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New potential condition on homoclinic orbits for a class of discrete Hamiltonian systems

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Abstract
In the present paper, we establish an existence criterion to guarantee that the second-order self-adjoint discrete Hamiltonian system $\triangle[p(n)\triangle u(n-1)] - L(n)u(n) + \nabla W(n, u(n)) = 0$ has a nontrivial homoclinic solution, which does not need periodicity and coercivity conditions on $L(n)$.
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1 Introduction

Consider the second-order self-adjoint discrete Hamiltonian system

$$\triangle\bigl[p(n)\triangle u(n-1)\bigr] - L(n)u(n) + \nabla W\bigl(n, u(n)\bigr) = 0, \tag{1.1}$$

where $n \in \mathbb{Z}$, $u \in \mathbb{R}^{\mathcal{N}}$, $\triangle u(n) = u(n+1) - u(n)$ is the forward difference, $p, L : \mathbb{Z} \to \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$ and $W : \mathbb{Z} \times \mathbb{R}^{\mathcal{N}} \to \mathbb{R}$.

Discrete Hamiltonian systems can be applied in many areas, such as physics, chemistry, and so on. For more discussions on discrete Hamiltonian systems, we refer the reader to [1, 2].

As usual, we say that a solution $u(n)$ of system (1.1) is homoclinic (to 0) if $u(n) \to 0$ as $n \to \pm\infty$. In addition, if $u(n) \not\equiv 0$ then $u(n)$ is called a nontrivial homoclinic solution.

The existence and multiplicity of homoclinic solutions of system (1.1) or its special forms have been investigated by many authors. Papers [3–8] deal with the periodic case where p, L and W are periodic in n or independent of n. In contrast, if the periodicity is lost, because of lack of compactness of the Sobolev embedding, up to our knowledge, all existence results require a coercivity condition on L:

$$\lim_{|n|\to\infty}\Bigl[\inf_{x\in\mathbb{R}^{\mathcal{N}},|x|=1}\bigl(L(n)x, x\bigr)\Bigr] = \infty. \tag{1.2}$$

For example, see [9–14]. In the above mentioned papers, except [14], L was always required to be positive definite.

In this paper, we derive an existence result which does not need periodicity and coercivity conditions on $L(n)$. To state our results precisely, we make the following assumptions.

(P) $p(n)$ is $\mathcal{N} \times \mathcal{N}$ real symmetric positive definite matrix for all $n \in \mathbb{Z}$.

(L) $L(n)$ is $\mathcal{N} \times \mathcal{N}$ real symmetric nonnegative definite matrix for all $n \in \mathbb{Z}$, and there exist a positive integer $N_0 \in \mathbb{Z}$ and $\beta > 0$ such that

$$\min_{x \in \mathbb{R}^{\mathcal{N}}, |x|=1} \big(L(n)x, x\big) \geq \beta, \quad |n| \geq N_0,$$

where here and in the sequel, $(\cdot,\cdot)$ denotes the standard inner product in $\mathbb{R}^{\mathcal{N}}$ and $|\cdot|$ is the induced norm.

(W1) $W(n,x)$ is continuously differentiable in x for every $n \in \mathbb{Z}$, $W(n,0) = 0$, $W(n,x) \geq 0$ for all $(n,x) \in \mathbb{Z} \times \mathbb{R}^{\mathcal{N}}$.

(W2) $\lim_{|x|\to 0} \frac{\nabla W(n,x)}{|x|} = 0$ uniformly for all $n \in \mathbb{Z}$.

(W3) $\lim_{|x|\to\infty} \frac{|W(n,x)|}{|x|^2} = \infty$ uniformly for all $n \in \mathbb{Z}$.

