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Strong convergence theorems for equilibrium problems and fixed point problems: A new iterative method, some comments and applications

Zhenhua He[1] and Wei-Shih Du[2*]

* Correspondence: wsdu@nknucc.nknu.edu.tw
[2]Department of Mathematics, National Kaohsiung Normal University, Kaohsiung 824, Taiwan
Full list of author information is available at the end of the article

Abstract

In this paper, we introduce a new approach method to find a common element in the intersection of the set of the solutions of a finite family of equilibrium problems and the set of fixed points of a nonexpansive mapping in a real Hilbert space. Under appropriate conditions, some strong convergence theorems are established. The results obtained in this paper are new, and a few examples illustrating these results are given. Finally, we point out that some 'so-called' mixed equilibrium problems and generalized equilibrium problems in the literature are still usual equilibrium problems.

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1 Introduction and preliminaries

Throughout this paper, we assume that H is a real Hilbert space with zero vector θ, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $|| \cdot ||$, respectively. The symbols $\mathbb{N}$ and $\mathbb{R}$ are used to denote the sets of positive integers and real numbers, respectively. Let K be a nonempty closed convex subset of H and $T : K \to H$ be a mapping. In this paper, the set of fixed points of T is denoted by $F(T)$. We use symbols $\to$ and $\rightharpoonup$ to denote strong and weak convergence, respectively.

For each point $x \in H$, there exists a unique nearest point in K, denoted by $P_K x$, such that

$$\| x - P_K x \| \leq \| x - y \|, \quad \forall\, y \in K.$$

The mapping P_K is called the *metric projection* from H onto K. It is well known that P_K satisfies

$$\langle x - y, P_K x - P_K y \rangle \geq \| P_K x - P_K y \|^2$$

for every $x, y \in H$. Moreover, $P_K x$ is characterized by the properties: for $x \in H$, and $z \in K$,

$$z = P_K(x) \Leftrightarrow \langle x - z, z - y \rangle \geq 0, \quad \forall y \in K.$$

Let f be a bi-function from $K \times K$ into $\mathbb{R}$. The classical equilibrium problem is to find $x \in K$ such that

$$f(x, y) \geq 0, \quad \forall y \in K. \tag{1.1}$$

Let $EP(f)$ denote the set of all solutions of the problem (1.1). Since several problems in physics, optimization, and economics reduce to find a solution of (1.1) (see, e.g., [1,2]), some authors had proposed some methods to find the solution of equilibrium problem (1.1); for instance, see [1-4]. We know that a mapping S is said to be nonexpansive mapping if for all $x, y \in K$, $||Sx - Sy|| \leq ||x - y||$. Recently, some authors used iterative method including composite iterative, CQ iterative, viscosity iterative etc. to find a common element in the intersection of $EP(f)$ and $F(S)$; see, e.g., [5-11].

Let I be an index set. For each $i \in I$, let f_i be a bi-function from $K \times K$ into $\mathbb{R}$. The system of equilibrium problem is to find $x \in K$ such that

$$f_i(x, y) \geq 0, \quad \forall y \in K \text{ and } \forall i \in I. \tag{1.2}$$

We know that $\bigcap_{i \in I} EP(f_i)$ is the set of all solutions of the system of equilibrium problem (1.2).

For each $i \in I$, if $f_i(x, y) = \langle A_i x, y - x \rangle$, where $A_i : K \to K$ is a nonlinear operator, then the problem (1.2) becomes the following system of variational inequality problem:

$$\text{Find an element } x \in K \text{ such that } \langle A_i x, y - x \rangle \geq 0, \quad \forall y \in K. \tag{1.3}$$

It is obvious that the problem (1.3) is a special case of the problem (1.2).

The following Lemmas are crucial to our main results.

Lemma 1.1 (Demicloseness principle [12]**)** *Let H be a real Hilbert space and K a closed convex subset of H. $S : K \to H$ is a nonexpansive mapping. Then the mapping I - S is demiclosed on K, where I is the identity mapping, i.e., $x_n \rightharpoonup x$ in K and $(I - S)x_n \to y$ implies that $x \in K$ and $(I - S)x = y$.*

Lemma 1.2 [13] *Let $\{x_n\}$and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in [0,1] with* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. *Suppose* $x_{n+1} = \beta_n y_n + (1 - \beta_n)x_n$ *for all integers* $n \geq 0$ *and* $\limsup_{n\to\infty}(||y_{n+1} - y_n|| - ||x_{n+1} - x_n||) \leq 0$, *then* $\lim_{n\to\infty} ||y_n - x_n|| = 0$.

Lemma 1.3 [5] *Let H be a real Hilbert space. Then the following hold.*

(a) $||x + y||^2 \leq ||y||^2 + 2\langle x, x + y\rangle$ *for all* $x, y \in H$;
(b) $||\alpha x + (1 - \alpha)y||^2 = \alpha||x||^2 + (1 - \alpha)\, ||y||^2 - \alpha(1 - \alpha)\, ||x - y||^2$ *for all* $x, y \in H$ *and* $\alpha \in \mathbb{R}$;
(c) $||x - y||^2 = ||x||^2 + ||y||^2 - 2\langle x, y\rangle$ *for all* $x, y \in H$.

