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Inclusions between weighted Orlicz spaces

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Abstract

Let Φ be a Young function and w be a weight. The weighted Orlicz space L_w^Φ is a natural generalization of the weighted Lebesgue space L_w^p ($1 \leq p \leq \infty$) and a characterization of an inclusion between weighted Lebesgue spaces is well known. In this study, we will investigate the inclusions between weighted Orlicz spaces $L_{w_1}^{\Phi_1}$ and $L_{w_2}^{\Phi_2}$ with respect to Young functions Φ_1, Φ_2 and weights w_1, w_2. Also, we give necessary and sufficient conditions for the equality of these two weighted Orlicz spaces under some conditions. Thereby, we obtain our result.

MSC: 46E30

Keywords: weighted Orlicz spaces; Young function; Lebesgue space; inclusion

1 Introduction and preliminaries

Generally Orlicz spaces are a natural generalization of the classical Lebesgue spaces L^p, $1 \leq p \leq \infty$ and there are many studies of Orlicz spaces in the literature (for example [1, 2]). In [3], the inclusion between L^p spaces is investigated with respect to the measure space (X, Σ, μ) and in [4] inclusions between Orlicz spaces are examined for a finite measure space and in general [5, 6]. Also, inclusions between weighted L^p spaces with respect to weights are studied in [7, 8] for a locally compact group with Haar measure. In this paper, we will investigate the inclusion between weighted Orlicz spaces $L_w^\Phi(X)$ with respect to a Young function Φ and a weight w for a general measure space. To this aim we will give the definition of a weighted Orlicz norm which depends on the usual Orlicz norm and we show that the inclusion map between the weighted Orlicz spaces is continuous. Also, we obtain the result that two weighted Orlicz spaces can be comparable with respect to Young functions for any measure space, although the weighted L^p spaces are not comparable with respect to the numbers p. Moreover, in the case of $X = \mathbb{R}^n$ we generalize some results in [7] to the weighted Orlicz spaces and we establish necessary and sufficient conditions on the weights w_1 and w_2 in order that $L_{w_1}^\Phi(\mathbb{R}^n) = L_{w_2}^\Phi(\mathbb{R}^n)$.

A non-zero function $\Phi : [0, +\infty) \to [0, +\infty]$ is called a Young function if Φ is convex and satisfies the conditions $\lim_{x \to 0^+} \Phi(x) = \Phi(0) = 0$ and $\lim_{x \to +\infty} \Phi(x) = +\infty$. We say that a Young function Φ satisfies the Δ_2 condition if there exists a $K > 0$ such that $\Phi(2x) \leq K\Phi(x)$ for all $x \geq 0$. Also, for a Young function Φ, the complementary Young function Ψ of Φ is given by

$$\Psi(y) = \sup\{xy - \Phi(x) : x \geq 0\}$$

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for $y \geq 0$. If Ψ is the complementary function of Φ, then Φ is the complementary of Ψ and (Φ, Ψ) is called a complementary pair of Young functions. We have the Young inequality for the complementary functions Φ and Ψ,

$$x \cdot y \leq \Phi(x) + \Psi(y) \quad (x, y \geq 0).$$

Let (X, Σ, μ) be a measure space. We will assume that μ is a σ-finite measure. Given a Young function Φ, the Orlicz space $L^\Phi(X, \Sigma, \mu)$ or simply $L^\Phi(X)$ is defined by

$$L^\Phi(X) = \left\{ f : X \to \mathbb{C} \,\Big|\, \int_X \Phi\big(\alpha \big|f(x)\big|\big)\, d\mu(x) < +\infty \text{ for some } \alpha > 0 \right\},$$

where f shows μ-equivalence classes of measurable functions. Then the Orlicz space is a Banach space under the (Orlicz) norm $\|\cdot\|_\Phi$ defined for $f \in L^\Phi(X)$ by

$$\|f\|_\Phi = \sup\left\{ \int_X \big|f(x)v(x)\big|\, d\mu(x) \,\Big|\, \int_X \Psi\big(\big|v(x)\big|\big)\, d\mu(x) \leq 1 \right\},$$

where Ψ is the complementary Young function of Φ.

For further information as regards Orlicz spaces, the reader is referred to [4–6].