(W4) $\widetilde{W}(n,x) := \frac{1}{2}(\nabla W(n,x), x) - W(n,x) \geq 0$, $\forall (n,x) \in \mathbb{Z} \times \mathbb{R}^{\mathcal{N}}$, and there exist $\varepsilon \in (0,1)$, $c_0 > 0$, and $R_0 > 0$ such that

$$\big(\nabla W(n,x), x\big) \leq \frac{\beta(1-\varepsilon)}{2}|x|^2, \quad \forall (n,x) \in \mathbb{Z} \times \mathbb{R}^{\mathcal{N}}, |x| \leq R_0$$

and

$$\big(\nabla W(n,x), x\big) \leq c_0|x|^2 \widetilde{W}(n,x), \quad \forall (n,x) \in \mathbb{Z} \times \mathbb{R}^{\mathcal{N}}, |x| \geq R_0.$$

Now, we are ready to state the main result of this paper.

Theorem 1.1 *Assume that p, L and W satisfy* (P), (L), (W1), (W2), (W3), *and* (W4). *If there exist $n_0 \in \mathbb{Z}$ and $x_0 \in \mathbb{R}^{\mathcal{N}}$ such that*

$$\beta \geq 2c_0 \sup_{s \geq 0} \left[\frac{s^2}{2} \big(\big(p(n_0) + p(n_0+1) + L(n_0)\big)x_0, x_0 \big) - W(n_0, sx_0) \right], \tag{1.3}$$

then system (1.1) *possesses a nontrivial homoclinic solution.*

In Theorem 1.1, we replace (L) and (W4) by the following assumptions:

(L′) $L(n)$ is $\mathcal{N} \times \mathcal{N}$ real symmetric nonnegative definite matrix for all $n \in \mathbb{Z}$, and it satisfies (1.2).

(W4′) $\widetilde{W}(n,x) := \frac{1}{2}(\nabla W(n,x), x) - W(n,x) \geq 0$, $\forall (n,x) \in \mathbb{Z} \times \mathbb{R}^{\mathcal{N}}$, and there exist $c_0 > 0$ and $R_0 > 0$ such that

$$\big(\nabla W(n,x), x\big) \leq c_0|x|^2 \widetilde{W}(n,x), \quad \forall (n,x) \in \mathbb{Z} \times \mathbb{R}^{\mathcal{N}}, |x| \geq R_0.$$

Then we have the following corollary immediately.

Corollary 1.2 *Assume that p, L and W satisfy* (P), (L′), (W1), (W2), (W3) *and* (W4′). *Then system* (1.1) *possesses a nontrivial homoclinic solution.*

Remark 1.3 If $W(n,x)$ satisfies the well-known global Ambrosetti-Rabinowitz superquadratic condition:

(AR) there exists $\mu > 2$ such that

$$0 < \mu W(n,x) \leq \big(\nabla W(n,x), x\big), \quad \forall (n,x) \in \mathbb{Z} \times \mathbb{R}^{\mathcal{N}} \setminus \{0\},$$

then there exists a constant $C_0 > 0$ such that

$$W(n,x) \geq C_0 |x|^{\mu}, \quad \forall (n,x) \in \mathbb{Z} \times \mathbb{R}^{\mathcal{N}}, |x| \geq 1;$$

moreover $\widetilde{W}(n,x) > 0$ for all $(n,x) \in \mathbb{Z} \times (\mathbb{R}^{\mathcal{N}} \setminus \{0\})$, and

$$\big(\nabla W(n,x), x\big) \leq \frac{2\mu}{\mu - 2} |x|^2 \widetilde{W}(n,x), \quad \forall (n,x) \in \mathbb{Z} \times \mathbb{R}^{\mathcal{N}}, |x| \geq 1.$$

In addition, by virtue of (W2), there exists $\beta_1 > 0$ such that

$$\big(\nabla W(n,x), x\big) \leq \frac{\beta_1}{2} |x|^2, \quad \forall (n,x) \in \mathbb{Z} \times \mathbb{R}^{\mathcal{N}}, |x| \leq 1.$$