Lemma 1.4. [14] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \lambda_n)a_n + \gamma_n, n \geq 0.$$

If

(i) $\lambda_n \in [0,1]$, $\sum_{n=0}^{\infty} \lambda_n = \infty$ *or, equivalently*, $\Pi_{n=0}^{\infty}(1-\lambda_n) = 0$;

(ii) $\limsup_{n\to\infty} \frac{\gamma_n}{\lambda_n} \le 0$ or $\sum_{n=0}^{\infty} |\gamma_n| < \infty$,

then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.5 [1] *Let K be a nonempty closed convex subset of H and F be a bi-function of K × K into* $\mathbb{R}$ *satisfying the following conditions.*

(A1) $F(x, x) = 0$ *for all* $x \in K$;

(A2) *F is monotone, that is,* $F(x, y) + F(y, x) \le 0$ *for all* $x, y \in K$;

(A3) *for each* $x, y, z \in K$,

$$\lim_{t\downarrow 0} F(tz + (1-t)x, y) \le F(x, y);$$

(A4) *for each* $x \in K$, $y \to F(x, y)$ *is convex and lower semi-continuous.Let* $r > 0$ *and* $x \in H$. *Then, there exists* $z \in K$ *such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \text{ for all } y \in K.$$

Lemma 1.6 [3] *Let K be a nonempty closed convex subset of H and let F be a bi-function of K × K into* R *satisfying (A1) - (A4). For* $r > 0$ *and* $x \in H$, *define a mapping* $T_r : H \to K$ *as follows:*

$$T_r(x) = \left\{ z \in K : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \ \forall\, y \in K \right\}$$

for all $x \in H$. *Then the following hold:*

(i) T_r *is single-valued;*

(ii) T_r *is firmly nonexpansive, that is, for any* $x, y \in H$,

$$\| T_r x - T_r y\|^2 \le \langle T_r x - T_r y, x - y\rangle;$$

(iii) $F(T_r) = EP(F)$;

(iv) $EP(F)$ *is closed and convex.*

2 Main results and their applications

Let $I = \{1, 2,..., k\}$ be a finite index set, where $k \in \mathbb{N}$. For each $i \in I$, let f_i be a bi-functions from $K \times K$ into $\mathbb{R}$ satisfying the conditions (A1)-(A4). Denote $T_{r_n}^i : H \to K$ by

$$T_{r_n}^i(x) = \left\{ z \in K : f_i(z, y) + \frac{1}{r_n}\langle y - z, z - x\rangle \ge 0, \ \forall\, y \in K \right\}.$$

For each $(i, n) \in I \times \mathbb{N}$, applying Lemmas 1.5 and 1.6, $T_{r_n}^i$ is a firmly nonexpansive single-valued mapping such that $F(T_{r_n}^i) = EP(f_i)$ is closed and convex. For each $i \in I$, let $u_n^i = T_{r_n}^i x_n$, $n \in \mathbb{N}$.

First, let us consider the following example.

Example A Let $f_i : [-1, 0]\times[-1,0] \to \mathbb{R}$ be defined by $f_i(x, y) = (1+x^{2i})(x - y)$, $i = 1, 2, 3$. It is easy to see that for any $i \in \{1, 2, 3\}$, $f_i(x, y)$ satisfies the conditions (A1)-(A4) and $\bigcap_{i=1}^{3} EP(f_i) = \{0\}$. Let $Sx = x^3$ and $gx = \frac{1}{2}x$, $\forall\ x \in [-1, 0]$ Then g is a $\frac{1}{2}$-contraction from K into itself and $S : K \to K$ is a nonexpansive mapping with $\left(\bigcap_{i=1}^{3} EP(f_i)\right)\bigcap F(S) = \{0\}$. Let $\lambda \in (0, 1)$, $\{r_n\} \subset [1, +\infty)$ and $\{\alpha_n\} \subset (0,1)$ satisfy the conditions (i) $\lim_{n\to\infty} \alpha_n = 0$, and (ii) $\sum_{n=1}^{\infty} \alpha_n = +\infty$, or equivalently, $\prod_{n=1}^{\infty}(1 - \alpha_n) = 0$; e.g., let $\lambda = \frac{1}{3}$, $\{\alpha_n\} \subset (0, 1)$ and $\{r_n\} \subset [1, +\infty)$ be given by

$$\alpha_n = \begin{cases} 0, & \text{if } n \text{ is even;} \\ \frac{1}{n}, & \text{if } n \text{ is odd.} \end{cases} \quad \text{and} \quad r_n = \begin{cases} 2, & \text{if } n \text{ is even;} \\ 2 - \frac{1}{n}, & \text{if } n \text{ is odd.} \end{cases}$$

Define a sequence $\{x_n\}$ by

$$\begin{cases} x_1 \in [-1, 0], \\ u_n^i = T_{r_n}^i x_n, \quad i = 1, 2, 3, \\ x_{n+1} = \alpha_n g(x_n) + (1 - \alpha_n)y_n, \\ y_n = (1 - \lambda)x_n + \lambda S z_n, \\ z_n = \dfrac{u_n^1 + u_n^2 + u_n^3}{3}, \quad \forall n \in \mathbb{N}. \end{cases} \tag{2.1}$$

Then the sequences $\{x_n\}$ and $\{u_n^i\}$, $i = 1, 2, 3$, defined by (2.1) all strongly converge to 0.

Proof

(a) By Lemmas 1.5 and 1.6, (2.1) is well defined.

(b) Let $K = [-1, 0]$. For each $i \in \{1, 2, 3\}$, define

$$L_i(y, z, v, r) = (z - y)\left[(1 + z^{2i}) - \frac{1}{r}(z - v)\right] \quad \forall y, z, v \in K, \forall r \geq 1.$$

We claim that for each $v \in K$ and any $i \in \{1, 2, 3\}$, there exists a unique $z = 0 \in K$ such that

$$(\mathcal{P}) \qquad L_i(y, z, v, r) \geq 0 \quad \forall y \in K, \forall r \geq 1$$

or, equivalently,

$$(1+z^{2i})(z-y) + \frac{1}{r}\langle y-z, z-v\rangle = (1+z^{2i})(z-y) + \frac{1}{r}(y-z)(z-v) \geq 0 \quad \forall y \in K, \forall r \geq 1.$$

Obviously, $z = 0$ is a solution of the problem $(\mathcal{P})$. On the other hand, there does not exist $z \in [-1, 0)$ such that $z - y \leq 0$ and $(1 + z^{2i}) - \frac{1}{r}(z - v) \leq 0$. So $z = 0$ is the unique solution of the problem $(\mathcal{P})$.

