Yu and Wang *Advances in Difference Equations* 2013, **2013**:124
http://www.advancesindifferenceequations.com/content/2013/1/124

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Existence of solutions for nonlinear second-order q-difference equations with first-order q-derivatives

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Abstract

In this paper, we establish the existence of solutions for a boundary value problem with the nonlinear second-order q-difference equation

$$\begin{cases} D_q^2 u(t) = f(t, u(t), D_q u(t)), & t \in I, \\ D_q u(0) = 0, & D_q u(1) = \alpha u(1). \end{cases}$$

The existence and uniqueness of solutions for the problem are proved by means of the Leray-Schauder nonlinear alternative and some standard fixed point theorems. Finally, we give two examples to demonstrate the use of the main results. The nonlinear team f contains $D_q u(t)$ in the equation.

Keywords: q-difference equations; Leray-Schauder nonlinear alternative; boundary value problem; fixed point theorem

1 Introduction

In this paper, we study the existence of solutions for a boundary value problem with nonlinear second-order q-difference equations

$$\begin{cases} D_q^2 u(t) = f(t, u(t), D_q u(t)), & t \in I, \\ D_q u(0) = 0, & D_q u(1) = \alpha u(1), \end{cases} \tag{1.1}$$

where $f \in C(I \times R^2, R)$, $I = \{q^n : n \in N\} \cup \{0, 1\}$, $q \in (0, 1)$, and $\alpha \neq 0$ is a fixed real number.

The q-difference equations initiated at the beginning of the twentieth century [1–4] is a very interesting field in difference equations. In the last few decades, it has evolved into a multidisciplinary subject and plays an important role in several fields of physics such as cosmic strings and black holes [5], conformal quantum mechanics [6], nuclear and high energy physics [7]. However, the theory of boundary value problems (BVPs) for nonlinear q-difference equations is still in the initial stages and many aspects of this theory need to be explored. To the best of our knowledge, for the BVPs of nonlinear q-difference equations, a few works were done; see [8–13] and the references therein. In particular, the study of BVPs for nonlinear q-difference equation with first-order q-difference is yet to be initiated.

The main aim of this paper is to develop some existence and uniqueness results for BVP (1.1). Our results are based on a variety of fixed point theorems such as the Banach contraction mapping principle, the Leray-Schauder nonlinear alternative and the Leray-Schauder continuous theorem. Some examples and special cases are also discussed.

2 Preliminary results

In this section, firstly, let us recall some basic concepts of q-calculus [14, 15].

Definition 2.1 For $0 < q < 1$, we define the q-derivative of a real-value function f as

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \qquad D_q f(0) = \lim_{t\to 0} D_q f(t).$$

Note that $\lim_{q\to 1^-} D_q f(t) = f'(t)$.

Definition 2.2 The higher-order q-derivatives are defined inductively as

$$D_q^0 f(t) = f(t), \qquad D_q^n f(t) = D_q D_q^{n-1} f(t), \quad n \in N.$$

For example, $D_q(t^k) = [k]_q t^{k-1}$, where k is a positive integer and the bracket $[k]_q = (q^k - 1)/(q-1)$. In particular, $D_q(t^2) = (1+q)t$.

Definition 2.3 The q-integral of a function f defined in the interval $[a,b]$ is given by

$$\int_a^x f(t)\,d_q t := \sum_{n=0}^{\infty} x(1-q)q^n f\bigl(xq^n\bigr) - af\bigl(q^n a\bigr), \quad x \in [a,b],$$

and for $a = 0$, we denote

$$I_q f(x) = \int_0^x f(t)\,d_q t = \sum_{n=0}^{\infty} x(1-q)q^n f\bigl(xq^n\bigr).$$

Then

$$\int_a^b f(t)\,d_q t = \int_0^b f(t)\,d_q t - \int_0^a f(t)\,d_q t.$$

Similarly, we have

$$I_q^0 f(t) = f(t), \qquad I_q^n f(t) = I_q I_q^{n-1} f(t), \quad n \in N.$$

Observe that

$$D_q I_q f(x) = f(x),$$

and if f is continuous at $x = 0$, then $I_q D_q f(x) = f(x) - f(0)$.