Remark 1.1 By using the Young inequality and the definition of the norm $\|\cdot\|_\Phi$, it is easy to see that a measurable function $f : X \to \mathbb{C}$ is in $L^\Phi(X)$ if and only if $\|f\|_\Phi < \infty$.

Now, let (X, Σ, μ) be a measure space and let Φ be a Young function. If w is a weight on X (*i.e.* $w : X \to (0, +\infty)$ is measurable function) then the weighted Orlicz spaces, denoted by $L^\Phi_w(X)$, are defined as follows:

$$L^\Phi_w(X) := \big\{ f \,|\, fw \in L^\Phi(X) \big\}.$$

If we define

$$\|f\|_{\Phi,w} := \|fw\|_\Phi$$

for all $f \in L^\Phi_w(X)$, then the function $\|\cdot\|_{\Phi,w}$ defines a norm on $L^\Phi_w(X)$, and it is called a weighted Orlicz norm.

For $w = 1$ the norm $\|\cdot\|_{\Phi,w}$ reduces to the usual Orlicz norm $\|\cdot\|_\Phi$ and we obtain the Orlicz space $(L^\Phi(X), \|\cdot\|_\Phi)$.

Let $1 \leq p < \infty$. Then, for the Young functions $\Phi(x) = \frac{x^p}{p}$, the space $L^\Phi_w(X)$ becomes the weighted Lebesgue space $L^p_w(X)$ and the norm $\|\cdot\|_{\Phi,w}$ is equivalent to the classical norm $\|\cdot\|_{p,w}$ in $L^p_w(X)$. In particular, if $p = 1$ then the complementary Young function of $\Phi(x) = x$ is

$$\Psi(x) = \begin{cases} 0, & 0 \leq x \leq 1, \\ +\infty, & x > 1, \end{cases} \tag{1}$$

and in this case $\|f\|_{\Phi,w} = \|f\|_{1,w}$ for all $f \in L^1_w(X)$, since $\int_X \Psi(|v(x)|)\, d\mu(x) \leq 1$ is true if and only if $|v(x)| \leq 1$ almost everywhere on X.

Also, if $p = +\infty$ then, for the Young function Φ given by (1), the space $L_w^{\Phi}(X)$ is equal to the space $L_w^{\infty}(X) = \{f : X \to \mathbb{C} | fw \in L^{\infty}(X)\}$. We have $\|f\|_{\Phi,w} = \|f\|_{\infty,w}$ for all $f \in L_w^{\infty}(X)$.

It can be shown that the weighted Orlicz space is also a Banach space by using the completeness of the usual Orlicz space.

Proposition 1.2 *$(L_w^{\Phi}(X), \|\cdot\|_{\Phi,w})$ is a Banach space.*

Proof To show that $(L_w^{\Phi}(X), \|\cdot\|_{\Phi,w})$ is a Banach space, take an arbitrary absolutely convergent series $\sum_{n=1}^{\infty} f_n$ in $L_w^{\Phi}(X)$. Then

$$\sum_{n=1}^{\infty} \|f_n w\|_{\Phi} = \sum_{n=1}^{\infty} \|f_n\|_{\Phi,w} < +\infty.$$

Thus, $\sum_{n=1}^{\infty} f_n w$ is absolutely convergent in the Orlicz space $L^{\Phi}(X)$. Since the Orlicz space $(L^{\Phi}(X), \|\cdot\|_{\Phi})$ is a Banach space, there exists a function $g \in L^{\Phi}(X)$ such that $\sum_{n=1}^{N} \|f_n w - g\|_{\Phi} \to 0$, $N \to +\infty$. If we set $f = \frac{g}{w}$ then $f \in L_w^{\Phi}(X)$ and

$$\left\| \sum_{n=1}^{N} f_n - f \right\|_{\Phi,w} = \left\| \left(\sum_{n=1}^{N} f_n \right) w - fw \right\|_{\Phi} \to 0,$$

where $N \to +\infty$. So the space $L_w^{\Phi}(X)$ becomes a Banach space. □

2 Main result

Let (X, Σ, μ) be a measure space, w_1 and w_2 be two weights on X and let Φ_1, Φ_2 be two Young functions. We will investigate the inclusion between the weighted Orlicz spaces $L_{w_1}^{\Phi_1}(X)$ and $L_{w_2}^{\Phi_2}(X)$. For this investigation we need some definitions.