(c) We notice that (2.1) is equivalent with (2.2), where

$$\begin{cases} x_1 \in [-1, 0], \\ f_i(u_n^i, y) + \dfrac{i}{r_n}\langle y - u_n^i, u_n^i - x_n\rangle \geq 0, \ \ \forall\ y \in K, \forall i = 1, 2, 3, \\ x_{n+1} = \alpha_n g(x_n) + (1 - \alpha_n)y_n, \\ y_n = (1 - \lambda)x_n + \lambda S z_n, \\ z_n = \dfrac{u_n^1 + u_n^2 + u_n^3}{3}, \ \ n \in \mathbb{N}. \end{cases} \tag{2.2}$$

It is easy to see that $\{x_n\} \subset [-1, 0]$, so, by (b), $u_n^1 = u_n^2 = u_n^3 = 0$ for all $n \in \mathbb{N}$. We need to prove $x_n \to 0$ as $n \to \infty$. Since $z_n = 0$ for all $n \in \mathbb{N}$, we have $y_n = (1 - \lambda)x_n$ and

$$x_{n+1} = \alpha_n g(x_n) + (1-\alpha_n)y_n = \frac{1}{2}\alpha_n x_n + (1-\alpha_n)(1-\lambda)x_n = \left[\left(1 - \frac{1}{2}\alpha_n\right) - (1-\alpha_n)\lambda\right] x_n \quad (2.3)$$

for all $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, from (2.3), we have

$$|x_{n+1}| = \left[\left(1 - \frac{1}{2}\alpha_n\right) - (1-\alpha_n)\lambda\right] |x_n| \le \left(1 - \frac{1}{2}\alpha_n\right) |x_n| . \quad (2.4)$$

Hence $\{|x_n|\}$ is a strictly deceasing sequence and $|x_n| \ge 0$ for all $n \in \mathbb{N}$. So $\lim_{n\to\infty} |x_n|$ exists.

On the other hand, for any $n, m \in \mathbb{N}$ with $n > m$, using (2.4), we obtain

$$\begin{aligned} |x_{n+1}| &\le \left(1 - \frac{1}{2}\alpha_n\right) |x_n| \\ &\le \left(1 - \frac{1}{2}\alpha_n\right)\left(1 - \frac{1}{2}\alpha_{n-1}\right) |x_{n-1}| \\ &\le \cdots \le \prod_{j=m}^{n} \left(1 - \frac{1}{2}\alpha_j\right) |x_m| , \end{aligned}$$

which implies $\limsup_{n\to\infty} |x_n| \le 0 \le \liminf_{n\to\infty} |x_n|$. Therefore $\{x_n\}$ strongly converges to 0. □

In this paper, motivated by the preceding *Example A*, we introduce a new iterative algorithm for the problem of finding a common element in the set of solutions to the system of equilibrium problem and the set of fixed points of a nonexpansive mapping. The following new strong convergence theorem is established in the framework of a real Hilbert space H.

Theorem 2.1 *Let K be a nonempty closed convex subset of a real Hilbert space H and $I = \{1, 2,..., k\}$ be a finite index set. For each $i \in I$, let f_i be a bi-function from $K \times K$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $S : K \to K$ be a nonexpansive mapping with $\Omega = \left(\bigcap_{i=1}^k EP(f_i)\right) \bigcap F(S) \neq \emptyset$. Let $\lambda, \rho \in (0, 1)$ and $g : K \to K$ is a ρ-contraction. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_1 \in K, \\ u_n^i = T_{r_n}^i x_n, \qquad \forall i \in I. \\ x_{n+1} = \alpha_n g(x_n) + (1-\alpha_n)y_n, \\ y_n = (1-\lambda)x_n + \lambda S z_n, \\ z_n = \dfrac{u_n^1 + \cdots + u_n^k}{k}, \quad \forall n \in \mathbb{N}. \end{cases} \qquad (D_H)$$

If the above control coefficient sequences $\{\alpha_n\} \subset (0, 1)$ and $\{r_n\} \subset (0, +\infty)$ satisfy the following restrictions:

(D1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = +\infty$ and $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n| = 0$;

(D2) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

then the sequences $\{x_n\}$ and $\{u_n^i\}$, for all $i \in I$, converge strongly to an element $c = P_\Omega g(c) \in \Omega$. The following conclusion is immediately drawn from Theorem 2.1.

Corollary 2.1 *Let K be a nonempty closed convex subset of a real Hilbert space H. Let f be a bi-function from $K \times K$ into $\mathbb{R}$ satisfying (A1)-(A4) and $S : K \to K$ be a nonexpansive mapping with $\Omega = EP(f) \cap F(S) \neq \emptyset$. Let $\lambda, \rho \in (0,1)$ and $g : K \to K$ is a ρ-contraction. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_1 \in K, \\ u_n = T_{r_n} x_n, \\ x_{n+1} = \alpha_n g(x_n) + (1-\alpha_n) y_n, \\ y_n = (1-\lambda) x_n + \lambda S u_n, \qquad \forall n \in \mathbb{N}. \end{cases}$$

If the above control coefficient sequences $\{\alpha_n\} \subset (0, 1)$ and $\{r_n\} \subset (0, +\infty)$ satisfy all the restrictions in Theorem 2.1, then the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to an element $c = P_\Omega g(c) \in \Omega$, respectively.

If $f_i(x, y) \equiv 0$ for all $(x, y) \in K \times K$ in Theorem 2.1 and all $i \in I$, then, from the algorithm (D_H), we obtain $u_n^i \equiv P_K(x_n)$, $\forall\, i \in I$. So we have the following result.

Corollary 2.2 *Let K be a nonempty closed convex subset of a real Hilbert space H. Let $S : K \to K$ be a nonexpansive mapping with $F(S) \neq \emptyset$. Let $\lambda, \rho \in (0, 1)$ and $g : K \to K$ is a ρ-contraction. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_1 \in K, \\ x_{n+1} = \alpha_n g(x_n) + (1-\alpha_n) y_n, \\ y_n = (1-\lambda) x_n + \lambda S P_K(x_n), \qquad \forall n \in \mathbb{N}. \end{cases}$$

If the above control coefficient sequences $\{\alpha_n\} \subset (0, 1)$ satisfy $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n| = 0$ and $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n| = 0$, then the sequences $\{x_n\}$ converge strongly to an element $c = P_\Omega g(c) \in F(S)$.

As some interesting and important applications of Theorem 2.1 for optimization problems and fixed point problems, we have the following.