In q-calculus, the product rule and integration by parts formula are

$$D_q(gh)(t) = D_q g(t) h(t) + g(qt) D_q h(t), \tag{2.1}$$

$$\int_0^x f(t) D_q g(t)\,d_q t = \bigl[f(t)g(t)\bigr]_0^x - \int_0^x D_q f(t) g(qt)\,d_q t. \tag{2.2}$$

Remark 2.4 In the limit $q \to 1^-$, the above results correspond to their counterparts in standard calculus.

Definition 2.5 $f : I \times R^2 \to R$ is called an S-Carathéodory function if and only if

(i) for each $(u,v) \in R^2$, $t \mapsto f(t,u,v)$ is measurable on I;
(ii) for a.e. $t \in I$, $(u,v) \mapsto f(t,u,v)$ is continuous on R^2;
(iii) for each $r > 0$, there exists $\varphi_r(t) \in L^1(I,R^+)$ with $t\varphi_r(t) \in L^1(I,R^+)$ on I such that $\max\{|u|,|v|\} \le r$ implies $|f(t,u,v)| \le \varphi_r(t)$, for a.e. I, where
$L^1(I,R^+) = \{u \in C_q : \int_0^1 u(t)\,d_qt \text{ exists}\}$, and normed by $\|u\|_{L^1} = \int_0^1 |u(t)|\,d_qt$ for all $u \in L^1(I,R^+)$.

Theorem 2.6 (Nonlinear alternative for single-valued maps [16]) *Let E be a Banach space, let C be a closed and convex subset of E, and let U be an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*
(ii) *there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $u = \lambda F(u)$.*

Lemma 2.7 *Let $y \in C[0,1]$, then the BVP*

$$\begin{cases} D_q^2 u(t) = y(t), & t \in I, \\ D_q u(0) = 0, & D_q u(1) = \alpha u(1), \end{cases} \tag{2.3}$$

has a unique solution

$$\begin{aligned} u(t) &= \int_0^t (t-qs)y(s)\,d_qs + \int_0^1 \left(\frac{1}{\alpha} - 1 + qs\right) y(s)\,d_qs \\ &= \int_0^1 G(t,s;q)y(s)\,d_qs, \end{aligned} \tag{2.4}$$

where

$$G(t,s;q) = \frac{1}{\alpha} \begin{cases} 1-\alpha+\alpha t, & s \le t, \\ 1-\alpha+\alpha qs, & t \le s. \end{cases} \tag{2.5}$$

Proof Integrating the q-difference equation from 0 to t, we get

$$D_q u(t) = \int_0^t y(s)\,d_qs + a_1. \tag{2.6}$$

Integrating (2.6) from 0 to t and changing the order of integration, we have

$$u(t) = \int_0^t (t-qs)y(s)\,d_qs + a_1 t + a_0, \tag{2.7}$$

where a_1, a_0 are arbitrary constants. Using the boundary conditions $D_qu(0) = 0, D_qu(1) = \alpha u(1)$ in (2.7), we find that $a_1 = 0$, and

$$a_1 = \int_0^1 \left(\frac{1}{\alpha} - 1 + qs\right) y(s)\,d_qs.$$

Substituting the values of a_0 and a_1 in (2.7), we obtain

$$u(t) = \int_0^t (t-qs)y(s)\,d_qs + \int_0^1 \left(\frac{1}{\alpha} - 1 + qs\right)y(s)\,d_qs.$$

This completes the proof. □

Remark 2.8 For $q \to 1$, equation (2.4) takes the form

$$u(t) = \int_0^t (t-qs)y(s)\,d_qs + a_1 t + a_0,$$

which is the solution of a classical second-order ordinary differential equation $u''(t) = y(t)$ and the associated form of Green's function for the classical case is

$$G(t,s) = \frac{1}{\alpha}\begin{cases} 1-\alpha+\alpha t, & s \le t, \\ 1-\alpha+\alpha s, & t \le s. \end{cases}$$

We consider the Banach space $C_q = C(I,R)$ equipped with the standard norm $\|u\| = \max\{\|u\|_\infty, \|D_q u\|_\infty\}$, and $\|\cdot\|_\infty = \sup\{\|\cdot\|, t \in I\}$, $u \in C_q$.