These show that (W3) and (W4) hold with $R_0 = 1$, $c_0 = 2\mu/(\mu - 2)$ and $\beta > \beta_1$. Let $p(n) = I_{\mathcal{N}}$ and $L(n) = \lambda n^2/(1 + n^2) I_{\mathcal{N}}$ and choose $n_0 = 0$ and $x_0 = (1, 0, \ldots, 0) \in \mathbb{R}^{\mathcal{N}}$. In view of Theorem 1.1, if

$$\lambda > \max\left\{ \frac{4\mu}{\mu - 2} \sup_{s \geq 0} \big[s^2 - W(0, s x_0)\big], \beta_1 \right\},$$

then system (1.1) possesses a nontrivial homoclinic solution.

Example 1.4 Let $p(n) = I_{\mathcal{N}}$, $L(n) = [1 + \lambda n^2/(1 + n^2)] I_{\mathcal{N}}$ and

$$W(n,x) = |x|^2 \ln\big(1 + |x|^2\big). \tag{1.4}$$

Then

$$\big(\nabla W(n,x), x\big) = 2|x|^2 \ln\big(1 + |x|^2\big) + \frac{2|x|^4}{1 + |x|^2}.$$

It is easy to see that $\widetilde{W}(n,x) \geq 0$ for all $(n,x) \in \mathbb{Z} \times \mathbb{R}^{\mathcal{N}}$, and

$$\big(\nabla W(n,x), x\big) \leq (2 \ln 2 + 1)|x|^2, \quad \forall (n,x) \in \mathbb{Z} \times \mathbb{R}^{\mathcal{N}}, |x| \leq 1,$$
$$\big(\nabla W(n,x), x\big) \leq 6|x|^2 \widetilde{W}(n,x), \quad \forall (n,x) \in \mathbb{Z} \times \mathbb{R}^{\mathcal{N}}, |x| \geq 1.$$

These show that (W3) and (W4) hold with $R_0 = 1$, $c_0 = 6$ and $\beta > 2(2 \ln 2 + 1)$. We choose $n_0 = 0$ and $x_0 = (1, 0, \ldots, 0) \in \mathbb{R}^{\mathcal{N}}$. Then

$$\begin{aligned}
&\sup_{s \geq 0} \left[\frac{s^2}{2} \big(\big(p(n_0) + p(n_0 + 1) + L(n_0)\big) x_0, x_0 \big) - W(n_0, s x_0) \right] \\
&\quad = \sup_{s \geq 0} \left[\frac{3s^2}{2} - s^2 \ln\big(1 + s^2\big) \right] < 6 - \ln 2.
\end{aligned}$$

In view of Theorem 1.1, if $\lambda \geq 12(6 - \ln 2)$, then system (1.1) possesses a nontrivial homoclinic solution.

2 Preliminaries

Throughout this section, we always assume that p and L satisfy (P) and (L). Let

$$S = \{\{u(n)\}_{n\in\mathbb{Z}} : u(n) \in \mathbb{R}^{\mathcal{N}}, n \in \mathbb{Z}\},$$
$$E = \left\{u \in S : \sum_{n\in\mathbb{Z}}\left[\left(p(n+1)\triangle u(n), \triangle u(n)\right) + \left(L(n)u(n), u(n)\right)\right] < +\infty\right\},$$

and for $u, v \in E$, let

$$\langle u, v\rangle = \sum_{n\in\mathbb{Z}}\left[\left(p(n+1)\triangle u(n), \triangle v(n)\right) + \left(L(n)u(n), v(n)\right)\right].$$

Then E is a Hilbert space with the above inner product, and the corresponding norm is

$$\|u\| = \left\{\sum_{n\in\mathbb{Z}}\left[\left(p(n+1)\triangle u(n), \triangle u(n)\right) + \left(L(n)u(n), u(n)\right)\right]\right\}^{1/2}, \quad u \in E.$$

