,t) = e^{\lambda t}\left(k_1 x^{m} + k_2 x^{m} \ln x\right),$$

  where k_i $(i = 1, 2)$ are constants.
- Case III: Conjugate complex roots (when $(A-B)^2 - 4A(C - M\lambda^\alpha - N\lambda^\beta) < 0$).

  If the roots of Equation (2.5) are the conjugate pair $m_1 = a + bi$ and $m_2 = a - bi$, then a solution of Equation (2.1) is

  $$u(x,t) = e^{\lambda t}\left(k_1 x^{a+bi} + k_2 x^{a-bi}\right),$$

  where k_i $(i = 1, 2)$ are constants.

In general,

$$u(x,t) = e^{\lambda t}\left[k_1 x^{\frac{A-B+\sqrt{(A-B)^2 - 4A(C - M\lambda^\alpha - N\lambda^\beta)}}{2A}} + k_2 x^{\frac{A-B-\sqrt{(A-B)^2 - 4A(C - M\lambda^\alpha - N\lambda^\beta)}}{2A}}\right]$$

forms a fundamental solution, where k_1, k_2 and λ are constants. □

Theorem 2.2 *The fractional partial differential equation*

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial u}{\partial x} + Cu = M\frac{\partial^\alpha u}{\partial t^\alpha} + N\frac{\partial^\beta u}{\partial t^\beta}, \tag{2.6}$$

with $u = u(x,t)$, $n - 1 < \alpha, \beta \leq n$, $n \in \mathbb{N}$ *and* A $(\neq 0)$, B, C, M, N *are constants, has its solutions of the form given by*

(a′)

$$u(x,t) = e^{\lambda t}\left[k_1 e^{(\frac{-B+\sqrt{B^2 - 4A(C - M\lambda^\alpha - N\lambda^\beta)}}{2A})x} + k_2 e^{(\frac{-B-\sqrt{B^2 - 4A(C - M\lambda^\alpha - N\lambda^\beta)}}{2A})x}\right], \tag{2.7}$$

when the discriminant $B^2 - 4A(C - M\lambda^\alpha - N\lambda^\beta) > 0$;

(b′)

$$u(x,t) = e^{\lambda t}\left[k_1 e^{mx} + k_2 x e^{mx}\right],$$

when the discriminant $B^2 - 4A(C - M\lambda^\alpha - N\lambda^\beta) = 0$, *and the roots* m_1, m_2 *of Equation* (2.10) *are repeated*; *that is*, $m_1 = m_2 = m$;

(c′)

$$u(x,t) = e^{\lambda t}\left[k_1 x^{(a+bi)x} + k_2 x^{(a-bi)x}\right],$$

when the discriminant $B^2 - 4A(C - M\lambda^\alpha - N\lambda^\beta) < 0$, *and* $a + bi$, $a - bi$ *are the conjugate pair roots of Equation* (2.10).

Proof The similarity between the forms of solutions of Equation (2.1) and solutions of a linear equation with constant coefficients of Equation (2.6) is not just a coincidence.

Suppose that $u(x,t) = e^{mx}e^{\lambda t}$. We have

$$\begin{aligned} &\frac{\partial u}{\partial x} = me^{mx}e^{\lambda t}, \\ &\frac{\partial^2 u}{\partial x^2} = m^2 e^{mx}e^{\lambda t}, \\ &\frac{\partial^\alpha u}{\partial t^\alpha} = \lambda^\alpha e^{mx}e^{\lambda t}, \quad \text{and} \\ &\frac{\partial^\beta u}{\partial t^\beta} = \lambda^\beta e^{mx}e^{\lambda t}. \end{aligned} \tag{2.8}$$

So that the given equation (2.6) becomes

$$Am^2 e^{mx}e^{\lambda t} + Bme^{mx}e^{\lambda t} + Ce^{mx}e^{\lambda t} - M\lambda^\alpha e^{mx}e^{\lambda t} - N\lambda^\beta e^{mx}e^{\lambda t} = 0. \tag{2.9}$$