Let w_1 and w_2 be two weights on X. If there exists a $c > 0$ such that

$$w_1(x) \leq c \cdot w_2(x)$$

for all $x \in X$, then we write $w_1 \preccurlyeq w_2$. If $w_1 \preccurlyeq w_2$ and $w_2 \preccurlyeq w_1$ then we say that w_1 and w_2 are equivalent and write $w_1 \approx w_2$. For example, $w_1(x) = (1 + |x|)$ and $w_2(x) = e^{|x|}$ are weights on $\mathbb{R}$ and it is clear that $w_1 \preccurlyeq w_2$ for $c = 1$.

Let Φ_1 and Φ_2 be two Young functions. Then we say that Φ_2 is stronger than Φ_1, $\Phi_1 \prec \Phi_2$ in symbols, if there exist a $c > 0$ and $T \geq 0$ (depending on c) such that $\Phi_1(x) \leq \Phi_2(cx)$ for all $x \geq T$. If $T = 0$ then we write $\Phi_1 \prec \Phi_2$ $(T = 0)$. If $\Phi_1 \prec \Phi_2$ and $\Phi_2 \prec \Phi_1$ then we write $\Phi_1 \backsimeq \Phi_2$. The same notation is valid for the case $(T = 0)$ [6].

Remark 2.1 It is clear that $\Phi_1 \prec \Phi_2$ $(T = 0)$ implies $\Phi_1 \prec \Phi_2$. But $\Phi_1 \prec \Phi_2$ is not sufficient to investigate the inclusion between weighted Orlicz spaces when the measure μ is not finite. So we need the condition $\Phi_1 \prec \Phi_2$ $(T = 0)$ for infinite measures. Also, if $\Phi_1 \prec \Phi_2$ $(T = 0)$ then $\Psi_2 \prec \Psi_1$ $(T = 0)$ for the complementary Young functions Ψ_1 and Ψ_2 of Φ_1 and Φ_2, respectively.

Example 2.2 $\Phi_1(x) = \cosh(x) - 1$, $x \geq 0$, and $\Phi_2(x) = e^x - x - 1$, $x \geq 0$, are Young functions and satisfy the inequality

$$\Phi_1(x) = \cosh(x) - 1 \leq e^{2x} - 2x - 1 = \Phi_2(2x)$$

for all $x \geq 0$. Thus, $\Phi_1 \prec \Phi_2$ $(T = 0)$ for $c = 2$.

Now we can give the following theorem for the inclusion between the weighted Orlicz spaces $L_{w_1}^{\Phi_1}(X)$ and $L_{w_2}^{\Phi_2}(X)$.

Theorem 2.3 *If* $\Phi_1 \prec \Phi_2$ $(T = 0)$ *and* $w_1 \preccurlyeq w_2$, *then* $L_{w_2}^{\Phi_2}(X) \subseteq L_{w_1}^{\Phi_1}(X)$.

Proof Suppose that $\Phi_1 \prec \Phi_2$ and $w_1 \preccurlyeq w_2$. Let $f \in L_{w_2}^{\Phi_2}(X)$. Then there exists $\alpha > 0$ such that

$$\int_X \Phi_2\bigl(\alpha \cdot w_2(x) \cdot \bigl|f(x)\bigr|\bigr)\, d\mu(x) < +\infty.$$

On the other hand, since $w_1 \preccurlyeq w_2$ and $\Phi_1 \prec \Phi_2$ $(T = 0)$ there exist numbers $c > 0$ and $c' > 0$ such that

$$w_1(x) \leq c' \cdot w_2(x) \quad \forall x \in X \tag{2}$$

and

$$\Phi_1(y) \leq \Phi_2(c \cdot y) \quad \forall y \geq 0. \tag{3}$$

If we set $\beta = \frac{\alpha}{c \cdot c'} > 0$ then from (2) we get

$$\Phi_1\bigl(\beta w_1(x) \cdot \bigl|f(x)\bigr|\bigr) \leq \Phi_1\bigl(\beta \cdot c' w_1(x) \cdot \bigl|f(x)\bigr|\bigr) = \Phi_1\left(\frac{\alpha}{c} \cdot w_2(x) \cdot \bigl|f(x)\bigr|\right)$$