Define an integral operator $T : C_p \to C_p$ by

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s;q) f\bigl(s,u(s),D_q u(s)\bigr)\,d_qs \\ &= \int_0^t (t-qs) f\bigl(s,u(s),D_q u(s)\bigr)\,d_qs \\ &\quad + \int_0^1 \left(\frac{1}{\alpha} - 1 + qs\right) f\bigl(s,u(s),D_q u(s)\bigr)\,d_qs, \quad t \in I, u \in C_q. \end{aligned} \tag{2.8}$$

Obviously, T is well defined and $u \in C_q$ is a solution of BVP (1.1) if and only if u is a fixed point of T.

3 Existence and uniqueness results

In this section, we apply various fixed point theorems to BVP (1.1). First, we give the uniqueness result based on Banach's contraction principle.

Theorem 3.1 *Let $f : I \times R^2 \to R$ be a continuous function, and there exists $L_1(t), L_2(t) \in C([0,1],[0,+\infty))$ such that*

$$\bigl|f(t,u_1,v_1) - f(t,u_2,v_2)\bigr| \le L_1(t)|u_1-u_2| + L_2(t)|v_1-v_2|, \quad t \in I, (u_1,v_1),(u_2,v_2) \in R^2.$$

In addition, suppose either

(H$_1$) $\Lambda < |\alpha|$ *for* $0 < |\alpha| < 1$, *or*
(H$_2$) $\Lambda < 1$ *for* $|\alpha| \ge 1$

holds, where $\Lambda = \max_{t\in[0,1]}\{L_1(t) + L_2(t)\}$. *Then BVP* (1.1) *has a unique solution.*

Proof Case 1: $|\alpha| < 1$. Let us set $\sup_{t\in I} |f(t,0,0)| = M_0$ and choose

$$r \geq \frac{M_0}{|\alpha|(1-\delta)}, \tag{3.1}$$

where δ is such that $\frac{\Lambda}{|\alpha|} \leq \delta \leq 1$. Now we show that $TB_r \subset B_r$, where $B_r = \{u \in C_q : \|u\| \leq r\}$. For each $u \in B_r$, we have

$$\begin{aligned}
\bigl|Tu(t)\bigr| &\leq \sup_{t\in I}\left|\int_0^t (t-qs)f\bigl(s,u(s),D_qu(s)\bigr)\,d_qs + \int_0^1 \left(\frac{1}{\alpha}-1+qs\right)f\bigl(s,u(s),D_qu(s)\bigr)\,d_qs\right| \\
&\leq \sup_{t\in I}\left|\int_0^t (t-qs)\bigl(\bigl|f\bigl(s,u(s),D_qu(s)\bigr)-f(s,0,0)\bigr| + \bigl|f(s,0,0)\bigr|\bigr)\,d_qs \right.\\
&\quad \left. + \int_0^1 \left(\frac{1}{\alpha}-1+qs\right)\bigl(\bigl|f\bigl(s,u(s),D_qu(s)\bigr)-f(s,0,0)\bigr| + \bigl|f(s,0,0)\bigr|\bigr)\,d_qs\right| \\
&\leq \sup_{t\in I}\left|\int_0^t (t-qs)\bigl(L_1(s)\bigl|u(s)\bigr| + L_2(s)\bigl|D_qu(s)\bigr| + \bigl|f(s,0,0)\bigr|\bigr)\,d_qs \right.\\
&\quad \left. + \int_0^1 \left(\frac{1}{\alpha}-1+qs\right)\bigl(L_1(s)\bigl|u(s)\bigr| + L_2(s)\bigl|D_qu(s)\bigr| + \bigl|f(s,0,0)\bigr|\bigr)\,d_qs\right| \\
&\leq \bigl(\Lambda\|u\| + M_0\bigr)\sup_{t\in I}\left|\int_0^t (t-qs)\,d_qs + \int_0^1 \left(\frac{1}{\alpha}-1+qs\right)d_qs\right| \\
&\leq \bigl(\Lambda\|u\| + M_0\bigr)\sup_{t\in I}\left\{\left|\frac{t^2}{1+q} + \frac{1}{\alpha} - 1 + \frac{q}{1+q}\right|\right\} \\
&\leq \bigl(\Lambda\|u\| + M_0\bigr)\frac{1}{|\alpha|} \leq (\Lambda r + M_0)\frac{1}{|\alpha|} \leq \left(\frac{\Lambda}{|\alpha|} + (1-\delta)\right)r \leq r,
\end{aligned}$$