If m_1 and m_2 are the two roots of the auxiliary equation

$$Am^2 + Bm + C - M\lambda^\alpha - N\lambda^\beta = 0, \tag{2.10}$$

then

$$m = m_1 = \frac{-B + \sqrt{B^2 - 4A(C - M\lambda^\alpha - N\lambda^\beta)}}{2A}$$

and

$$m = m_2 = \frac{-B - \sqrt{B^2 - 4A(C - M\lambda^\alpha - N\lambda^\beta)}}{2A}.$$

The analysis of three cases is similar to Theorem 2.1, we can obtain each solution of the forms as follows:

$u(x,t) = e^{\lambda t}(k_1 e^{m_1 x} + k_2 e^{m_2 x})$ with $m_1 \neq m_2$, two distinct real roots,
$u(x,t) = e^{\lambda t}(k_1 e^{mx} + k_2 x e^{mx})$ with $m_1 = m_2 = m$, repeated real roots, and
$u(x,t) = e^{\lambda t}(k_1 e^{(a+ib)x} + k_2 e^{(a-ib)x})$ with the conjugate complex roots. □

Remark The constant λ in Equations (2.2) and (2.7) can be solved directly by constant initial value and constant boundary values (or by the numerical methods).

Corollary 2.1 *The fractional partial differential equation*

$$Ax^2\frac{\partial^2 u}{\partial x^2} + Bx\frac{\partial u}{\partial x} + C = M\frac{\partial^\alpha u}{\partial t^\alpha}, \tag{2.11}$$

with $u = u(x,t)$, $n-1 < \alpha \leq n$, $n \in \mathbb{N}$ *and* A $(\neq 0)$, B, C, M *are constants, has its solutions of the form given by*

(a″)

$$u(x,t) = e^{\lambda t}\left[k_1 x^{\frac{A-B+\sqrt{(A-B)^2-4A(C-M\lambda^\alpha)}}{2A}} + k_2 x^{\frac{A-B-\sqrt{(A-B)^2-4A(C-M\lambda^\alpha)}}{2A}}\right], \tag{2.12}$$

when the discriminant $(A-B)^2 - 4A(C - M\lambda^\alpha) > 0$;

(b″)

$$u(x,t) = e^{\lambda t}\left[k_1 x^m + k_2 x^m \ln x\right],$$

when the discriminant $(A-B)^2 - 4A(C - M\lambda^\alpha) = 0$, *and the roots* m_1, m_2 *of Equation* (2.5) *with* $N = 0$ *are repeated; that is,* $m_1 = m_2 = m$;

(c″)

$$u(x,t) = e^{\lambda t}\left[k_1 e^{a+bi} + k_2 e^{a-bi}\right],$$

when the discriminant $(A-B)^2 - 4A(C - M\lambda^\alpha) < 0$, *and* $a + bi$, $a - bi$ *are the conjugate pair roots of Equation* (2.5) *with* $N = 0$.

Corollary 2.2 *The fractional partial differential equation*

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial u}{\partial x} + Cu = M\frac{\partial^\alpha u}{\partial t^\alpha}, \tag{2.13}$$

with $u = u(x,t)$, $n-1 < \alpha \leq n$, $n \in \mathbb{N}$ *and* A $(\neq 0)$, B, C, M *are constants, has its solutions of the form given by*

(a‴)

$$u(x,t) = e^{\lambda t}\left[k_1 e^{(\frac{-B+\sqrt{B^2-4A(C-M\lambda^\alpha)}}{2A})x} + k_2 e^{(\frac{-B-\sqrt{B^2-4A(C-M\lambda^\alpha)}}{2A})x}\right], \tag{2.14}$$

when the discriminant $B^2 - 4A(C - M\lambda^\alpha) > 0$;