for all $x \in X$, since the Young function Φ_1 is increasing. Then we obtain

$$\Phi_1\left(\frac{\alpha}{c} \cdot w_2(x) \cdot \bigl|f(x)\bigr|\right) \leq \Phi_2\left(c \cdot \frac{\alpha}{c} \cdot w_2(x) \cdot \bigl|f(x)\bigr|\right) = \Phi_2\bigl(\alpha \cdot w_2(x) \cdot \bigl|f(x)\bigr|\bigr)$$

for all $x \in X$ from (3). So,

$$\int_X \Phi_1\bigl(\beta w_1(x) \cdot \bigl|f(x)\bigr|\bigr)\, dx \leq \int_X \Phi_2\bigl(\alpha \cdot w_2(x)\bigl|f(x)\bigr|\bigr)\, dx < +\infty.$$

Thus $f \in L_{w_1}^{\Phi_1}(X)$. □

Remark 2.4 The converse of Theorem 2.3 is not true in general. The following example shows this.

Example 2.5 Let (X, Σ, μ) be a measure space with $\mu(X) < +\infty$ and let w be a weight on X. If $1 < p_1 < p_2 < +\infty$, then for the Young functions

$$\Phi_1(x) = \frac{x^{p_1}}{p_1}, \qquad \Phi_2(x) = \frac{x^{p_2}}{p_2}, \quad x \geq 0,$$

the weighted Orlicz spaces $L_w^{\Phi_1}(X)$ and $L_w^{\Phi_2}(X)$ become the weighted Lebesgue spaces $L_w^{p_1}(X)$ and $L_w^{p_2}(X)$, respectively. Here, $L_w^{p_2}(X) \subseteq L_w^{p_1}(X)$. But Φ_2 is not stronger than Φ_1 for $T = 0$. Indeed, if we assume that $\Phi_1 \prec \Phi_2$ $(T = 0)$, then there exists $c > 0$ such that $\Phi_1(x) \leq \Phi_2(cx)$ for all $x \geq 0$. This says that for all $x > 0$, $\frac{x^{p_1}}{p_1} \leq \frac{(cx)^{p_2}}{p_2}$ and so $\frac{1}{x^{p_2-p_1}} \leq \frac{p_1 \cdot c^{p_2}}{p_2}$ for all $x \geq 0$. Then we get a contradiction, if we pass to the limit that x goes to zero, since $p_2 - p_1 > 0$.

Now, let μ be a finite measure. If we take $\Phi_1 \prec \Phi_2$ instead of $\Phi_1 \prec \Phi_2$ $(T = 0)$ in Theorem 2.3, then by using similar techniques as in [5, Theorem 3.17.1], we get the following proposition.

Proposition 2.6 *Let $\mu(X) < \infty$. If $\Phi_1 \prec \Phi_2$ and $w_1 \preccurlyeq w_2$; then $L_{w_2}^{\Phi_2}(X) \subseteq L_{w_1}^{\Phi_1}(X)$.*

By considering Theorem 2.3 we derive the following corollary.

Corollary 2.7 *If $\Phi_1 \backsimeq \Phi_2$ $(T = 0)$ (or $\Phi_1 \backsimeq \Phi_2$ when $\mu(X) < \infty$) and $w_1 \approx w_2$ then $L_{w_1}^{\Phi_1}(X) = L_{w_2}^{\Phi_2}(X)$.*

Before we investigate the inclusions between the weighted Orlicz spaces with respect to the Young function and weight, respectively, we will show that, if $L_{w_2}^{\Phi_2}(X) \subseteq L_{w_1}^{\Phi_1}(X)$ then the inclusion map $i : (L_{w_2}^{\Phi_2}(X), \|\cdot\|_{\Phi_2,w_2}) \to (L_{w_1}^{\Phi_1}(X), \|\cdot\|_{\Phi_1,w_1})$ is continuous.