As usual, for $1 \le s < +\infty$, set

$$l^s(\mathbb{Z}, \mathbb{R}^{\mathcal{N}}) = \left\{u \in S : \sum_{n\in\mathbb{Z}}|u(n)|^s < +\infty\right\}$$

and

$$l^\infty(\mathbb{Z}, \mathbb{R}^{\mathcal{N}}) = \left\{u \in S : \sup_{n\in\mathbb{Z}}|u(n)| < +\infty\right\},$$

and their norms are defined by

$$\|u\|_s = \left(\sum_{n\in\mathbb{Z}}|u(n)|^s\right)^{1/s}, \quad \forall u \in l^s(\mathbb{Z}, \mathbb{R}^{\mathcal{N}});$$
$$\|u\|_\infty = \sup_{n\in\mathbb{Z}}|u(n)|, \quad \forall u \in l^\infty(\mathbb{Z}, \mathbb{R}^{\mathcal{N}}),$$

respectively.

Lemma 2.1 *Suppose that* (L) *is satisfied. Then*

$$\|u\|_\infty \le \frac{1}{\sqrt{\beta}}\|u\| + \sum_{|s|\le N_0-1}|\triangle u(s)|, \quad u \in E, \tag{2.1}$$

and

$$\|u\|_\infty \le \max\left\{\sqrt{\frac{2}{\beta}}, \sqrt{\frac{2N_0}{\alpha}}\right\}\|u\|, \quad u \in E, \tag{2.2}$$

where $\alpha = \min_{|n|\le N_0, |x|=1}(p(n)x, x)$.

Proof Since $u \in E$, it follows that $\lim_{|n|\to\infty} |u(n)| = 0$. Hence, there exists $n_* \in \mathbb{Z}$ such that $\|u\|_\infty = |u(n_*)|$. There are two possible cases.

Case (i). $|n_*| \geq N_0$. According to (L), one has

$$\|u\|_\infty^2 = \left|u(n_*)\right|^2 \leq \frac{1}{\beta} \sum_{|s|\geq N_0} \left(L(s)u(s), u(s)\right) \leq \frac{1}{\beta}\|u\|.$$

Case (ii). $|n_*| < N_0$. Without loss of generality, we can assume that $n_* \geq 0$, then

$$\begin{aligned}
\|u\|_\infty &\leq \left|u(N_0)\right| + \sum_{s=n_*}^{N_0-1} \left|\triangle u(s)\right| \\
&\leq \left[\frac{1}{\beta} \sum_{|s|\geq N_0} \left(L(s)u(s), u(s)\right)\right]^{1/2} + \sqrt{\frac{N_0}{\alpha}} \left(\sum_{s=n_*}^{N_0-1} \alpha\left|\triangle u(s)\right|^2\right)^{1/2} \\
&\leq \sqrt{2}\left[\frac{1}{\beta} \sum_{|s|\geq N_0} \left(L(s)u(s), u(s)\right) + \frac{N_0}{\alpha} \sum_{s=n_*}^{N_0-1} \left(p(s+1)\triangle u(s), \triangle u(s)\right)\right]^{1/2} \\
&\leq \max\left\{\sqrt{\frac{2}{\beta}}, \sqrt{\frac{2N_0}{\alpha}}\right\}\|u\|.
\end{aligned} \tag{2.3}$$

Cases (i) and (ii) imply that (2.1) and (2.2) hold. □

Now we define a functional Φ on E by

$$\Phi(u) = \frac{1}{2} \sum_{n\in\mathbb{Z}} \left[\left(p(n+1)\triangle u(n), \triangle u(n)\right) + \left(L(n)u(n), u(n)\right)\right] - \sum_{n\in\mathbb{Z}} W\left(n, u(n)\right). \tag{2.4}$$

For any $u \in E$, there exists an $n_1 \in \mathbb{N}$ such that $|u(n)| \leq 1$ for $|n| \geq n_1$. Hence, under assumptions (P), (L), (W1), and (W2), the functional Φ is of class $C^1(E, \mathbb{R})$. Moreover,

$$\Phi(u) = \frac{1}{2}\|u\|^2 - \sum_{n\in\mathbb{Z}} W(n, u), \quad \forall u \in E \tag{2.5}$$

and

$$\left\langle \Phi'(u), v\right\rangle = \langle u, v\rangle - \sum_{n\in\mathbb{Z}} \left(\nabla W(n, u), v\right), \quad \forall u, v \in E. \tag{2.6}$$

Furthermore, the critical points of Φ in E are solutions of system (1.1) with $u(\pm\infty) = 0$, see [5, 6].