(b‴)

$$u(x,t) = e^{\lambda t}\left[k_1 e^{mx} + k_2 x e^{mx}\right],$$

when the discriminant $B^2 - 4A(C - M\lambda^\alpha) = 0$, *and the roots* m_1, m_2 *of Equation* (2.10) *with* $N = 0$ *are repeated; that is,* $m_1 = m_2 = m$;

(c‴)

$$u(x,t) = e^{\lambda t}\left[k_1 e^{(a+bi)x} + k_2 e^{(a-bi)x}\right],$$

when the discriminant $B^2 - 4A(C - M\lambda^\alpha) < 0$, *and* $a + bi$, $a - bi$ *are the conjugate pair roots of Equation* (2.10) *with* $N = 0$.

3 Examples

Example 3.1 If the two-dimensional harmonic equation $\nabla_1^2 u = 0$ is transformed to plane polar coordinates r and θ, defined by $x = r\cos\theta$, $y = r\sin\theta$, it takes the form

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \tag{3.1}$$

then it has solutions of the form

$$u(r,\theta) = \left(k_1 r^{i\lambda} + k_2 r^{-i\lambda}\right)e^{\lambda\theta},$$

where k_1, k_2 and λ are constants.

Solution Equation (3.1) is coincident to

$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0.$$

We have the solution

$$u(r,\theta) = \left(k_1 r^{i\lambda} + k_2 r^{-i\lambda}\right)e^{\lambda\theta}$$

by taking $A = 1$, $B = 1$, $C = 0$, $M = -1$, $N = 0$ and $\alpha = 2$ in Theorem 2.1. □

Example 3.2 The fractional partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} = \frac{\partial^\alpha u}{\partial t^\alpha} \quad \text{with } n-1 < \alpha \le n, n \in \mathbb{N}.$$

Solution Putting $A = 1$, $B = 1$, $C = 0$ and $M = 1$ in Corollary 2.2, we obtain the solution

$$u(x,t) = e^{\lambda t}\left[c_1 e^{(\frac{-1+\sqrt{1+4\lambda^\alpha}}{2})x} + c_2 e^{(\frac{-1-\sqrt{1+4\lambda^\alpha}}{2})x}\right],$$

where the discriminant $1 + 4\lambda^\alpha > 0$.

If $1 + 4\lambda^\alpha = 0$,

$$u(x,t) = e^{-(\frac{1}{2}x+\frac{1}{64}t)}(c_1 + c_2 x).$$

The analysis of the case $1 + 4\lambda^\alpha < 0$ is similar to Theorem 2.2. □

Example 3.3 The fractional partial differential equation

$$x^2\frac{\partial^2 u}{\partial x^2} - x\frac{\partial u}{\partial x} = -2\frac{\partial^{\frac{1}{2}} u}{\partial t^{\frac{1}{2}}} \quad \text{with } u(1,t) = u(e,t) = 0.$$

Solution Putting $A = 1$, $B = -1$, $C = 0$ and $M = -2$ and $\alpha = \frac{1}{2}$ in Corollary 2.1, we obtain the solution

$$\begin{aligned} u(x,t) &= xe^{\lambda t}\left[k_1 \cos\frac{\sqrt{4\times 2\lambda^{\frac{1}{2}} - 4}}{2}\cdot \ln x + k_2 \sin\frac{\sqrt{4\times 2\lambda^{\frac{1}{2}} - 4}}{2}\cdot \ln x\right] \\ &= xe^{\lambda t}\left[k_1 \cos\sqrt{2\lambda^{\frac{1}{2}} - 1}\cdot \ln x + k_2 \sin\sqrt{2\lambda^{\frac{1}{2}} - 1}\cdot \ln x\right], \end{aligned}$$

$$u(1,t) = k_1 e^{\lambda t} = 0 \quad \text{implies} \quad k_1 = 0.$$

Then

$$u(x,t) = k_2 x e^{\lambda t} \sin\sqrt{2\lambda^{\frac{1}{2}} - 1} \cdot \ln x,$$

$$u(e,t) = k_2 e^{\lambda t+1} \sin\sqrt{2\lambda^{\frac{1}{2}} - 1} = 0 \quad \text{implies that} \quad \sqrt{2\lambda^{\frac{1}{2}} - 1} = n\pi, \quad n \in \mathbb{Z}.$$

That is, $\lambda = \frac{(n^2\pi^2+1)^2}{4}$.