Proposition 2.8 *Let (X, Σ, μ) be a measure space. If Φ_1, Φ_2 are Young functions and w_1, w_2 are weights, then $L_{w_2}^{\Phi_2}(X) \subseteq L_{w_1}^{\Phi_1}(X)$ if and only if there exists a $c > 0$ such that $\|f\|_{\Phi_1,w_1} \leq c \cdot \|f\|_{\Phi_2,w_2}$ for all $f \in L_{w_2}^{\Phi_2}(X)$.*

Proof For the sufficiency part of the proof assume that the condition given in the theorem is true. Conversely, suppose that $L_{w_2}^{\Phi_2}(X) \subseteq L_{w_1}^{\Phi_1}(X)$ is true. Let $f \in L_{w_2}^{\Phi_2}(X)$. Then $\|f\|_{\Phi_2,w_2} < +\infty$ and $\|f\|_{\Phi_1,w_1} < +\infty$. Then we can define

$$\big|\|f\|\big| = \|f\|_{\Phi_1,w_1} + \|f\|_{\Phi_2,w_2}$$

for all $f \in L_{w_2}^{\Phi_2}(X)$, and hence $(L_{w_2}^{\Phi_2}(X), |\|\cdot\||)$ becomes a normed space.

Moreover, the norm convergence of (f_n) in the Orlicz space $L^{\Phi}(X)$ implies the convergence almost everywhere of some subsequence (f_{n_k}) on X [6]. So, a similar assertion holds for the weighted Orlicz space since a weight w is strictly positive. Then, by using similar techniques as in the Lebesgue space case, it can be shown that the normed space $(L_{w_2}^{\Phi_2}(X), |\|\cdot\||)$ is a Banach space.

If we consider the mapping $T : (L_{w_2}^{\Phi_2}(X), |\|\cdot\||) \to (L_{w_2}^{\Phi_2}(X), \|\cdot\|_{\Phi_2,w_2})$, $Tf = f$ for all $f \in L_{w_2}^{\Phi_2}(X)$. Then from the closed graph theorem there exists a $c > 0$ such that

$$\|f\|_{\Phi_1,w_1} \leq \big|\|f\|\big| \leq c \cdot \|f\|_{\Phi_2,w_2}$$

for all $f \in L_{w_2}^{\Phi_2}(X)$. □

We can summarize the above results in the following corollary.

Corollary 2.9 *Let (X, Σ, μ) be a measure space, let Φ_1, Φ_2 be two Young functions, and let w_1, w_2 be two weights. If we have the conditions*
(i) $\Phi_1 \prec \Phi_2$ $(T = 0)$ *and* $w_1 \preccurlyeq w_2$,
(ii) $L^{\Phi_2}_{w_2}(X) \subseteq L^{\Phi_1}_{w_1}(X)$,
(iii) *there exists* $c > 0$ *such that* $\|f\|_{\Phi_1,w_1} \leq c \cdot \|f\|_{\Phi_2,w_2}$ *for all* $f \in L^{\Phi_2}_{w_2}(X)$,
then we have (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii).

Remark 2.10 Note that a similar result can be shown if we change $\Phi_1 \prec \Phi_2$ $(T = 0)$ by $\Phi_1 \prec \Phi_2$ for a finite measure space.

Now, let w be a weight on X and fix $w_1 = w_2 = w$. We will investigate the inclusions between the weighted Orlicz spaces $L^{\Phi_1}_w(X)$ and $L^{\Phi_2}_w(X)$ with respect to Young functions Φ_1 and Φ_2.

Since a weight w satisfies $w \approx w$, we get the following consequences of Theorem 2.3.

Corollary 2.11 *Let Φ_1 and Φ_2 be two Young functions and let w be a weight. If* $\Phi_1 \prec \Phi_2$ $(T = 0)$, *then* $L^{\Phi_2}_w(X) \subseteq L^{\Phi_1}_w(X)$.

Remark 2.12 Corollary 2.11 shows that the weighted Orlicz spaces $L^{\Phi_1}_w(X)$ and $L^{\Phi_2}_w(X)$ can be comparable with respect to Young functions Φ_1 and Φ_2, although the weighted Lebesgue spaces $L^{p_1}_w(X)$ and $L^{p_2}_w(X)$ cannot be comparable with respect to $p_1, p_2 \in [1, \infty)$ in general (for instance $X = \mathbb{R}$ with Lebesgue measure μ).

Remark 2.13 The converse of the Corollary 2.11 is not true in general (see Example 2.5).

On the other hand, if we consider the finite measure space then by using a similar method as in [5] we can derive the following corollary.