Application (I) of Theorem 2.1 We will give an iterative algorithm for the following optimization problem with a nonempty common solution set:

$$\min_{x\in K} h_i(x), \quad i \in \{1, 2, \dots, k\}, \qquad (OP)$$

where $h_i(x)$, $i \in \{1, 2,..., k\}$, are convex and lower semi-continuous functions defined on a closed convex subset K of a Hilbert space H (for example, $h_i(x) = x^i$, $x \in K := [0, 1]$, $i \in \{1, 2,..., k\}$).

If we put $f_i(x, y) = h_i(y) - h_i(x)$, $i \in \{1, 2,..., k\}$, then $\bigcap_{i=1}^k EP(f_i)$ is the common solution set of the problem (OP), where $\bigcap_{i=1}^k EP(f_i)$ denote the common solution set of the following equilibrium:

$$\text{Find } x \in K \quad \text{such that } f_i(x, y) \geq 0, \quad \forall\, y \in K \quad \text{and } \forall\, i \in \{1, 2, \dots, k\}.$$

For $i \in \{1, 2,..., k\}$, it is obvious that the $f_i(x, y)$ satisfies the conditions (A1)-(A4). Let $S = I$ (identity mapping), then from (D_H), we have the following algorithm

$$\begin{cases} h_i(y) - h_i(u_n^i) + \dfrac{1}{r_n}\langle y - u_n^i, u_n^i - x_n\rangle \geq 0, \ \ \forall\, y \in K \ \text{ and } \forall\, i \in \{1, 2, \dots, k\}, \\ x_{n+1} = \alpha_n g(x_n) + (1-\alpha_n) y_n, \\ y_n = (1-\lambda) x_n + \lambda z_n, \\ z_n = \dfrac{u_n^1 + \cdots + u_n^k}{k}, \ n \geq 1. \end{cases} \tag{2.5}$$

where $x_1 \in K, \lambda \in (0, 1), g : K \to K$ is a ρ-contraction. From Theorem 2.1, we know that $\{x_n\}$ and $\{u_n^i\}$, $i \in \{1,2,..., k\}$, generated by (2.5), strongly converge to an element of $\bigcap_{i=1}^k EP(f_i)$ if the coefficients $\{\alpha_n\}$ and $\{r_n\}$ satisfy the conditions of Theorem 2.1.

Application (II) of Theorem 2.1 Let $H, K, I, \lambda, \rho, g$ be the same as Theorem 2.1. Let $A_1, A_2,..., A_k : K \to K$ be k nonlinear mappings with $\bigcap_{i=1}^{k} F(A_i) \neq \emptyset$. For any $i \in I$, put $f_i(x, y) = \langle x - A_i x, y - x\rangle, \forall x, y \in K$. Since $\bigcap_{i=1}^k EP(f_i) = \bigcap_{i=1}^k F(A_i)$, we have $\bigcap_{i=1}^{k} EP(f_i) \neq \emptyset$. Let $S = I$ (identity mapping) in the algorithm (D_H). Then the sequences $\{x_n\}$ and $\{u_n^i\}$, defined by the algorithm (D_H), converge strongly to a common fixed point of $\{A_1, A_2,..., A_k\}$, respectively.

The following result is important in this paper.

Lemma 2.1 *Let H be a real Hilbert space. Then for any $x_1, x_2,... x_k \in H$ and $a_1, a_2,..., a_k \in [0,1]$ with $\sum_{i=1}^k a_i = 1$, $k \in \mathbb{N}$, we have*

$$\left\| \sum_{i=1}^{k} a_i x_i \right\|^2 = \sum_{i=1}^{k} a_i \| x_i\|^2 - \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} a_i a_j \| x_i - x_j\|^2. \tag{2.6}$$

Proof It is obvious that (2.6) is true if $a_j = 1$ for some j, so it suffices to show that (2.6) is true for $a_j \neq 1$ for all j. The proof is by mathematic induction on k. Clearly, (2.6) is true for $k = 1$. Let $x_1, x_2 \in H$ and $a_1, a_2 \in [0,1]$ with $a_1 + a_2 = 1$. By Lemma 1.3, we obtain

$$\| a_1x_1 + a_2x_2\|^2 = a_1 \| x_1\|^2 + a_2 \| x_2\|^2 - a_1a_2 \| x_1 - x_2\|^2,$$

which means that (2.6) hold for $k = 2$. Suppose that (2.6) is true for $k = l \in \mathbb{N}$. Let $x_1, x_2,..., x_l, x_{l+1} \in H$ and $a_1, a_2,..., a_l, a_{l+1} \in [0, 1)$ with $\sum_{i=1}^{l+1} a_i = 1$. Let $y = \sum_{i=2}^{l+1} \frac{a_i}{1-a_1} x_i$. Then applying the induction hypothesis we have

$$\begin{aligned}
\left\| \sum_{i=1}^{l+1} a_i x_i \right\|^2 &= \| a_1x_1 + (1-a_1)y\|^2 \\
&= a_1 \| x_1\|^2 + (1-a_1) \| y\|^2 - a_1(1-a_1) \| x_1 - y\|^2 \\
&= \sum_{i=1}^{l+1} a_i \| x_i\|^2 - \frac{1}{1-a_1} \sum_{i=2}^{l} \sum_{j=i+1}^{l+1} a_i a_j \| x_i - x_j\|^2 \\
&\quad - a_1(1-a_1) \left\| \sum_{i=2}^{l+1} \frac{a_i}{1-a_1} (x_i - x_1) \right\|^2 \\
&= \sum_{i=1}^{l+1} a_i \| x_i\|^2 - \frac{1}{1-a_1} \sum_{i=2}^{l} \sum_{j=i+1}^{l+1} a_i a_j \| x_i - x_j\|^2 - a_1(1-a_1) \sum_{i=2}^{l+1} \frac{a_i}{1-a_1} \| x_1 - x_i\|^2 \\
&\quad + a_1(1-a_1) \sum_{i=2}^{l} \sum_{j=i+1}^{l+1} \frac{a_i}{1-a_1} \frac{a_j}{1-a_1} \| x_i - x_j\|^2 \\
&= \sum_{i=1}^{l+1} a_i \| x_i\|^2 - \frac{1}{1-a_1} \sum_{i=2}^{l} \sum_{j=i+1}^{l+1} a_i a_j \| x_i - x_j\|^2 \\
&\quad - \sum_{i=2}^{l+1} a_1 a_i \| x_1 - x_i\|^2 + \frac{a_1}{1-a_1} \sum_{i=2}^{l} \sum_{j=i+1}^{l+1} a_i a_j \| x_i - x_j\|^2 \\
&= \sum_{i=1}^{l+1} a_i \| x_i\|^2 - \sum_{i=2}^{l+1} a_1 a_i \| x_1 - x_i\|^2 - \sum_{i=2}^{l} \sum_{j=i+1}^{l+1} a_i a_j \| x_i - x_j\|^2 \\
&= \sum_{i=1}^{l+1} a_i \| x_i\|^2 - \sum_{i=1}^{l} \sum_{j=i+1}^{l+1} a_i a_j \| x_i - x_j\|^2.
\end{aligned}$$