and

$$\begin{aligned}
\bigl|D_qTu(t)\bigr| \leq \bigl|D_qTu(t)\bigr| &\leq \sup_{t\in I}\left|\int_0^t f\bigl(s,u(s),D_qu(s)\bigr)\,d_qs\right| \\
&\leq \sup_{t\in I}\int_0^t \bigl(\bigl|f\bigl(s,u(s),D_qu(s)\bigr)-f(s,0,0)\bigr| + \bigl|f(s,0,0)\bigr|\bigr)\,d_qs \\
&\leq \sup_{t\in I}\int_0^t \bigl(L_1(s)\bigl|u(s)\bigr| + L_2(s)\bigl|D_qu(s)\bigr| + \bigl|f(s,0,0)\bigr|\bigr)\,d_qs \\
&\leq \bigl(\Lambda\|u\| + M_0\bigr) \leq \bigl(\Lambda r + |\alpha|(1-\delta)r\bigr) \leq \left(\frac{\Lambda}{|\alpha|} + (1-\delta)\right)r \leq r.
\end{aligned}$$

Hence, we obtain that $\|Tu\| \leq r$, so $TB_r \subset B_r$.

Now, for $u, v \in C_q$ and for each $t \in I$, we have

$$\begin{aligned}
\bigl|Tu(t) - Tv(t)\bigr| &\leq \sup_{t\in I}\bigl|Tu(t) - Tv(t)\bigr| \\
&\leq \sup_{t\in I}\left|\int_0^t (t-qs)\bigl|f\bigl(s,u(s),D_qu(s)\bigr) - f\bigl(s,v(s),D_qv(s)\bigr)\bigr|\,d_qs \right.\\
&\quad \left. + \int_0^1 \left(\frac{1}{\alpha}-1+qs\right)\bigl|f\bigl(s,u(s),D_qu(s)\bigr) - f\bigl(s,v(s),D_qv(s)\bigr)\bigr|\,d_qs\right| \\
&\leq \sup_{t\in I}\left|\int_0^t (t-qs)\bigl(L_1(s)\bigl|u(s)-v(s)\bigr| + L_2(s)\bigl|D_qu(s) - D_qv(s)\bigr|\bigr)\,d_qs\right.
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^1\left(\frac{1}{\alpha}-1+qs\right)\left(L_1(s)\left|u(s)-v(s)\right|+L_2(s)\left|D_qu(s)-D_qv(s)\right|\right)d_qs\Bigg| \\
&\le \Lambda\sup_{t\in I}\left\{\left|\frac{t^2}{1+q}+\frac{1}{\alpha}-1+\frac{q}{1+q}\right|\right\}\|u-v\| \\
&\le \frac{\Lambda}{|\alpha|}\|u-v\|<\|u-v\|,
\end{aligned}
$$

