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POSITIVE SOLUTIONS OF THE DIOPHANTINE EQUATION

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ABSTRACT. Integral solutions of $x^3 + \lambda y + 1 - xyz = 0$ are observed for all integral λ. For $\lambda = 2$ the 13 solutions of the equation in positive integers are determined. Solutions of the equation in positive integers were previously determined for the case $\lambda = 1$.

KEY WORDS AND PHRASES. Diophantine equation, cubic, positive solution.

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1. INTRODUCTION.

The Diophantine equation

$$x^3 + \lambda y + 1 - xyz = 0 \tag{1}$$

is always satisfied by the positive triple $(2\lambda + 1, 2, 2\lambda^2 + 2\lambda + 1)$. For $\lambda = 1$, S. P. Mohanty [1] has given all 9 positive solutions of this equation and in a sequel [2] has given all integral solutions of this equation. In this paper we determine all of the 13 positive solutions of

$$x^3 + 2y + 1 - xyz = 0. \tag{2}$$

Equation (2) has an infinite number of integral solutions. For example, (-1,0,z), (-1,y,-2) are solutions of (2). In general (-1, 0, z) and $(-1, y, -\lambda)$ satisfy (1).

THEOREM. There are only a finite number of solutions of (2) in positive integers.

PROOF. As in [1] we write the given equation (2) as an equivalent system. If (x,y,z) satisfies (2), then $x|2y + 1$ and $y|x^3 + 1$. Conversely, if x, y are positive integers for which $x|2y + 1$, $y|x^3 + 1$, then $xy|x^3 + 2y + 1$ hence for some positive z one has $x^3 + 2y + 1 - xyz = 0$.

Hereafter, we focus attention on the system $x|2y + 1$, $y|x^3 + 1$. If (x, y) are positive integers for which these statements prevail, then there are positive integers r, s for which

$$rx = 2y + 1 \tag{3}$$

$$sy = x^3 + 1 \tag{4}$$

Eliminating y from (3), (4), one has

$$s(rx - 1) = 2x^3 + 2 \tag{5}$$

which may be written as

$$x(sr - 2x^2) = s + 2 \tag{6}$$

Let $n = sr - 2x^2$, a positive integer, to secure $xn = s + 2$ from (6).
Then

$$2x^2 = sr - n = (xn - 2)r - n = rnx - (2r + n). \tag{7}$$

The extremes of this equation imply $2x < rn$ from which we gain the existence of a postive integer for which

$$rn = 2x + k \tag{8}$$

Combining (7), (8) we have

$$xk = 2r + n \tag{9}$$

and finally, that

$$(n - 2)(r - 1) + (x - 1)(k - 2) = 4. \tag{10}$$

If we write

$$A = (n - 2)(r - 1)$$

$$B = (x - 1)(k - 2)$$

then (10) becomes $A + B = 4$. We continue the proof by considering the cases $A < 0$, $B < 0$, $A = 0$, $B = 0$, and then the case where A, B are both positive.

Case $A < 0$. For this case, $n = 1$ and $B > 0$ (in particular, $k > 2$).
From (10),

$$x = 1 + \frac{r + 3}{k - 2} \tag{11}$$

From (8), with $n = 1$,

$$x = 1 + \frac{2x + k + 3}{k - 2}$$

and hence

$$x = \frac{2k + 1}{k - 4} = 2 + \frac{9}{k - 4}$$

Thus, $k - 4 \mid 9$ and $k = 5, 7, 13, 3, 1, -5$. For $k = -5$, y is negative; for $k = 1, 3$, x is negative. Given k, $x = (2k + 1)/(k - 4)$, $r = 2x + k$ and $y = (rx - 1)/2$. Starting this sequence with $k = 5, 7, 13$ one secures $(x, y) = (5, 42), (11, 148), (3, 28)$, respectively.

Case $B < 0$. This case implies $k = 1$ and $A > 0$ (in particular, $n > 2$). For $k = 1$, (8), (9) becomes $rn = 2x + 1$ and $x = 2r + n$. If we eliminate n from these equations, we secure

$$(r - 2)\, x = 2r^2 + 1 \tag{12}$$

The case $r = 1$ is included below (Case $A = 0$). $r = 2$ imples $x = 4 + n$ by (9). Since $2y = rx - 1 = 2n + 7$, y is not an integer and so no solution results from $r = 2$. We now consider $r > 2$ and write

$$x = \frac{2r^2 + 1}{r - 2} = 2r + 4 + \frac{9}{r - 2}$$

from which we infer that $r = 3, 5, 11, 1, -1, -7$. For the last three values, $x < 0$. For $r = 3, 5, 11$ we calculate $x = (2r^2 + 1)/(r - 2)$, $y = (rx - 1)/2$ to secure, respectively, the pairs $(x, y) = (19, 28), (17, 42), (27, 148)$.

Case $A = 0$. In this case, $B = 4$. Since $B = 4$, $(x, k) = (2, 6), (3, 4), (5, 3)$. Since $A = 0$, either $r = 1$ or $n = 2$. If $r = 1$ we recall that $2y = x - 1$ (from (3)) hence $(x, y) = (3, 1), (5, 2)$ result as solutions ($x = 2$ does not give an integral y). If $n = 2$ we compute r from $2r = 2x + k$ (equation (8)) and then compute y from $2y = rx - 1$ to secure one usable $r(=5)$ from which the solution $(x, y) = (3, 7)$ results.

Case $B = 0$. This case is similar to $A = 0$ and gives three pairs $(x, y) = (5, 7), (1, 2), (1, 1)$.

Case $A > 0$ and $B > 0$. This gives three subcases. (a) $A = 1$, $B = 3$; (b) $A = B = 2$; (c) $A = 3$, $B = 1$. Clearly, these cases yield a finite number of solutions since, in particular, x and r are bounded and, because of (3), y may be determined from them.

For (a) we have $(n - 2)(r - 1) = 1$, $(x - 1)(k - 2) = 3$. Thus, $n = 3$ and $r = 2$. None of the possible pairs $(x, k) = (2, 5), (4, 3)$ gives an integral y.

For (b), $(n - 2)(r - 1) = (x - 1)(k - 2) = 2$. Thus $(n, r) = (4, 2), (3, 3)$ and $(x, k) = (2, 4), (3, 3)$. The pair $r = 3$, $x = 3$ yields the only solution, $(x, y) = (3, 4)$.

Similarly, for (c) one secures no solution.

This concludes the proof of the theorem.

We conclude by giving the complete set of positive triples (x,y,z) for which (2) is satisfied: (1,1,4), (1,2,3), (3,1,10), (3,4,3), (3,7,3), (3,28,1), (5,2,13), (5,7,4), (5,42,1), (11,148,1), (17,42,7), (19,28,13), (27,148,5).

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