Let $e = \{e(n)\}_{n\in\mathbb{Z}} \in E$ with $e(n_0) = x_0$ and $e(n) = 0 \in \mathbb{R}^{\mathcal{N}}$ for $n \neq n_0$.

Lemma 2.2 *Suppose that* (L), (W1) *and* (W2) *are satisfied. Then*

$$\sup\left\{\Phi(se) : s \geq 0\right\} \leq \sup_{s\geq 0}\left[\frac{s^2}{2}\left(\left(p(n_0) + p(n_0+1) + L(n_0)\right)x_0, x_0\right) - W(n_0, sx_0)\right]. \tag{2.7}$$

Proof From (2.4) and the definition of e, we get

$$\begin{aligned}\Phi(se) &= \frac{s^2}{2}\sum_{n\in\mathbb{Z}}\bigl[\bigl(p(n+1)\triangle e(n),\triangle e(n)\bigr)+\bigl(L(n)e(n),e(n)\bigr)\bigr]-\sum_{n\in\mathbb{Z}}W\bigl(n,se(n)\bigr)\\ &= \frac{s^2}{2}\bigl[\bigl(p(n_0+1)\triangle e(n_0),\triangle e(n_0)\bigr)+\bigl(p(n_0)\triangle e(n_0-1),\triangle e(n_0-1)\bigr)\\ &\quad+\bigl(L(n_0)e(n_0),e(n_0)\bigr)\bigr]-W\bigl(n_0,se(n_0)\bigr)\\ &= \frac{s^2}{2}\bigl(\bigl(p(n_0)+p(n_0+1)+L(n_0)\bigr)x_0,x_0\bigr)-W(n_0,sx_0).\end{aligned} \tag{2.8}$$

Now the conclusion of Lemma 2.1 follows by (2.8). □

Applying the mountain-pass lemma without the (PS) condition, by standard arguments, we can prove the following lemma.

Lemma 2.3 *Let* $W(n,x)\geq 0$, $\forall(n,x)\in\mathbb{Z}\times\mathbb{R}^{\mathcal{N}}$. *Suppose that* (P), (L), (W1), (W2) *and* (W3) *are satisfied. Then there exist a constant* $c\in(0,\sup_{s\geq 0}\Phi(se)]$ *and a sequence* $\{u_k\}\subset E$ *satisfying*

$$\Phi(u_k)\to c,\qquad \bigl\|\Phi'(u_k)\bigr\|\bigl(1+\|u_k\|\bigr)\to 0. \tag{2.9}$$

Lemma 2.4 *Suppose that* (P), (L), (W1), (W2), (W3), *and* (W4) *are satisfied. Then any sequence* $\{u_k\}\subset E$ *satisfying*

$$\Phi(u_k)\to c>0,\qquad \bigl\langle\Phi'(u_k),u_k\bigr\rangle\to 0 \tag{2.10}$$

is bounded in E.