Hence, the equality (2.6) is also true for $k = l + 1$. This completes the induction. □

3 Proof of Theorem 2.1

We will proceed with the following steps.

Step 1: There exists a unique $c \in \Omega \subset H$ such that $P_\Omega g(c) = c$.

Since $P_\Omega g$ is a ρ-contraction on H, Banach contraction principle ensures that there exists a unique $c \in H$ such that $c = P_\Omega g(c) \in \Omega$.

Step 2: We prove that the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{u_n^i\}$, $\forall i \in I$, are all bounded.

First, we notice that (D_H) is equivalent with (Z_H), where

$$\begin{cases} x_1 \in K \\ f_1(u_n^1, y) + \dfrac{1}{r_n}\langle y - u_n^1, u_n^1 - x_n\rangle \geq 0, \quad \forall\, y \in K, \\ f_2(u_n^2, y) + \dfrac{1}{r_n}\langle y - u_n^2, u_n^2 - x_n\rangle \geq 0, \quad \forall\, y \in K, \\ \vdots \\ f_k(u_n^k, y) + \dfrac{1}{r_n}\langle y - u_n^k, u_n^k - x_n\rangle \geq 0, \quad \forall\, y \in K, \\ x_{n+1} = \alpha_n g(x_n) + (1-\alpha_n)y_n, \\ y_n = (1-\lambda)x_n + \lambda S z_n, \\ z_n = \dfrac{u_n^1 + \cdots + u_n^k}{k}, \ n \in \mathbb{N}. \end{cases} \qquad (Z_H)$$

For each $i \in I$, we have

$$||u_n^i - c|| \ = \ ||T_{r_n}^i x_n - T_{r_n}^i c|| \ \leq \ ||x_n - c||, \ \forall\, n \in \mathbb{N}. \tag{3.1}$$

For any $n \in \mathbb{N}$, from (Z_H) we have

$$\| z_n - c \| \leq \| x_n - c \|$$

and

$$\| y_n - c \| \leq \| x_n - c \| \ . \tag{3.2}$$

Since g is a ρ-contraction, it follows from (3.2) that

$$\begin{aligned} \| x_{n+1} - c \| &\leq \alpha_n \left\|g(x_n) - c\right\| + (1-\alpha_n)\left\|y_n - c\right\| \\ &\leq \alpha_n \left\|g(x_n) - g(c)\right\| + \alpha_n \left\|g(c) - c\right\| + (1-\alpha_n)\left\|y_n - c\right\| \\ &\leq \alpha_n \rho \left\|x_n - c\right\| + \alpha_n \left\|g(c) - c\right\| + (1-\alpha_n)\left\|x_n - c\right\| \\ &= \left[1 - \alpha_n(1-\rho)\right]\left\|x_n - c\right\| + \alpha_n(1-\rho)\frac{\left\|g(c) - c\right\|}{1-\rho} \\ &\leq \max\left\{\| x_n - c \|, \frac{\| g(c) - c \|}{1-\rho}\right\}, \qquad \text{for } n \in \mathbb{N}. \end{aligned}$$

By induction, we obtain

$$\| x_n - c \| \leq \max\left\{\| x_1 - c \|, \frac{\| g(c) - c \|}{1-\rho}\right\} \text{for all } n \in \mathbb{N},$$

which shows that $\{x_n\}$ is bounded. Also, we know that $\{y_n\}$, $\{z_n\}$ and $\{u_n^i\}$, $\forall i \in I$, are all

bounded.

Step 3: We prove $\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0$.

For each $i \in I$, since $u_{n-1}^i, u_n^i \in K$, from (Z_H), we have

$$f_i(u_n^i, u_{n-1}^i) + \frac{1}{r_n}\langle u_{n-1}^i - u_n^i, u_n^i - x_n\rangle \geq 0, \tag{3.3}$$

and

$$f_i(u_{n-1}^i, u_n^i) + \frac{1}{r_{n-1}}\langle u_n^i - u_{n-1}^i, u_{n-1}^i - x_{n-1}\rangle \geq 0. \tag{3.4}$$

By (3.3) and (3.4) and (A2),

$$\begin{aligned} 0 &\leq r_n\left[f_i(u_n^i, u_{n-1}^i) + f_i(u_{n-1}^i, u_n^i)\right] + \langle u_{n-1}^i - u_n^i, u_n^i - x_n - \frac{r_n}{r_{n-1}}(u_{n-1}^i - x_{n-1})\rangle \\ &\leq \langle u_{n-1}^i - u_n^i, u_n^i - x_n - \frac{r_n}{r_{n-1}}(u_{n-1}^i - x_{n-1})\rangle, \end{aligned}$$

which implies

$$\langle u_{n-1}^i - u_n^i, u_{n-1}^i - u_n^i + x_n - x_{n-1} + x_{n-1} - u_{n-1}^i + \frac{r_n}{r_{n-1}}(u_{n-1}^i - x_{n-1})\rangle \leq 0. \tag{3.5}$$