Corollary 2.14 *Let $\mu(X) < \infty$ and w be a fixed weight on X. If Φ_1, Φ_2 are Young functions, then* $\Phi_1 \prec \Phi_2$ *if and only if* $L^{\Phi_2}_w(X) \subseteq L^{\Phi_1}_w(X)$.

Remark 2.15 If we combine this corollary with the result mentioned in Remark 2.10 then we get the following.

Corollary 2.16 *Let Φ_1, Φ_2 be two Young functions. If w is a weight then the following statements are equivalent for a finite measure space.*
(i) $\Phi_1 \prec \Phi_2$.
(ii) $L^{\Phi_2}_w(X) \subseteq L^{\Phi_1}_w(X)$.
(iii) *There exists* $c > 0$ *such that* $\|f\|_{\Phi_1,w} \leq c \cdot \|f\|_{\Phi_2,w}$ *for all* $f \in L^{\Phi_2}_w(X)$.

Let $1 < p_1, p_2 < \infty$ be two numbers. If we take $\Phi_1(x) = \frac{x^{p_1}}{p_1}$, $\Phi_2(x) = \frac{x^{p_2}}{p_2}$ in Corollary 2.16, we get the following well-known result.

Corollary 2.17 *Let $1 < p_1, p_2 < \infty$. The following statements are equivalent for a finite measure space.*
(i) $p_1 < p_2$.
(ii) $L^{p_2}_w(X) \subseteq L^{p_1}_w(X)$.
(iii) *There exists* $c > 0$ *such that* $\|f\|_{p_1,w} \leq c \cdot \|f\|_{p_2,w}$ *for all* $f \in L^{p_2}_w(X)$.

Remark 2.18 If we combine this corollary with the result mentioned in Remark 2.10 then we get the following.

Now, let Φ be a Young function and let w_1, w_2 be two weights. Fixing $\Phi_1 = \Phi_2 = \Phi$, we will study the inclusion between the weighted Orlicz spaces $L^{\Phi}_{w_1}(X)$ and $L^{\Phi}_{w_2}(X)$ with respect to the weights w_1 and w_2.

We get the following corollaries of Theorem 2.3, since $\Phi \prec \Phi$.

Corollary 2.19 *If $w_1 \preccurlyeq w_2$, then $L^{\Phi}_{w_2}(X) \subseteq L^{\Phi}_{w_1}(X)$.*

Corollary 2.20 *If $w_1 \approx w_2$ then $L^{\Phi}_{w_2}(X) = L^{\Phi}_{w_1}(X)$.*

We will show that the converse of the Corollary 2.19 is true when $X = \mathbb{R}^n$ and μ is the Lebesgue measure on $\mathbb{R}^n$. To do this, we will assume

$$w(x+y) \le w(x) \cdot w(y), \quad \text{for all } x, y \in \mathbb{R}^n, \tag{4}$$

for the weight w, then by using similar techniques in [7, 8], it is easy to see that the weighted Orlicz spaces $L^{\Phi}_{w}(\mathbb{R}^n)$ have the following properties.

Lemma 2.21 *Let w be a weight on $\mathbb{R}^n$ satisfying the condition (4). If Φ is a continuous Young function satisfying the Δ_2 condition, then:*

(i) *For all $f \in L^{\Phi}_{w}(\mathbb{R}^n)$ and for all $x \in \mathbb{R}^n$ $L_x f \in L^{\Phi}_{w}(\mathbb{R}^n)$ and $\|L_x f\|_{\Phi,w} \le w(x)\|f\|_{\Phi,w}$.*
(ii) *If $f \in L^{\Phi}_{w}(\mathbb{R}^n)$, then the map $x \mapsto L_x f$ from $\mathbb{R}^n$ to $L^{\Phi}_{w}(\mathbb{R}^n)$ is continuous.*
(iii) *If $f \in L^{\Phi}_{w}(\mathbb{R}^n)$ and $f \neq 0$, then there exists a $c > 0$ (depends on f) such that*

$$\frac{1}{c} \cdot w(x) \le \|L_x f\|_{\Phi,w} \le c \cdot w(x).$$

Now, we can give the necessary and sufficient conditions for the inclusion between the weighted Orlicz spaces $L^{\Phi}_{w_1}(\mathbb{R}^n)$ and $L^{\Phi}_{w_2}(\mathbb{R}^n)$.