and

$$
\begin{aligned}
\left|D_qTu(t)-D_qTv(t)\right| &\le \sup_{t\in I}\left|D_qTu(t)-D_qTv(t)\right| \\
&\le \sup_{t\in I}\left|\int_0^t\left|f\left(s,u(s),D_qu(s)\right)-f\left(s,v(s),D_qv(s)\right)\right|d_qs\right| \\
&\le \sup_{t\in I}\left|\int_0^t\left(L_1(s)\left|u(s)-v(s)\right|+L_2(s)\left|D_qu(s)-D_qv(s)\right|\right)d_qs\right| \\
&\le \Lambda\|u-v\| \le \frac{\Lambda}{|\alpha|}\|u-v\|<\|u-v\|.
\end{aligned}
$$

Therefore, we obtain that $\|Tu-Tv\|<\|u-v\|$, so T is a contraction. Thus, the conclusion of the theorem follows by Banach's contraction mapping principle.

Case 2: $|\alpha|\ge 1$. It is similar to the proof of case 1. This completes the proof of Theorem 3.1. □

Corollary 3.2 *Assume that* $f: I\times R^2\to R$ *is a continuous function and there exist two positive constants* L_1, L_2 *such that*

$$
\left|f(t,u_1,v_1)-f(t,u_2,v_2)\right|\le L_1|u_1-u_2|+L_2|v_1-v_2|,\quad t\in I,(u_1,v_1),(u_2,v_2)\in R^2.
$$

In addition, suppose either

(H_3) $L_1+L_2<|\alpha|$ *for* $0<|\alpha|<1$, *or*
(H_4) $L_1+L_2<1$ *for* $|\alpha|\ge 1$

holds. Then BVP (1.1) *has a unique solution.*

Corollary 3.3 *Assume that* $f: I\times R^2\to R$ *is a continuous function and there exist two functions* $L_1(t),L_2(t)\in L^1(I,R^+)$ *such that*

$$
\left|f(t,u_1,v_1)-f(t,u_2,v_2)\right|\le L_1|u_1-u_2|+L_2|v_1-v_2|,\quad t\in I,(u_1,v_1),(u_2,v_2)\in R^2.
$$

In addition, suppose either

(H_5) $A+Bq|\alpha|<|\alpha|$ *for* $0<|\alpha|<1$, *or*
(H_6) $A<1$ *for* $|\alpha|\ge 1$

holds, where

$$
A=\int_0^1\left[L_1(s)+L_2(s)\right]d_qs,\qquad B=\int_0^1 s\left[L_1(s)+L_2(s)\right]d_qs.
$$

Then BVP (1.1) *has a unique solution.*

Proof It is similar to the proof of Theorem 3.1. □

The next existence result is based on the Leray-Schauder nonlinear alternative theorem.

Lemma 3.4 *Let $f : I \times R^2 \to R$ be an S-Carathéodory function. Then $T : C_q \to C_q$ is completely continuous.*

Proof The proof consists of several steps.

(i) T maps bounded sets into bounded sets in C_q.

Let $B_r = \{u \in C_q : \|u\| \leq r\}$ be a bounded set in C_q and $u \in B_r$. Then we have

$$\begin{aligned} \left|Tu(t)\right| &\leq \int_0^t |t-qs|\left|f\left(s,u(s),D_qu(s)\right)\right| d_qs + \int_0^1 \left|\frac{1}{\alpha} - 1 + qs\right| \left|f\left(s,u(s),D_qu(s)\right)\right| d_qs \\ &\leq \left(1+\frac{1}{|\alpha|}\right)\int_0^1 \varphi_r(s)\, d_qs = \left(1+\frac{1}{|\alpha|}\right)\|\varphi_r\|_{L^1}, \end{aligned}$$

and

$$\left|D_qTu(t)\right| \leq \int_0^t \left|f\left(s,u(s),D_qu(s)\right)\right| d_qs \leq \int_0^1 \varphi_r(s)\, d_qs = \|\varphi_r\|_{L^1}.$$

Thus $\|Tu\| \leq \max\{\|Tu\|_\infty, \|D_qTu\|_\infty\} \leq (1+\frac{1}{|\alpha|})\|\varphi_r\|_{L^1}$.