It follows from (3.5) that

$$\| u_n^i - u_{n-1}^i \| \leq \| x_n - x_{n-1} \| + \left|\frac{r_n - r_{n-1}}{r_{n-1}}\right| \| x_{n-1} - u_{n-1}^i \| \quad \text{for all } n \in \mathbb{N}. \tag{3.6}$$

Let $M := \frac{1}{k}\sum_{i=1}^k \| x_{n-1} - u_{n-1}^i \| < \infty$. For any $n \in \mathbb{N}$, since $z_n = \frac{1}{k}(u_n^1 + \cdots + u_n^k)$, by (3.6), we have

$$\| z_n - z_{n-1} \| \leq \frac{1}{k}\sum_{i=1}^k \| u_n^i - u_{n-1}^i \| \leq \| x_n - x_{n-1} \| + M\left|\frac{r_n - r_{n-1}}{r_{n-1}}\right|. \tag{3.7}$$

Set

$$v_n = \frac{x_{n+1} - (1-\beta_n)x_n}{\beta_n}, \tag{3.8}$$

where $\beta_n = 1 - (1 - \lambda)(1 - \alpha_n)$, $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$,

$$x_{n+1} - x_n = \beta_n(v_n - x_n) \tag{3.9}$$

and

$$v_n = \frac{\alpha_n g(x_n) + \lambda(1-\alpha_n)Sz_n}{\beta_n}. \tag{3.10}$$

For any $n \in \mathbb{N}$, since

$$\begin{aligned} v_{n+1} - v_n &= \frac{\alpha_{n+1}g(x_{n+1})}{\beta_{n+1}} - \frac{\alpha_n g(x_n)}{\beta_n} - \frac{\lambda(1-\alpha_n)Sz_n}{\beta_n} + \frac{\lambda(1-\alpha_{n+1})Sz_{n+1}}{\beta_{n+1}} \\ &= \frac{\alpha_{n+1}g(x_{n+1})}{\beta_{n+1}} - \frac{\alpha_n g(x_n)}{\beta_n} - \frac{\lambda(1-\alpha_n)(Sz_n - Sz_{n+1})}{\beta_n} - \lambda(\frac{1-\alpha_n}{\beta_n} - \frac{1-\alpha_{n+1}}{\beta_{n+1}})Sz_{n+1}, \end{aligned}$$

by (3.7), it follows that

$$\begin{aligned}\| v_{n+1}-v_n \| - \| x_{n+1}-x_n \| \le\ & \frac{\alpha_{n+1} \| g(x_{n+1}) \|}{\beta_{n+1}} + \frac{\alpha_n \| g(x_n) \|}{\beta_n} + \frac{\lambda(1-\alpha_n) \| z_n - z_{n+1} \|}{\beta_n} \\ & + \left| \frac{1-\alpha_n}{\beta_n} - \frac{1-\alpha_{n+1}}{\beta_{n+1}} \right| \| Sz_{n+1} \| - \| x_{n+1}-x_n \| \\ \le\ & \frac{\alpha_{n+1} \| g(x_{n+1}) \|}{\beta_{n+1}} + \frac{\alpha_n \| g(x_n) \|}{\beta_n} + \left[\frac{\lambda(1-\alpha_n)}{\beta_n} - 1 \right] \| x_{n+1}-x_n \| \\ & + \frac{M}{\beta_n} \left| \frac{r_{n+1}-r_n}{r_n} \right| + \left| \frac{1-\alpha_n}{\beta_n} - \frac{1-\alpha_{n+1}}{\beta_{n+1}} \right| \| Sz_{n+1} \| .\end{aligned}$$

From this and (D1), (D2), we get

$$\limsup_{n\to\infty} \{ \| v_{n+1}-v_n \| - \| x_{n+1}-x_n \| \} \le 0. \tag{3.11}$$

By Lemma 1.2 and (3.11),

$$\lim_{n\to\infty} \| v_n - x_n \| = 0. \tag{3.12}$$

Owing to (3.9) and (3.12), we obtain

$$\lim_{n\to\infty} \| x_{n+1} - x_n \| = 0. \tag{3.13}$$

Step 4: We show $\lim_{n\to\infty} \| Su_n^i - u_n^i \| = 0$.
By (3.6), (3.13) and (D2), we have

$$\lim_{n\to\infty} \| u_{n+1}^i - u_n^i \| = 0, \quad \forall i \in I.$$

From (Z_H), we get

$$\lim_{n\to\infty} \| x_{n+1} - y_n \| = \lim_{n\to\infty} \alpha_n \| g(x_n) - y_n \| = 0. \tag{3.14}$$

Since $||x_n - y_n|| \le ||x_n - x_{n+1}|| + ||x_{n+1} - y_n||$, by (3.13) and (3.14),

$$\lim_{n\to\infty} \| y_n - x_n \| = 0,$$

which implies that

$$\lim_{n\to\infty} \| Sz_n - x_n \| = \lim_{n\to\infty} \frac{1}{\lambda} \| y_n - x_n \| = 0.$$

By Lemma 1.6,

$$\| u_n^i - c \|^2 = \| T_{r_n}^i x_n - T_{r_n}^i c \|^2 \le \langle T_{r_n}^i x_n - T_{r_n}^i c, x_n - c \rangle = \frac{1}{2} \left\{ \| u_n^i - c \|^2 + \| x_n - c \|^2 - \| u_n^i - x_n \|^2 \right\},$$

which yields that

$$\| u_n^i - c \|^2 \le \| x_n - c \|^2 - \| u_n^i - x_n \|^2. \tag{3.15}$$

From (3.15) and Lemma 2.1,

$$\| z_n - c \|^2 = \left\| \sum_{i=1}^k \frac{1}{k} \left(u_n^i - c \right) \right\|^2 \le \frac{1}{k} \sum_{i=1}^k \| u_n^i - c \|^2 \le \| x_n - c \|^2 - \frac{1}{k} \sum_{i=1}^k \| u_n^i - x_n \|^2.$$