Theorem 2.22 *Let Φ be a continuous Young function satisfying Δ_2 condition and w_1, w_2 be two weights on $\mathbb{R}^n$ satisfying the condition (4). Then $L^{\Phi}_{w_2}(\mathbb{R}^n) \subseteq L^{\Phi}_{w_1}(\mathbb{R}^n)$ if and only if $w_1 \preccurlyeq w_2$.*

Proof If $w_1 \preccurlyeq w_2$, then it is clear that $L^{\Phi}_{w_2}(\mathbb{R}^n) \subseteq L^{\Phi}_{w_1}(\mathbb{R}^n)$ from Corollary 2.19. Conversely, assume that $L^{\Phi}_{w_2}(\mathbb{R}^n) \subseteq L^{\Phi}_{w_1}(\mathbb{R}^n)$. Then, by Proposition 2.8, there exists a $d > 0$ such that $\|f\|_{\Phi,w_1} \le d \cdot \|f\|_{\Phi,w_2}$ for all $f \in L^{\Phi}_{w_2}(\mathbb{R}^n)$. If we fix $f \in L^{\Phi}_{w_2}(\mathbb{R}^n) \subseteq L^{\Phi}_{w}(\mathbb{R}^n)$ then from Lemma 2.21(iii), there exist $c_1, c_2 > 0$ such that

$$\frac{1}{c_1} \cdot w_1(x) \le \|L_x f\|_{\Phi,w_1} \le c_1 \cdot w_1(x)$$

and

$$\frac{1}{c_2} \cdot w_2(x) \le \|L_x f\|_{\Phi,w_2} \le c_2 \cdot w_2(x)$$

for all $x \in \mathbb{R}^n$. So, we get the inequality

$$\frac{1}{c_1} \cdot w_1(x) \le \|L_x f\|_{\Phi,w_1} \le d \cdot \|f\|_{\Phi,w_2} \le c_2 \cdot d \cdot w_2(x)$$

for all $x \in \mathbb{R}^n$. This shows that $w_1(x) \le c \cdot w_2(x)$ for all $x \in \mathbb{R}^n$ where $c = c_1 \cdot c_2 \cdot d > 0$. Thus $w_1 \preccurlyeq w_2$. □

The following is an easy consequence of Theorem 2.22.

Corollary 2.23 $L^{\Phi}_{w_2}(\mathbb{R}^n) = L^{\Phi}_{w_1}(\mathbb{R}^n)$ *if and only if* $w_1 \approx w_2$.

Also, if we combine Theorem 2.22 and Corollary 2.19 we obtain the following corollary.

Corollary 2.24 *Let* Φ *be a continuous Young function satisfying the* Δ_2 *condition. If* w_1 *and* w_2 *are weights satisfying condition* (4), *then the following statements are equivalent.*

(i) $w_1 \preccurlyeq w_2$.
(ii) $L^{\Phi}_{w_2}(\mathbb{R}^n) \subseteq L^{\Phi}_{w_1}(\mathbb{R}^n)$.
(iii) *There exists a* $c > 0$ *such that* $\|f\|_{\Phi,w_1} \le c \cdot \|f\|_{\Phi,w_2}$ *for all* $f \in L^{\Phi}_{w_2}(\mathbb{R}^n)$.

Let $1 < p < \infty$. If we take $\Phi(x) = \frac{x^p}{p}$ in Corollary 2.24, then Φ satisfies the Δ_2 condition so we get the following well-known result.

Corollary 2.25 *Let* $1 < p < \infty$ *and* w_1, w_2 *be two weights satisfying condition* (4), *then the following statements are equivalent.*

(i) $w_1 \preccurlyeq w_2$.
(ii) $L^{p}_{w_2}(\mathbb{R}^n) \subseteq L^{p}_{w_1}(\mathbb{R}^n)$.
(iii) *There exists a* $c > 0$ *such that* $\|f\|_{p,w_1} \le c \cdot \|f\|_{p,w_2}$ *for all* $f \in L^{p}_{w_2}(\mathbb{R}^n)$.

Competing interests
The author declares that he has no competing interests.

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