(ii) T maps bounded sets into equicontinuous sets of C_q.

Let $r_1, r_2 \in I$, $r_1 < r_2$, and let B_r be a bounded set of C_q as before. Then, for $u \in B_r$, we have

$$\begin{aligned} \left|Tu(r_2) - Tu(r_1)\right| &= \left|\int_0^{r_2} (r_2 - qs) f\left(s,u(s),D_qu(s)\right) d_qs \right. \\ &\quad \left. - \int_0^{r_1} (r_1 - qs) f\left(s,u(s),D_qu(s)\right) d_qs\right| \\ &= \left|\int_0^{r_1} (r_2 - r_1) f\left(s,u(s),D_qu(s)\right) d_qs \right. \\ &\quad \left. + \int_{r_1}^{r_2} (r_2 - qs) f\left(s,u(s),D_qu(s)\right) d_qs\right| \\ &\leq \int_0^{r_1} |r_2 - r_1|\varphi_r(s)\, d_qs + \int_{r_1}^{r_2} |r_2 - qs|\varphi_r(s)\, d_qs \to 0 \\ &\quad (r_2 - r_1 \to 0), \end{aligned}$$

and

$$\begin{aligned} \left|d_qTu(r_2) - D_qTu(r_1)\right| &= \left|\int_0^{r_2} f\left(s,u(s),D_qu(s)\right) d_qs - \int_0^{r_1} f\left(s,u(s),D_qu(s)\right) d_qs\right| \\ &= \left|\int_{r_1}^{r_2} f\left(s,u(s),D_qu(s)\right) d_qs\right| \leq \int_{r_1}^{r_2} \varphi_r(s)\, d_qs \to 0 \\ &\quad (r_2 - r_1 \to 0). \end{aligned}$$

As a consequence of the Arzelá-Ascoli theorem, we can conclude that $T : C_q \to C_q$ is completely continuous. This proof is completed. □

Theorem 3.5 *Let $f : I \times R^2 \to R$ be an S-Carathéodory function. Suppose further that there exists a real number $M > 0$ such that*

$$\frac{|\alpha|M}{(1+|\alpha|)\|\varphi_r\|_{L^1}} > 1$$

holds, where

$$\|\varphi_r\|_{L^1} = \int_0^1 \varphi_r(s)\, d_q s \neq 0.$$

Then BVP (1.1) *has at least one solution.*

Proof In view of Lemma 3.4, we obtain that $T : C_q \to C_q$ is completely continuous. Let $\lambda \in (0,1)$ and $u = \lambda Tu$. Then, for $t \in I$, we have

$$\begin{aligned} \bigl|u(t)\bigr| &= \bigl|\lambda Tu(t)\bigr| \\ &\le \int_0^t |t-qs|\bigl|f\bigl(s,u(s),D_q u(s)\bigr)\bigr|\, d_q s \\ &\quad + \int_0^1 \left|\frac{1}{\alpha} - 1 + qs\right|\bigl|f\bigl(s,u(s),D_q u(s)\bigr)\bigr|\, d_q s \\ &\le \left(1+\frac{1}{|\alpha|}\right)\int_0^1 \varphi_r(s)\, d_q s = \left(1+\frac{1}{|\alpha|}\right)\|\varphi_r\|_{L^1}, \end{aligned}$$

and

$$\bigl|D_q u(t)\bigr| = \bigl|D_q \lambda Tu(t)\bigr| \le \int_0^t \bigl|f\bigl(s,u(s),D_q u(s)\bigr)\bigr|\, d_q s \le \int_0^1 \varphi_r(s)\, d_q s = \|\varphi_r\|_{L^1}.$$