Since

$$\begin{aligned}\| x_{n+1} - c\|^2 &\le \alpha_n \| g(x_n) - c\|^2 + (1-\alpha_n) \| y_n - c\|^2\\ &\le \alpha_n \| x_n - c\|^2 + 2\alpha_n \mathcal{L} + (1-\alpha_n) \| y_n - c\|^2\\ &\le [1-\lambda(1-\alpha_n)] \| x_n - c\|^2 + 2\alpha_n \mathcal{L} + \lambda(1-\alpha_n) \| z_n - c\|^2\end{aligned}$$

where

$$\mathcal{L} = \max\{2 \| g(c) - c \| \| x_n - c \|, \| g(c) - c\|^2\} < \infty,$$

We have

$$\frac{1-\alpha_n}{k}\lambda \sum_{i=1}^{k} \| u_n^i - x_n\|^2 \le \| x_n - c\|^2 - \| x_{n+1} - c\|^2 + 2\alpha_n\mathcal{L} \le (\| x_n - c \| + \| x_{n+1} - c \|) \| x_n - x_{n+1} \| + 2\alpha_n \mathcal{L} \quad (3.16)$$

Letting $n \to \infty$ in the inequality (3.16), we obtain

$$\lim_{n\to\infty} \| u_n^i - x_n \| = 0, \quad \forall i \in I. \quad (3.17)$$

Furthermore, it is easy to prove that

$$\lim_{n\to\infty} \| z_n - x_n \| = \lim_{n\to\infty} \| u_n^i - z_n \| = 0 \quad \forall i \in I.$$

For any $i \in I$, since

$$\| Su_n^i - u_n^i \| \le \| Su_n^i - Sz_n \| + \| Sz_n - x_n \| + \| x_n - u_n^i \| ,$$

it implies

$$\lim_{n\to\infty} \| Su_n^i - u_n^i \| = 0. \quad (3.18)$$

Step 5: Prove $\limsup_{n\to\infty} \langle g(c) - q, x_n - c\rangle \le 0$.

Take a subsequence $\{x_{n_\ell}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle g(c) - c, x_n - c\rangle = \lim_{\ell\to\infty} \langle g(c) - c, x_{n_\ell} - c\rangle. \quad (3.19)$$

Since $\{x_{n_\ell}\}$ is bounded, there exists a subsequence of $\{x_{n_\ell}\}$ which is still denoted by $\{x_{n_\ell}\}$ such that $x_{n_\ell} \rightharpoonup z$ as $\ell \to \infty$. Notice that for each $i \in I$, $\lim_{\ell\to\infty} \| u_{n_\ell}^i - x_{n_\ell} \| = 0$ by (3.17), so we also have $u_{n_\ell}^i \rightharpoonup z$ as $\ell \to \infty$, $\forall\ i \in I$.

We want to show $z \in \Omega$. First, we show that $z \in F(S)$. In fact, since $\lim_{\ell\to\infty} \| (I-S)u_{n_\ell}^i \| = \lim_{\ell\to\infty} \| Su_{n_\ell}^i - u_{n_\ell}^i \| = 0$ and $u_{n_\ell}^i \rightharpoonup z$ as $\ell \to \infty$, by Lemma 1.1, we have $(I - S)z = \theta$ or, equivalently, $z \in F(S)$.

For each $i \in I$, since $f_i(u_{n_\ell}^i, y) + \frac{1}{r_{n_\ell}} \langle y - u_{n_\ell}^i, u_{n_\ell}^i - x_{n_\ell}\rangle \ge 0$, $\forall\ y \in K$, it follows from (A2) that

$$\frac{1}{r_{n_\ell}} \langle y - u_{n_\ell}^i, u_{n_\ell}^i - x_{n_\ell}\rangle \ge f_i(y, u_{n_\ell}^i) + f_i(u_{n_\ell}^i, y) + \frac{1}{r_{n_\ell}} \langle y - u_{n_\ell}^i, u_{n_\ell}^i - x_{n_\ell}\rangle \ge f_i(y, u_{n_\ell}^i),$$

and hence

$$\langle y - u_{n_\ell}^i, \frac{u_{n_\ell}^i - x_{n_\ell}}{r_{n_\ell}}\rangle \ge f_i(y, u_{n_\ell}^i), \quad \forall y \in K.$$

Applying (3.17) and (A4),

$$f_i(y, z) \leq 0, \quad \forall y \in K. \tag{3.20}$$

Let $y \in K$ be given. Put $y_t = ty + (1 - t)z$, $t \in (0, 1)$. Then $y_t \in K$ and $f_i(y_t, z) \leq 0$ for all $i \in I$. By (A1) and (A4), we get

$$0 = f_i(y_t, y_t) \leq tf_i(y_t, y) + (1 - t)f_i(y_t, z) \leq tf_i(y_t, y) \quad \forall i \in I.$$

For any $i \in I$, by (A3), we have

$$f_i(z, y) \geq \lim_{t \downarrow 0} f_i(ty + (1 - t)z, y) = \lim_{t \downarrow 0} f_i(y_t, y) \geq 0. \tag{3.21}$$

Hence, from (3.21), $z \in \bigcap_{i=1}^{k} EP(f_i)$. Therefore, we proved $z \in \Omega = (\bigcap_{i=1}^{k} EP(f_i)) \bigcap F(S)$. On the other hand, by (3.19), we obtain

$$\limsup_{n \to \infty} \langle g(c) - c, x_n - c \rangle = \langle g(c) - c, z - c \rangle \leq 0. \tag{3.22}$$

Step 6: Finally, we prove $\{x_n\}$ and $\{u_n^i\}$, for all $i \in I$, converge strongly to $c = P_\Omega g(c) \in \Omega$.