Thus, $u(x,t) = k_2 x e^{\frac{(n^2\pi^2+1)^2}{4}t} \sin n\pi \cdot \ln x$, $n \in \mathbb{Z}$.

If the discriminant $2\lambda^{\frac{1}{2}} - 1 = 0$, the solution is trivial. If the discriminant $2\lambda^{\frac{1}{2}} - 1 < 0$, then the solution is

$$u(x,t) = k_1 e^{\frac{(n^2+\pi^2)^2 t}{4}} \left(x^{1+in\pi} - x^{1-in\pi}\right), \quad n \in \mathbb{N}. \qquad \square$$

Example 3.4 The fractional partial differential equation

$$\frac{\partial^{\frac{2}{3}} u}{\partial t^{\frac{2}{3}}} = \frac{1}{4}\frac{\partial^2 u}{\partial x^2} \quad \text{with } u(0,t) = e^{-2t}.$$

Solution Putting $\alpha = \frac{2}{3}$, $A = \frac{1}{4}$, $B = 0$, $C = 0$ and $M = 1$ in Corollary 2.2, the discriminant is $\lambda^{\frac{2}{3}} = (\lambda^{\frac{1}{3}})^2 \geq 0$, but $\lambda = 0$ leads to a contradiction, hence there are different real roots $m_1 = 2\lambda^{\frac{1}{3}}$ and $m_2 = -2\lambda^{\frac{1}{3}}$, so that we have

$$u(x,t) = e^{\lambda t}\left(k_1 e^{2\lambda^{\frac{1}{3}} x} + k_2 e^{-2\lambda^{\frac{1}{3}} x}\right).$$

By the boundary condition $u(0,t) = e^{-2t}$, we obtain $\lambda = -2$ and $k_1 + k_2 = 1$. So,

$$u(x,t) = e^{-2t} e^{2\sqrt[3]{2}x} - 2k_1 e^{-2t} \sinh\left(2\sqrt[3]{2}x\right) = e^{-2t} e^{-2\sqrt[3]{2}x} + 2k_2 e^{-2t} \sinh\left(2\sqrt[3]{2}x\right),$$

and the particular solution is

$$u(x,t) = e^{-2t} e^{2\sqrt[3]{2}x} \quad \left(\text{or } u(x,t) = e^{-2t} e^{-2\sqrt[3]{2}x}\right).$$

If we apply the Laplace transform to $\phi(t) = u(0,t) = e^{-2t}$, then $\mathcal{L}\{\phi(t)\}(s) = \frac{1}{s+2}$ and $\tilde{u}(x,s) = \frac{1}{s+2} e^{-2xs^{\frac{1}{3}}}$. Using the residue theorem,

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi i} \int_C e^{st} \frac{1}{s+2} e^{-2xs^{\frac{1}{3}}}\, ds \\ &= \frac{1}{2\pi i} \cdot 2\pi i \lim_{s \to -2} (s+2) \frac{e^{-2xs^{\frac{1}{3}}} e^{st}}{s+2} \\ &= e^{-2t} e^{2\sqrt[3]{2}x}. \end{aligned}$$

The solution obtained by the method of Laplace transform and the residue theorem is a coincidence, which is our result above. $\square$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
SDL carried out the molecular genetic studies, participated in the sequence alignment and drafted the manuscript. CHL carried out the immunoassays. SMS participated in the sequence alignment.

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