Proof To prove the boundedness of $\{u_k\}$, arguing by contradiction, suppose that $\|u_k\|\to\infty$. Let $v_k=u_k/\|u_k\|$. Then $\|v_k\|=1$. By virtue of (2.5), (2.6), and (2.10), we have

$$\Phi(u_k)-\frac{1}{2}\bigl\langle\Phi'(u_k),u_k\bigr\rangle=\sum_{n\in\mathbb{Z}}\widetilde{W}(n,u_k)=c+o(1). \tag{2.11}$$

If $\delta:=\limsup_{k\to\infty}\|v_k\|_\infty=0$, then it follows from (L), (W4) and (2.11) that

$$\begin{aligned}\sum_{|u_k|<R_0}\bigl|\bigl(\nabla W(n,u_k),u_k\bigr)\bigr| &\leq \frac{\beta}{2}\sum_{|u_k|<R_0}|u_k|^2\leq\frac{\beta}{2}\sum_{|s|\geq N_0}\bigl|u_k(s)\bigr|^2+\frac{\beta}{2}\sum_{|s|<N_0}\bigl|u_k(s)\bigr|^2\\ &\leq \frac{1}{2}\|u_k\|^2+N_0\beta\|u_k\|^2\|v_k\|_\infty^2\leq\left[\frac{1}{2}+o(1)\right]\|u_k\|^2\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}\sum_{|u_k|\geq R_0}\frac{|(\nabla W(n,u_k),u_k)|}{\|u_k\|^2} &\leq c_0\sum_{|u_k|\geq R_0}|v_k|^2\widetilde{W}(n,u_k)\leq c_0\|v_k\|_\infty^2\sum_{|u_k|\geq R_0}\widetilde{W}(n,u_k)\\ &\leq c_0(c+1)\|v_k\|_\infty^2\to 0,\quad k\to\infty.\end{aligned} \tag{2.13}$$

Combining (2.12) with (2.13) and using (2.5) and (2.10), we have

$$\begin{aligned}1+o(1) &\leq \frac{\|u_k\|^2 - \langle \Phi'(u_k), u_k\rangle}{\|u_k\|^2} \leq \sum_{n\in\mathbb{Z}} \frac{|(\nabla W(n,u_k),u_k)|}{\|u_k\|^2} \\ &= \sum_{|u_k|<R_0} \frac{|(\nabla W(n,u_k),u_k)|}{\|u_k\|^2} + \sum_{|u_k|\geq R_0} \frac{|(\nabla W(n,u_k),u_k)|}{\|u_k\|^2} \leq \frac{1}{2} + o(1). \end{aligned} \tag{2.14}$$

This contradiction shows that $\delta > 0$.

Going if necessary to a subsequence, we may assume the existence of $n_k \in \mathbb{Z}$ such that

$$|v_k(n_k)| = \|v_k\|_\infty > \frac{\delta}{2}.$$

Let $w_k(n) = v_k(n + n_k)$, then

$$|w_k(0)| > \frac{\delta}{2}, \quad \forall k \in \mathbb{N}. \tag{2.15}$$

Now we define $\tilde{u}_k(n) = u_k(n+n_k)$. Then $\tilde{u}_k(n)/\|u_k\| = w_k(n)$ and $\|w_k\|_2 = \|v_k\|_2$. Passing to a subsequence, we have $w_k \rightharpoonup w$ in $l^2(\mathbb{Z}, \mathbb{R}^{\mathcal{N}})$, then $w_k(n) \to w(n)$ for all $n \in \mathbb{Z}$. Clearly, (2.15) implies that $w(0) \neq 0$.

It is obvious that $w(n) \neq 0$ implies $\lim_{k\to\infty} |\tilde{u}_k(n)| = \infty$. Hence, it follows from (2.5), (2.10), and (W3) that

$$\begin{aligned}0 &= \lim_{k\to\infty} \frac{c+o(1)}{\|u_k\|^2} = \lim_{k\to\infty} \frac{\Phi(u_k)}{\|u_k\|^2} \\ &= \lim_{k\to\infty} \left[\frac{1}{2} - \sum_{n\in\mathbb{Z}} \frac{W(n,u_k)}{|u_k|^2} |v_k|^2\right] \\ &= \lim_{k\to\infty} \left[\frac{1}{2} - \sum_{n\in\mathbb{Z}} \frac{W(n+k_n,\tilde{u}_k)}{|\tilde{u}_k|^2} |w_k|^2\right] \\ &\leq \frac{1}{2} - \liminf_{k\to\infty} \sum_{n\in\mathbb{Z}} \frac{W(n+k_n,\tilde{u}_k)}{|\tilde{u}_k|^2} |w_k|^2 \\ &= -\infty,\end{aligned}$$