From (Z_H) and (a) of Lemma 1.3, we have

$$\begin{aligned} ||x_{n+1} - c||^2 &\leq (1 - \alpha_n)^2 ||y_n - c||^2 + 2\alpha_n \langle g(x_n) - g(c) + g(c) - c, x_{n+1} - c \rangle \\ &\leq (1 - \alpha_n)^2 ||x_n - c||^2 + 2\alpha_n \rho ||x_n - c|| \, ||x_{n+1} - c|| + 2\alpha_n \langle g(c) - c, x_{n+1} - c \rangle \\ &\leq (1 - 2\alpha_n + \alpha_n^2) ||x_n - c||^2 + 2\alpha_n \rho ||x_n - c|| \, ||x_n - x_{n+1}|| + 2\alpha_n \rho ||x_n - c||^2 \\ &\quad + 2\alpha_n \langle g(c) - c, x_{n+1} - c \rangle \\ &= (1 - 2(1 - \rho)\alpha_n) ||x_n - c||^2 + \alpha_n^2 ||x_n - c||^2 + 2\alpha_n \rho ||x_n - c|| \, ||x_n - x_{n+1}|| \\ &\quad + 2\alpha_n \langle g(c) - c, x_{n+1} - c \rangle \\ &= (1 - 2(1 - \rho)\alpha_n) ||x_n - c||^2 + \alpha_n^2 ||x_n - c||^2 + 2\alpha_n \rho ||x_n - c|| \, ||x_n - x_{n+1}|| \\ &\quad + 2\alpha_n \langle g(c) - c, x_{n+1} - c \rangle. \end{aligned} \tag{3.23}$$

For any $n \in \mathbb{Z}$, let

$$a_n = ||x_n - c||^2,$$
$$b_n = \alpha_n ||x_n - c||^2 + 2\rho ||x_n - c|| \, ||x_n - x_{n+1}|| + 2\langle g(c) - c, x_{n+1} - c \rangle,$$
$$\lambda_n = 2(1 - \rho)\alpha_n,$$

and

$$\gamma_n = \alpha_n b_n.$$

From (3.23), we have

$$a_{n+1} \leq (1 - \lambda_n) a_n + \gamma_n, \quad \forall n \in \mathbb{N}.$$

It is easy to verify that all conditions of Lemma 1.4 are satisfied. Hence, applying Lemma 1.4, we obtain $\lim_{n \to \infty} a_n = 0$ which implies

$$\lim_{n \to \infty} ||x_n - c|| = 0,$$

or equivalence, $\{x_n\}$ strongly converges to c. By (3.17), we can prove that for any $i \in I$, $\{u_n^i\}$ strongly converges to c. The proof of Theorem 2.1 is completed. □

4 Further remarks

Let K be a nonempty closed convex subset of H and f be a bi-function of $K \times K$ into $\mathbb{R}$.

Remark 4.1 Recently, some authors introduced the following mixed equilibrium problem (MEP, for short) (see [15-17] and references therein) and generalized equilibrium problem (GEP, for short) (see [18-20] and references therein):

(a) Mixed equilibrium problem [15-17]:

$$\text{Find an element } x \in C \text{ such that } f(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \qquad \text{(MEP)}$$

where $\phi : C \to \mathbb{R}$ is a real-valued function.

(b) Generalized equilibrium problem [18-20]:

$$\text{Find an element } x \in C \text{ such that } f(x, y) + (Ax, y - x) \geq 0, \quad \forall y \in C. \qquad \text{(GEP)}$$

where $A : C \to H$ is a nonlinear operator.

In [15-17], the authors gave some iterative methods for finding the solution of MEP when the bi-function $f(x, y)$ admits the conditions (A1)-(A4) and the real-valued function ϕ satisfies the following condition:

(A5) $\phi : C \to \mathbb{R}$ is a proper lower semi-continuous and convex function.

However, in this case, we argue that the problem MEP is still the equilibrium problem (1.1). In fact, if we put $f_1(x, y) = f(x, y)$, $f_2(x, y) = \phi(y) - \phi(x)$ and $F(x, y) = f_1(x, y) + f_2(x, y)$ for each $(x, y) \in C \times C$, then $f_1(x, y)$ satisfies the conditions (A1)-(A4), $f_2(x, y)$ satisfies the condition (A5) and the function ϕ must satisfy the conditions (A1)-(A4). This shows that for each $(x, y) \in C \times C$, $F(x, y)$ satisfies the conditions (A1)-(A4). So, when we study the solution of MEP, we only need to study the solution of the equilibrium (1.1). This also shows that some "so-called" mixed equilibrium problem studied in [15-17] is still the equilibrium problem (1.1).

Remark 4. 2 Let us recall some well-known definitions. A mapping $T : C \to C$ is said to be

(1) *v-expansive* if $||Tx - Ty|| \geq v||x - y||$ for all $x, y \in C$. In particular, if $v = 1$, then T is called *expansive*.

(2) *v-strongly monotone* if there exists a constant $v > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq v||x - y||^2, \quad \forall x, y \in C.$$

Clearly, any v-strongly monotone mapping is v-expansive.

(3) *u-inverse strongly monotone* if there exists a constant $u > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq u||Tx - Ty||^2, \quad \forall x, y \in C.$$

(4) *L-Lipschitz continuous* if $||Tx - Ty|| \leq L||x - y||$ for all $x, y \in C$. In particular, if $L = 1$, then T is called *nonexpansive*.

It is easy to see that a u-inverse strongly monotone operator is $\frac{1}{u}$-Lipschitz continuous.

For the problem GEP, if the nonlinear operator $A : C \to H$ is a u-inverse strongly monotone operator and the bi-function $f(x, y)$ admits the conditions (A1)-(A4), we argue that the problem GEP is still the problem (1.1) and so it is indeed not a generalization. In fact, if A is a u-inverse strongly monotone operator from C into H, then A is a continuous operator. So, we obtain easily that the function $(x, y) \to \langle Ax, y - x\rangle$, $\forall x, y \in C$, satisfies the conditions (A1)-(A4). Hence, if we put $F(x, y) = f(x, y) + \langle Ax, y - x\rangle \geq 0$, then the problem GEP studied in [18-20] is still the problem (1.1).

5 Conclusion

The problem MEP studied in [15-17] and the problem GEP studied in [18-20] are still the problem (1.1) studied in the literature [5-11,21-24] and others.

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Author details
[1]Department of Mathematics, Honghe University, Yunnan, 661100, China [2]Department of Mathematics, National Kaohsiung Normal University, Kaohsiung 824, Taiwan

Authors' contributions
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