Hence, consequently,

$$\frac{|\alpha|\|u\|}{(1+|\alpha|)\|\varphi_r\|_{L^1}} \le 1.$$

Therefore, there exists $M > 0$ such that $\|u\| \neq M$. Let us set $U = \{u \in C_q : \|u\| < M\}$. Note that the operator $T : \overline{U} \to C_q$ is completely continuous (which is known to be compact restricted to bounded sets). From the choice of U, there is no $U \in \partial U$ such that $u = \lambda Tu$ for some $\lambda \in (0,1)$. Consequently, by Theorem 2.6, we deduce that T has a fixed point $u \in \overline{U}$ which is a solution of problem (1.1). This completes the proof. □

The next existence result is based on the Leray-Schauder continuation theorem.

Theorem 3.6 *Let $f : I \times R^2 \to R$ be an S-Carathéodory function. Suppose further that there exist functions $p(t), q(t), r(t) \in L^1(I, R^+)$ with $tp(t) \in L^1(I, R^+)$ such that*

$$\bigl|f(t,u,v)\bigr| \le p(t)|u| + q(t)|v| + r(t), \quad \textit{for a.e. } t \in I \textit{ and } (u,v) \in R^2.$$

Then BVP (1.1) *has at least one solution provided* $(N+1)P+P_1+Q<1$, *where* $N=\max\{\frac{1}{\alpha},1-\frac{1}{\alpha}\}$.

Proof We consider the space

$$P=\{u\in C_q : D_qu(0)=0, D_qu(1)=\alpha u(1)\}$$

and define the operator $T_1 : P\times[0,1]\to P$ by

$$T_1(u,\lambda)=\lambda Tu=\lambda\int_0^1 G(t,s;q)f\bigl(s,u(s),D_qu(s)\bigr)\,d_qs,\quad t\in I. \tag{3.2}$$

Obviously, we can see that $P\subset C_q$. In view of Lemma 3.4, it is easy to know that for each $\lambda\in[0,1]$, $T_1(u,\lambda)$ is completely continuous in P. It is clear that $u\in P$ is a solution of BVP (1.1) if and only if u is a fixed point of $T_1(\cdot,1)$. Clearly, $T_1(u,0)=0$ for each $u\in P$. If for each $\lambda\in[0,1]$ the fixed points of $T_1(\cdot,1)$ in P belong to a closed ball of P independent of λ, then the Leray-Schauder continuation theorem completes the proof.

Next we show that the fixed point of $T_1(\cdot,1)$ has *a priori* bound M, which is independent of λ. Assume that $u=T_1(u,\lambda)$, and set

$$P=\int_0^1 p(s)\,d_qs,\qquad P_1=\int_0^1 sp(s)\,d_qs,$$
$$Q=\int_0^1 q(s)\,d_qs,\qquad R=\int_0^1 r(s)\,d_qs.$$

By (2.5), it is clear that $|G(t,s;q)|\le N$ for each $\alpha\neq 0$. For any $u\in P$, we have

$$\begin{aligned}\bigl|u(t)\bigr|&=\left|u(1)-\int_t^1 D_qu(s)\,d_qs\right|\le\left|\frac{1}{\alpha}D_qu(1)\right|+\left|\int_t^1 D_qu(s)\,d_qs\right|\\&\le(N+1-t)\|D_qu\|_\infty\le(N+1+t)\|D_qu\|_\infty,\quad t\in I,\end{aligned}$$

and so it holds that

$$\begin{aligned}\|D_qu\|_\infty&\le\bigl\|\lambda f\bigl(s,u(s),D_qu(s)\bigr)\bigr\|_{L^1}\le\bigl\|f\bigl(s,u(s),D_qu(s)\bigr)\bigr\|_{L^1}\\&\le\bigl\|p(t)\bigl|u(s)\bigr|+q(t)\bigl|D_qu(s)\bigr|+r(s)\bigr\|_{L^1}\\&\le\bigl((N+1)P+P_1+Q\bigr)\|D_qu\|_\infty+R;\end{aligned}$$