which is a contradiction. Thus $\{u_k\}$ is bounded in E. □

3 Proof of theorem

Proof of Theorem 1.1 Applying Lemmas 2.3 and 2.4, we deduce that there exists a bounded sequence $\{u_k\} \subset E$ satisfying (2.9). By Lemma 2.2 and (1.3), one has

$$c \leq \frac{\beta}{2c_0}. \tag{3.1}$$

Going if necessary to a subsequence, we can assume that $u_k \rightharpoonup \bar{u}$ in E and $\Phi'(u_k) \to 0$. Next, we prove that $\bar{u} \neq 0$.

Arguing by contradiction, suppose that $\bar{u} = 0$, *i.e.* $u_k \rightharpoonup 0$ in E, and so $u_k(n) \to 0$ for every $n \in \mathbb{Z}$. Hence,

$$\|u_k\|_2^2 = \sum_{|n|\geq N_0} \left|u_k(n)\right|^2 + \sum_{|n|<N_0} \left|u_k(n)\right|^2 \leq \frac{1}{\beta}\|u_k\|^2 + o(1). \tag{3.2}$$

According to (W4) and (3.2), one gets

$$\sum_{|u_k|<R_0} \left(\nabla W(n,u_k), u_k\right) \leq \frac{\beta(1-\varepsilon)}{2} \sum_{|u_k|<R_0} |u_k|^2 \leq \frac{1-\varepsilon}{2}\|u_k\|^2 + o(1). \tag{3.3}$$

By virtue of (2.5), (2.6), and (2.9), we have

$$\Phi(u_k) - \frac{1}{2}\left\langle \Phi'(u_k), u_k\right\rangle = \sum_{n\in\mathbb{Z}} \widetilde{W}(n,u_k) = c + o(1). \tag{3.4}$$

Using (W4), (2.1), (3.1), (3.2), and (3.4), we obtain

$$\begin{aligned}
\sum_{|u_k|\geq R_0} \left(\nabla W(n,u_k), u_k\right) &\leq c_0 \sum_{|u_k|\geq R_0} |u_k|^2 \widetilde{W}(n,u_k) \\
&\leq c_0 \|u_k\|_\infty^2 \sum_{|u_k|\geq R_0} \widetilde{W}(n,u_k) \\
&\leq c_0 c \|u_k\|_\infty^2 + o(1) \\
&\leq c_0 c \left(\frac{1}{\sqrt{\beta}}\|u_k\| + \sum_{|s|\leq N_0-1} \left|\triangle u_k(s)\right| \right)^2 + o(1) \\
&= \frac{c_0 c}{\beta}\|u_k\|^2 + o(1) \\
&\leq \frac{1}{2}\|u_k\|^2 + o(1),
\end{aligned} \tag{3.5}$$

which, together with (2.6), (2.9), and (3.3), yields

$$\begin{aligned}
o(1) &= \left\langle \Phi'(u_k), u_k\right\rangle = \|u_k\|^2 - \sum_{n\in\mathbb{Z}} \left(\nabla W(n,u_k), u_k\right) \\
&\geq \frac{\varepsilon}{2}\|u_k\|^2 + o(1),
\end{aligned} \tag{3.6}$$

resulting in the fact that $\|u_k\| \to 0$. Consequently, it follows from (W1), (2.5), and (2.9) that

$$0 < c = \lim_{k\to\infty} \Phi(u_k) = \Phi(0) = 0.$$

This contradiction shows $\bar{u} \neq 0$. By standard arguments, we easily prove that $\bar{u}$ is a nontrivial solution of (1.1). □

Competing interests

The author declares that they have no competing interests.

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