therefore,

$$\|D_qu\|_\infty\le\frac{R}{1-((N+1)P+P_1+Q)}:=M_1.$$

At the same time, we have

$$\begin{aligned}\bigl|u(t)\bigr|&\le\lambda\left|\int_0^1 G(t,s,q)f\bigl(s,u(s),D_qu(s)\bigr)\,d_qs\right|\\&\le N\int_0^1\bigl|f\bigl(s,u(s),D_qu(s)\bigr)\bigr|\,d_qs\end{aligned}$$

$$\leq N\int_0^1 \big(p(t)\big|u(s)\big| + q(t)\big|D_q u(s)\big| + r(s)\big)\, d_q s$$
$$\leq NP\|u\|_\infty + NQ\|D_q u\|_\infty + NR, \quad t \in I,$$

and so

$$\|u\|_\infty \leq \frac{N(QM_1 + R)}{1 - NP} := M_2.$$

Set $M = \max\{M_1, M_2\}$, which is independent of λ. So, BVP (1.1) has at least one solution. This completes the proof. □

4 Example

Example 4.1 Consider the following BVP:

$$\begin{cases} D_q^2 u(t) = e^t + \frac{1}{4}\sin(u(t)) + \frac{1}{8}\tan^{-1}(D_q u(t)), & t \in I, \\ D_q u(0) = 0, \quad D_q u(1) = \frac{1}{2}u(1). \end{cases} \tag{4.1}$$

Here, $f(t, u(t), D_q u(t)) = e^t + \frac{1}{4}\sin(u(t)) + \frac{1}{8}\tan^{-1}(D_q u(t))$, $q = \frac{1}{2}$, $\alpha = \frac{1}{2}$. Clearly, $|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq \frac{1}{4}|u_1 - u_2| + \frac{1}{8}|v_1 - v_2|$. Then $L_1 = \frac{1}{4}$, $L_2 = \frac{1}{8}$ and $L_1 + L_2 = \frac{3}{8} < \alpha$. By Corollary 3.2, we obtain that BVP (4.1) has a unique solution.

Example 4.2 Consider the following BVP:

$$\begin{cases} D_q^2 u(t) = \frac{t}{20}\sin(u(t)) + \frac{1}{(2+\sqrt{2})\sqrt{t}}(D_q u(t))^\beta + \frac{3}{2}t, & t \in I, \\ D_q u(0) = 0, \quad D_q u(1) = \frac{1}{2}u(1). \end{cases} \tag{4.2}$$

Here, $q = \frac{1}{2}$, $0 < \beta < 1$. It is obvious that $|f(t, u, v)| \leq \frac{t}{20}|u| + \frac{1}{(2+\sqrt{2})\sqrt{t}}|v| + \frac{3}{2}t$, where $p(t) = \frac{t}{20}$, $q(t) = \frac{1}{(2+\sqrt{2})\sqrt{t}}$, $r(t) = \frac{3}{2}t$. Then $P = \frac{1}{30}$, $P_1 = \frac{1}{35}$, $Q = \frac{1}{2}$, $R = 1$, so $(N + 1)P + P_1 + Q = \frac{22}{35} < 1$. By Theorem 3.6, we obtain that BVP (4.2) has at least one solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors, CLY and JFW contributed to each part of this work equally and read and approved the final version of the manuscript.

Acknowledgements
Research supported by the Natural Science Foundation of China (10901045), (11201112), the Natural Science Foundation of Hebei Province (A2009000664), (A2011208012) and the Foundation of Hebei University of Science and Technology (XL201047), (XL200757).

Received: 16 October 2012 Accepted: 15 April 2013 Published: 2 May 2013

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doi:10.1186/1687-1847-2013-124
Cite this article as: Yu and Wang: **Existence of solutions for nonlinear second-order q-difference equations with first-order q-derivatives.** *Advances in Difference Equations* 2013 **2013**:124.